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Integral Inequalities for Generalized Preinvex Functions

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Abstract. The objective of this paper is to derive Fejér inequality for generalized preinvex functions. Some new integral inequalities related to Hermite-Hadamard type for the class of preinvex functions, whose second derivative at certain powers, are established. It is shown that our results contain several known results as applications. The technique of this paper may enhance further research in this field.

AMS (MOS) Subject Classification Codes: 26D15; 26D10, 90C23
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1. Introduction

Recently, the theory of convex functions has recieved much attention. A significant and important class of convex function is that of preinvex functions, which was developed by Ben-Israel and Mond [2]. It has been shown in [2] that preinvex functions and invex sets may not be convex functions and convex sets. Recently many authors have established a

number of inequalities by using the technique of [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 28, 29, 30, 31, 32, 33, 34] and proved Fejér type inequality, see [6, 8, 14, 17, 18, 27, 31].
Inspired by the recent development in this field, Noor et al. [19] investigated a class of $(\hbar_1, \hbar_2)$-preinvex functions involving two nonnegative arbitrary functions $\hbar_1$ and $\hbar_2$, which is called $(\hbar_1, \hbar_2)$-preinvex function or generalized preinvex functions. Consequently, the concept of inequalities for generalized preinvex functions have been studied in [19]. The purpose of this paper is to derive some new classes of Hermite-Hadamard-Fejér inequalities for twice differentiable generalized preinvex functions. For suitable choice of the arbitrary functions $\hbar_1$ and $\hbar_2$, one can obtain new results for preinvex functions and their variant forms such as midpoint, Trapezoidal and Simpson type inequalities for twice differentiable generalized preinvex function. From our results, a wide class of new inequalities for generalized preinvex functions can be obtained as special cases. Some of the results obtained in this work may be viewed as generalizations and refinement of the well known results.

2. Notations and Preliminaries

Let $\mathcal{W}$ be a nonempty set in $\mathbb{R}^n$. Let $\varphi : \mathcal{W} \longrightarrow \mathbb{R}$ be a continuous function and $\varrho(\cdot,\cdot) : \mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{R}^n$ be an arbitrary continuous bifunction. For the sake of simplicity, we always suppose that $\mathcal{W} = [\mu, \mu + \varrho(\nu, \mu)]$.
We recall some results and concepts, which are essential in the derivation of our main results.

Definition 2.1. [2, 34]. *"A set $\mathcal{W} \subset \mathbb{R}$ is said to be an invex set, if and only if, there exists a bifunction $\varrho(\cdot,\cdot)$ such that*

$$\mu + \kappa\varrho(\nu, \mu) \in \mathcal{W}, \quad \forall \mu, \nu \in \mathcal{W},\ \kappa \in\ [0, 1]."$$

We now consider a new class of preinvex function with respect to two arbitrary nonnegative functions.

Definition 2.2. [19]. *"Let $\hbar_1, \hbar_2 : (0, 1) \subseteq J \longrightarrow \mathbb{R}$ be two nonnegative functions. A function $\varphi : \mathcal{W} \longrightarrow \mathbb{R}$ is said to be generalized preinvex function with respect to $\hbar_1$ and $\hbar_2$, if*

$$\varphi(\mu + \kappa\varrho(\nu, \mu)) \le \hbar_1(1-\kappa)\hbar_2(\kappa)\varphi(\mu) + \hbar_1(\kappa)\hbar_2(1-\kappa)\varphi(\nu), \forall \mu, \nu \in \mathcal{W}, \kappa \in [0, 1]." \quad (2.\ 1)$$

For $\kappa = \frac{1}{2}$, we have the Jensen type generalized preinvex functions.

$$\varphi\left(\frac{2\mu + \varrho(\nu, \mu)}{2}\right) \le \hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)[\varphi(\mu) + \varphi(\nu)], \quad \forall \mu, \nu \in \mathcal{W},\ \kappa \in [0, 1] \quad (2.\ 2)$$

Some special cases of generalized preinvex functions as follows:

(I). If $\hbar_1(1-\kappa) = \kappa, \quad \hbar_2(\kappa) = \psi(\kappa)$ and $\varrho(\nu, \mu) = \nu - \mu$, in Definition 2.2, then we have a Definition due to Dragomir [7].

(II). Let $s \in [0, 1]$ be a real number. If $\hbar_1(\kappa) = \kappa^s$ and $\hbar_2(\kappa) = \kappa^s$ in Definition 2.2, then

Definition 2.3. [4]. *"A function $\varphi : \mathcal{W} \longrightarrow \mathbb{R}$ is said to be s-preinvex function, if*

$$\varphi(\mu + \kappa\varrho(\nu, \mu)) \leq \kappa^s(1-\kappa)^s[\varphi(\mu) + \varphi(\nu)], \quad \forall \mu, \nu \in \mathcal{W}, \kappa \in [0,1]."$$

(III). If $\hbar_1(1-\kappa)\hbar_2(\kappa) + \hbar_2(1-\kappa)\hbar_1(\kappa) = 1$ in Definition 2.2, then

Definition 2.4. [30]. *A function $\varphi : \mathcal{W} \longrightarrow \mathbb{R}$ is said to be Toader like generalized preinvex function with respect to $\varrho(\cdot, \cdot)$ if*

$$\varphi(\mu + \kappa\varrho(\nu, \mu)) \leq \varphi(\mu) + \hbar_1(\kappa)\hbar_2(1-\kappa)[\varphi(\nu) - \varphi(\mu)], \forall \mu, \nu \in \mathcal{W}, \ \kappa \in [0,1].$$

We would like to emphasize that for appropriate and suitable choice of the non-negative functions $\hbar_1$ and $\hbar_2$, one can obtain a variety of new classes of preinvex and convex functions as special cases of the generalized preinvex functions. This shows that the generalized preinvex(generalized preinvex) functions are quite general and unified ones. See also [20, 25, 28, 29, 30, 31, 32, 33].

