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Rational Bernstein Collocation Method for Solving the Steady Flow of a Third Grade Fluid in a Porous Half Space

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Abstract. In the current work, a mathematical model which describes the steady flow of a third grade fluid in a porous half space is investigated numerically. An approximate expression for solution of the governing non-linear two point boundary value problem on semi-infinite domain is developed as a combination of rational Bernstein functions. A spectral collocation method based on the rational Bernstein functions is introduced and implemented to find numerical solution of the governing problem. The efficiency and accuracy of the proposed numerical technique is illustrated through the figures and tables. The effects of variations of various embedded parameters on the fluid velocity profile have been investigated graphically.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Rational Bernstein functions; Third grade fluid; Porous space; Semi-infinite domain; Spectral method.

1. Introduction

In recent years, many researchers have been focusing on studying and analyzing the non-Newtonian fluids models due to their widely applications for describing several phenomena in the natural sciences and engineering. Various practical fluid flows such as, salt solutions and molten polymers, certain crude oils, greases, paint and industrial wastes have been categorized as non-Newtonian fluids. Many mathematical models have recently been introduced and developed to describe and simulate several types of non-Newtonian fluid flows. In the current study, we focus on a special and interesting mathematical model which describes the steady flow of a non-Newtonian fluid in a porous medium. The proposed model has been introduced by Hayat et al. [14, 15], they have generalized the mathematical model of a second grade non-Newtonian fluid in porous medium

$$(\bigtriangledown p)_x = -\frac{\mu\varphi}{k}(1+\frac{\alpha_1}{\mu}\frac{\partial}{\partial t})u,$$

to the following modified Darcy's Law for a third grade fluid

$$(\bigtriangledown p)_x = -\frac{\varphi}{k}[\mu u + \alpha_1\frac{\partial u}{\partial t} + 2\beta_3(\frac{\partial u}{\partial y})^2 u]. \quad (1.\ 1)$$

In the above equations u, μ and p respectively denote the fluid velocity, dynamic viscosity and the pressure, α_1, β_3 are material constants and k and φ, respectively represent the permeability and porosity of the porous half space which occupies the region $y > 0$. Also they defined non dimensional fluid velocity f and the coordinate z

$$z = \frac{V_0}{\nu}y,\ f(z) = \frac{u}{V_0}, \quad (1.\ 2)$$

where $V_0 = u(0)$, $\nu = \frac{\mu}{\rho}$ represents the kinematic viscosity and ρ denotes the fluid density. Using the presented suitable similarity transformations, the steady state flow of a third grade fluid in a porous half space has been transformed to the following nonlinear boundary value problem(see [14] for more details)

$$\frac{d^2f}{dz^2} + b_1(\frac{df}{dz})^2\frac{d^2f}{dz^2} - b_2 f(\frac{df}{dz})^2 - cf = 0, \quad (1.\ 3)$$

$$f(0) = 1,\ f(z) \to 0\ as\ z \to \infty, \quad (1.\ 4)$$

where parameters are as follow:

$$b_1 = \frac{6\beta_3 V_0^4}{\mu\nu^2}, \quad b_2 = \frac{2\beta_3\varphi V_0^4}{\mu\nu}, \quad c = \frac{\varphi\nu^2}{kV_0^2}.$$

It is clear the parameters are not independent, since

$$b_2 = \frac{b_1 c}{3}.$$

Now our interest is to solve the nonlinear boundary problem (1. 3) with boundary conditions (1. 4). In [14] authors have been employed the Homotopy analysis method (HAM) for solving the governing problem. Faiz Ahmad [1] estimated a simple analytical solution for the problem. Very recently various semi-analytical and numerical techniques such as, Hankel-padè method [2, 3], spectral method based on the Modified Generalized Laguerre [19], rational Legendre Tau method [6] and radial basis collocation method [16, 23] have been formulated and used to solve the problem (1. 3)-(1. 4).

As was observed, non-Newtonian fluid mathematical models can be reduced to a nonlinear ordinary differential equation on a semi-infinite domain by using proper similarity transforms. Clearly finding the analytical solutions of these types of problems are very interesting and important for researchers. However, obtaining analytical solutions for such problems is not often an easy task. So, several numerical techniques have been recently introduced and developed to deal with these types of problems. Specially, spectral methods are very efficient and powerful numerical techniques for solving differential equations in bounded domains or with complex boundary conditions. In recent years, several types of spectral methods have been developed and successfully applied to deal with differential problems in unbounded domains. The commonly used techniques are spectral methods based on the orthogonal polynomial on unbounded domains such as Laguerre or Hermite polynomials [12, 11]. Another of most widely used spectral methods with domain truncation strategy. These techniques truncate unbounded domains to bounded intervals and employ classical orthogonal function to solve the problem supplemented with artificial boundary conditions [8]. Moreover, there is an very effective class of spectral methods for solving such problems based on the rational approximations. In [9], a new system of orthogonal functions, rational Chebyshev functions, has been defined which forms an orthogonal basis system on the semi-infinite domain. Also in [13], authors introduced rational Legendre basis functions and used them for solving differential equations on the half line. Newly, this class of numerical methods has been developed and widely employed to deal with many types of Newtonian and non-Newtonian fluids problems [20, 25, 5, 28, 21, 4].

Bernstein polynomials enjoy considerable popularity in many scientific branches because of their many useful properties [17]. Specially, the Bernstein functions are very powerful tools in computer-aided design applications [26]. Recently, some spectral methods based on the Bernstein polynomials have been introduced and employed to solve some types of boundary value problems [7, 27]. In [24] authors have been formulated a spectral method based on the Bernstein polynomials to approximate the fractional heat and wave-like equations. Numerical techniques based on the Bernstein polynomials are interesting alternatives for dealing with differential and integral equations due to their simple implementation. Because Bernstein polynomials are not orthogonal, some of orthogonal basis transformations could be used when it is necessary [10, 22]. In the current work rational Bernstein basis functions would be introduced and then a collocation method based on the new basis functions is formulated and employed to solve the steady flow problem of a third grade fluid in a porous half space.

This paper is arranged as follows: In Section 2 we describe relation between Legendre and Bernstein polynomials. In section 3, we present rational Legendre and introduce a new basis functions called rational Bernstein function. A collocation technique based on the rational Bernstein functions would be formulated and used for approximating the steady flow problem of a third grade fluid in a porous half space in section 4. In Section 5 we investigate behavior of the governing model via tables and figures. Finally, concluding remarks will be reported in Section 6.

2. Legendre and Bernstein basis functions

In this section some preliminary definitions of the Bernstein and Legendre functions which are required for establishing our main results, are reviewed. Moreover some results which describe the transformations between these two basis functions are explained.

2.1. **Bernstein functions.** The classical Bernstein polynomials of degree n on the interval $[0,1]$ are defined as

$$\chi_{r,n}(t) = \begin{pmatrix} n \\ r \end{pmatrix} t^r(1-t)^{n-r}, \quad r = 0,1,\ldots,n,\ t \in [0,1]. \tag{2. 5}$$

The set of Bernstein polynomials, $\{\chi_{0,\,n}(t)\,,\,\chi_{1,\,n}(t)\,,\,\cdots\,,\,\chi_{n,\,n}(t)\}$ forms a complete basis for space of polynomials over the interval $[0,1]$. The Bernstein polynomials satisfy the symmetry, $\chi_{r,n}(t) = \chi_{n-r,n}(1-t)$, non-negativity, $\chi_{r,n}(t) \geq 0$ and partitions of unity, $\sum_{r=0}^{n} \chi_{r,n}(t) = 1$, properties for all $t \in [0,1]$. Moreover, the Bernstein functions can be extended over an arbitrary interval $[\alpha,\beta]$ by mapping original domain into $[0,1]$ using $t = \frac{x-\alpha}{\beta-\alpha}$, so relation (2. 5) is converted to:

$$\chi_{r,n}(x) = \begin{pmatrix} n \\ r \end{pmatrix} \frac{(x-\alpha)^r(\beta-x)^{n-r}}{(\beta-\alpha)^n}, \quad r = 0,1,\ldots,n,\ x \in [\alpha,\beta]. \tag{2. 6}$$

For any real-valued continuous function $f(x)$ on the interval $[\alpha,\beta]$, the $n-th$ Bernstein polynomial for $f(x)$ is defined by:

$$B_n(f;x) = \sum_{i=0}^{n} f(\alpha + \frac{(\beta-\alpha)i}{n})\chi_{r,n}(x), \quad x \in [\alpha,\beta]. \tag{2. 7}$$

Theorem 1. Let $f \in C^k[\alpha,\beta]$, for some positive integer k, then $B_n^{(k)}(f;x)$ tends to $f^{(k)}(x)$ uniformly as $n \to \infty$.
Proof. See [18].

2.2. **Legendre functions.** The classical Legendre polynomials are orthogonal basis functions with respect to the weighting function $w(x) = 1$ on the interval $[-1,1]$. Moreover, the set of shifted Legendre functions on $[0,1]$ are generated by the following recurrence relation:

$$\begin{aligned} L_0(t) &= 1, \\ L_1(t) &= 2t-1, \\ L_{n+1}(t) &= \frac{2n+1}{n+1}(2t-1)L_n(t) - nL_{n-1}(t), \quad n = 2,3,\ldots. \end{aligned}$$

The shifted Legendre functions $\{L_n(t)\}$ satisfy the following orthogonality property:

$$\int_0^1 L_n(t)L_m(t)dt = \begin{cases} \frac{1}{2n+1} & n = m \\ 0 & n \neq m \end{cases}$$

Every function $f(t) \in L^2([0,1])$ can be approximated by using a linear combination of the shifted Legendre functions $\{L_n(t)\}_{n\geq 0}$ as follow:

$$f(t) = \sum_{n=0}^{\infty} l_n L_n(t),$$

where the Legendre coefficients $\{l_n\}$ can be easily computed by using:

$$l_n = (2n+1)\int_0^1 L_n(t)f(t)dt, \quad n = 0,1,2,\ldots.$$

2.3. **Transformation between Legendre and Bernstein functions.** Here, the relationship of transformations between the Legendre and Bernstein basis function will be reviewed. For this purpose, let $P_n(t)$ be a polynomial of degree n over interval $[0,1]$, then $P_n(t)$ can be expanded as:

$$P_n(t) = \sum_{i=0}^{n} \mu_i L_i(t) = \sum_{r=0}^{n} \gamma_r \chi_{r,n}(t)$$

where $\{L_i(t)\}_{i=0}^{n}$ and $\{\chi_{r,n}(t)\}_{r=0}^{n}$ are the shifted Legendre basis functions on $[0,1]$ and the classical Bernstein functions, respectively. The transformation of the classical Bernstein basis functions into the shifted Legendre polynomial basis of degree k can be given as follow:

Theorem 2. The shifted Legendre polynomial $L_k(t)$ over interval $[0,1]$ can be expressed in the Bernstein basis functions $\chi_{r,k}(t)$ of degree k as

$$L_k(t) = \sum_{r=0}^{k} (-1)^{r+k} \begin{pmatrix} k \\ r \end{pmatrix} \chi_{r,n}(t), \qquad (2.\ 8)$$

Proof: See [10].

3. Bernstein functions on unbounded intervals

As observed in the previous section the well-known Bernstein polynomials are defined only on the bounded interval, so the classical Bernstein functions should be extended to approximate the function with unbounded domain. In this section the well-known Bernstein polynomials would be extended and a new set of basis functions, called rational Bernstein functions, would be introduced. For this purpose, firstly the rational Legendre functions and their properties are briefly reviewed. Then by using the relationship of transformations between the Legendre and Bernstein basis functions, rational Bernstein functions are defined.

3.1. **Rational Legendre functions.** The well-known rational Legendre polynomials on $[0,\infty)$, denoted by $R_{l,n}(x)$, are defined as follows [8, 9]:

$$R_{l,0}(x) = 1,\ R_{l,1}(x) = \frac{x-L}{x+L},$$

$$R_{l,n+1}(x) = (\frac{2n+1}{n+1})(\frac{x-L}{x+L})R_{l,n}(x) - (\frac{n}{n+1})R_{l,n-1}(x),\ n \geq 1, \qquad (3.\ 9)$$

where L is a constant parameter. Let $w(x) = \frac{2L}{(x+L)^2}$ denotes a non-negative, integrable, real-valued function over the interval $\Lambda = [0,\infty]$. We define

$$L^2_w(\Lambda) = \{\nu :\ \Lambda \to \mathbb{R}|\ \nu \text{ is measurable and } \parallel \nu \parallel_w < \infty\},$$

where

$$\parallel \nu \parallel_w = (\int_0^\infty \mid \nu(x) \mid^2 w(x)dx)^{1/2}$$

is a norm induced by the following inner product:

$$(u,\nu)_w = \int_0^\infty u(x)\nu(x)w(x)dx. \qquad (3.\ 10)$$

Thus $\{R_{l,n}(x)\}_{n\geq 0}$ denotes a system which is mutually orthogonal the inner products defined in relation (3. 10), i.e.,

$$(R_{l,n}, R_{l,m})_w = \frac{2}{2n+1}\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta function. This system is complete in $L^2_w(\Lambda)$. For any function $u \in L^2_w(\Lambda)$ the following expansion holds

$$u(x) = \sum_{k=0}^{+\infty} a_k R_{l,k},$$

with

$$a_k = \frac{(u, R_{l,k})_w}{\| R_{l,k} \|_w}.$$

The a_k are the discrete expansion coefficients associated with the family $\{R_{l,k}\}$.

3.2. **Rational Bernstein basis functions.** In the current section, firstly, we introduce rational Bernstein functions on the semi-infinite interval $[0, \infty)$ by using a suitable algebraic map on the classical Bernstein polynomials. Then based on the relationship of transformations between the Legendre and Bernstein functions, we show that a set of rational Bernstein functions forms a basis functions.
So, for $z \in [0, \infty)$, $t \in [0, 1]$ and every fixed positive constant L, the following algebraic relation:

$$t = \frac{z}{z+L}, \tag{3. 11}$$

maps the semi-infinite interval $[0, \infty)$ into $[0, 1]$. Now by substituting the above algebraic map into the relation (2. 5), the rational Bernstein functions, $R\chi_{r,n}(z)$, are defined as follow:

$$\begin{aligned} R\chi_{r,n}(z) &= \binom{n}{r} (\frac{z}{z+L})^r (1 - \frac{z}{z+L})^{n-r} \\ &= \binom{n}{r} \frac{z^r L^{n-r}}{(z+L)^n}, \quad r = 0, 1, \ldots, n, \ z \in [0, \infty). \end{aligned} \tag{3. 12}$$

The behavior of these new rational functions for $n = 3$ and $L = 1$ are plotted in figure 1.

The constant parameter L sets the length scale of the mapping. Boyd in [8] offered guidelines for optimizing the map parameter L where $L > 0$ for some orthogonal rational functions. Numerical results depend smoothly on constant parameter L, and therefore are not very sensitive to L, so the error varies very slowly with L around the minimum. A little trial and error is usually sufficient to find a value that is nearly optimum. In general, there is no way to avoid a small amount of trial and error in choosing L when solving problems on an unbounded domain. Experience and the asymptotic approximations of Boyd can help, but some experimentation is always necessary as explained in his paper [9].
Clearly by substituting the mapping operator in the Theorem 2, one can find a similar transformations between the rational Legendre basis functions and new rational Bernstein functions on semi-infinite domain $[0, \infty)$. So clearly it can be concluded that the new rational Bernstein functions form basis functions.

In the next section an efficient numerical technique based on the the new rational functions is formulated and employed to investigated the behaviour of a third grade steady fluid flow in a porous half space.

3.3. **Formulation of rational Bernstein collocation method for solving the problem.** In this section a rational Bernstein collocation method is formulated and used to solve the governing two point boundary problem (1. 3) on the semi-infinite domain $[0,\infty)$ with boundary conditions (1. 4). For this purpose, the solution of the problem (1. 3) is approximated on the $span\{B\chi_{0,n}(z), B\chi_{1,n}(z), \ldots, B\chi_{n,n}(z)\}$ as follow:

$$f_n(z) = \sum_{r=0}^{n} \lambda_r B\chi_{r,n}(z), \tag{3. 13}$$

where $\{\lambda_r\}_{r=0}^n$ is a set of unknown coefficients should be determined. Substituting the proposed solution (3. 13) in governing problem (1. 3), the following residual function is obtained:

$$Res(z) \;=\; \frac{d^2 f_n}{dz^2} + b_1(\frac{df_n}{dz})^2\frac{d^2 f_n}{dz^2} - b_2 f_n(\frac{df_n}{dz})^2 - cf_n. \tag{3. 14}$$

Now based on the collocation method, the unknown coefficients in relation (3. 13) can be calculated by taking the residual function equal to zero at proper collocation points and enforcing the boundary conditions (1. 4). In our implementation the following shifted Chebyshev-Gauss-Radau points, $\{\varsigma_r\}_{r=0}^n$ are used as the collocation nodes:

$$\varsigma_r = \frac{Lx_r + L}{2(1-x_r)}, \quad r = 0,\ldots,n, \tag{3. 15}$$

where $\{x_r\}_{r=0}^n$ are standard Chebyshev-Gauss-Radau points,

$$x_r = -\cos(\frac{2r\pi}{2n+1}), \quad r = 0,\ldots,n$$

So we obtain the following nonlinear system of algebraic equations with $n+1$ equations and $n+1$ unknown coefficients $\{\lambda_r\}_{r=0}^n$:

$$\begin{aligned} Res(\varsigma_r) &= 0, \quad r = 1,2,\ldots,n-1,\\ f_n(0) &= 1, \quad f_n(\varsigma_n) = 0. \end{aligned} \tag{3. 16}$$

The above nonlinear system of algebraic equations can be solved by Newton method for the unknown coefficients.

4. Numerical Results and discussion

In this section, the proposed numerical technique is used to investigate the behaviour of a third grade steady fluid flow in a porous half space. In table 1 the computed results for non-dimensional parameter $f'(0)$ are reported. Moreover to confirm the performance of the proposed technique, our results are compared with other available results. The results are obtained by letting $n = 20$ and $L = 5$. The presented results as Table 1 show a good agreement between our approximate solutions and other numerical results. To investigate the convergence of the method, the obtained results for the norm-2 of the residual function, $\|Res(z)\|_2 = (\int_0^\infty |Res(z)|^2 dz)^{\frac{1}{2}}$, and also $f'(0)$ by letting $b_1 = 0.6,\ b_2 = 0.1,\ c = 0.5,\ L = 5$ and for various values of n

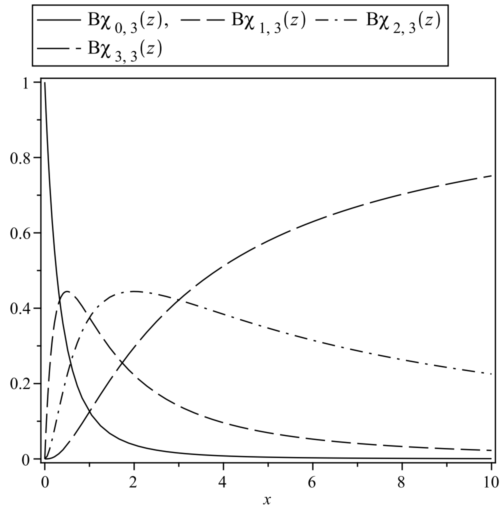

FIGURE 1. Graph of rational Bernstein functions for $n = 3$ and $L = 1$.

TABLE 1. Approximate results for $f^{'}(0)$ for various values of b_1 and c, ($b_2 = \frac{b_1 c}{3}$) .

b_1	c	Rational Bernstein collocation method	Shooting method [6]	Rational Legendre Tau method [6]
0.3	0.5	-0.69127903	-0.691280	-0.691493
0.6		-0.67830161	-0.678301	-0.678511
0.9		-0.66732656	-0.667327	-0.667528
0.6	0.3	-0.53330129	-0.533303	-0.533545
	0.6	-0.73800751	-0.738008	-0.738116
	0.9	-0.88746735	-0.887467	-0.887350
	1.2	-1.00865268	-1.008653	-1.008516

are reported in table 2. The results show that accuracy of the method increases by increasing the number of collocation nodes, n and also the the unknown value $f^{'}(0)$ is approximated with high accuracy. In figure 2 the effect of model parameter b_1 on profile of $f(z)$, for fixed value $c = 1.5$ is illustrated. The results show that increasing or decreasing the value of b_2 has no sensible effect on profiles of $f(z)$. Moreover the effect of c on profile of $f(z)$, for fixed value $b_1 = 0.5$ is demonstrated in figure 3. From figure 3 it is evident that for fixed value of b_1, the profiles of $f(z)$ decrease by increasing the values of c.

5. CONCLUSION

In this study the Bernstein polynomials has been extended on semi-infinite interval and rational Bernstein functions have been introduced. An efficient numerical

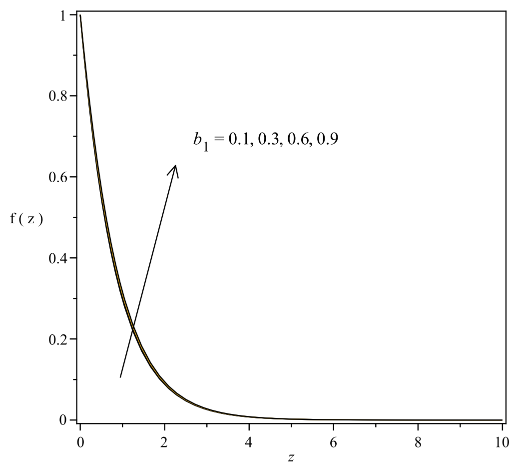

FIGURE 2. Profiles of $f(z)$ for various values of b_1 and $c = 1.5$.

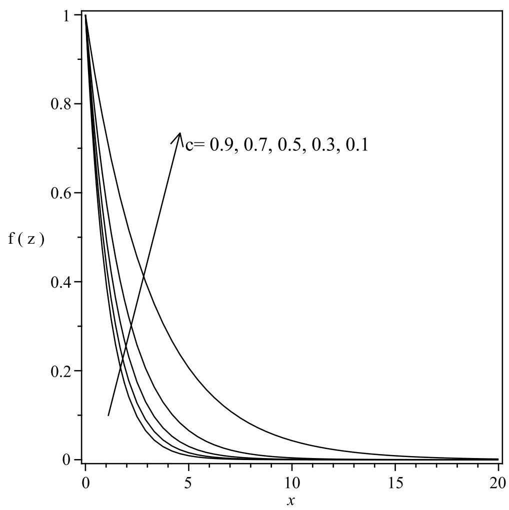

FIGURE 3. Profiles of $f(z)$ for various values of c and $b_1 = 0.5$.

technique, rational Bernstein collocation method, is formulated and employed to investigate the behaviour of the third grade steady fluid flow in a porous half space. The convergence and accuracy of the method is demonstrated through some numerical results. The effect of the model parameters b_1, b_2 and c on the solution profile has been investigated. The presented results through the figures and tables confirm the performance and accuracy of the proposed method.

TABLE 2. Numerical results for norm-2 of the residual function and $f^{'}(0)$ by letting $b_1 = 0.6,\ b_2 = 0.1,\ c = 0.5,\ L = 5$ and several values of n.

n	$\|Res(z\|_2)$	$f^{'}(0)$
4	5.712×10^{-2}	-0.68343944
8	7.869×10^{-3}	-0.67807809
12	2.156×10^{-3}	-0.67827420
16	2.380×10^{-5}	-0.67830334
20	1.548×10^{-7}	-0.67830161
[2]	$-$	-0.67830162

6. ACKNOWLEDGMENT

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