Definition 2.5. [26]. *"Two functions φ and ψ are said to be similarly ordered (φ is ψ-monotone), if and only if,*

$$\langle \varphi(\mu) - \varphi(\nu), \psi(\mu) - \psi(\nu) \rangle \geq 0, \quad \forall \ \mu, \nu \in \mathbb{R}^n." \tag{2. 3}$$

We also need the following assumption, which is due to Mohan and Neogy [13].

Condition *C*. The bifunction $\varrho(\cdot, \cdot)$ satisfies the assumptions,

$$\varrho(\mu, \mu + \kappa\varrho(\nu, \mu)) = -\kappa\varrho(\nu, \mu), \quad \forall \ \mu, \nu \in \mathcal{W}, \ \kappa \in [0,1]. \tag{2. 4}$$

$$\varrho(\nu, \mu + \kappa\varrho(\nu, \mu)) = (1-\kappa)\varrho(\nu, \mu), \quad \forall \ \mu, \nu \in \mathcal{W}, \ \kappa \in [0,1]. \tag{2. 5}$$

We are viewing the functions which are known as Gamma function and Beta function respectively.

$$\begin{aligned} \Gamma(\mu) &= \int_0^\infty e^{-\kappa}\kappa^{\mu-1}d\kappa, \\ \mathbb{B}(\mu, \nu) &= \int_0^1 \kappa^{\mu-1}(1-\kappa)^{\nu-1}d\kappa = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}, \quad \mu, \nu > 0 \end{aligned}$$

We need the following result, which can be proved using the technique of Ozdemir [25].

Lemma 2.6. *"Let $\varphi : \mathcal{W} \longrightarrow \mathbb{R}$ be a differentiable preinvex function on the interior$\mathcal{W}^\circ$ of $\mathcal{W}$ with $\varrho(\nu,\mu) > 0$. If $\varphi^{''} \in \mathcal{L}[\mu, \mu+\varrho(\nu,\mu)]$ is generalized preinvex function, then*

$$\varphi\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) - \frac{1}{\varrho(\nu,\mu)} \int\limits_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)dx$$

$$= -\frac{(\varrho(\nu,\mu))^2}{16}\Big[\int\limits_0^1 \kappa^2\varphi^{''}\Big(\kappa\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) + (1-\kappa)\mu\Big)d\kappa$$

$$+ \int\limits_0^1 (\kappa-1)^2\varphi^{''}\Big((1-\kappa)\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) + \kappa(\mu+\varrho(\nu,\mu)\Big)d\kappa\Big]."$$

3. Main results

In this section, we establish our main results.

Theorem 3.1. *Let φ and ψ be two similarly ordered generalized preinvex functions. Then their product $\varphi\psi$ is also a generalized preinvex function provided that*

$$\hbar_1(1-\kappa)\hbar_2(\kappa) + \hbar_2(1-\kappa)\hbar_1(\kappa) \le 1.$$

Proof. Let φ and ψ be similar ordered generalized preinvex functions. Then

$$\begin{aligned}
&\varphi(\mu+\kappa\varrho(\nu,\mu))\psi(\mu+\kappa\varrho(\nu,\mu))\\
&\le [\hbar_1(1-\kappa)\hbar_2(\kappa)\varphi(\mu) + \hbar_2(1-\kappa)\hbar_1(\kappa)\varphi(\nu)][\hbar_1(1-\kappa)\hbar_2(\kappa)\psi(\mu) + \hbar_2(1-\kappa)\hbar_1(\kappa)\psi(\nu)].\\
&= [\hbar_1(1-\kappa)\hbar_2(\kappa)]^2\varphi(\mu)\psi(\mu) + [\hbar_2(1-\kappa)\hbar_1(\kappa)]^2\varphi(\nu)\psi(\nu)\\
&\quad + \hbar_1(1-\kappa)\hbar_2(\kappa)\hbar_2(1-\kappa)\hbar_1(\kappa)[\varphi(\mu)\psi(\nu) + \varphi(\nu)\psi(\mu)]\\
&\le [\hbar_1(1-\kappa)\hbar_2(\kappa)]^2\varphi(\mu)\psi(\mu) + [\hbar_2(1-\kappa)\hbar_1(\kappa)]^2\varphi(\nu)\psi(\nu)\\
&\quad + \hbar_1(1-\kappa)\hbar_2(\kappa)\hbar_2(1-\kappa)\hbar_1(\kappa)[\varphi(\mu)\psi(\mu) + \varphi(\nu)\psi(\nu)]\\
&= [\hbar_1(1-\kappa)\hbar_2(\kappa)\varphi(\mu)\psi(\mu) + \hbar_2(1-\kappa)\hbar_1(\kappa)\varphi(\nu)\psi(\nu)][\hbar_1(1-\kappa)\hbar_2(\kappa) + \hbar_2(1-\kappa)\hbar_1(\kappa)]\\
&\le [\hbar_1(1-\kappa)\hbar_2(\kappa)]\varphi(\mu)\psi(\mu) + [\hbar_1(\kappa)\hbar_2(1-\kappa)]\varphi(\nu)\psi(\nu), \qquad (3.\ 6)
\end{aligned}$$

where we have used (2. 3) and the fact that $\hbar_1(1-\kappa)\hbar_2(\kappa) + \hbar_2(1-\kappa)\hbar_1(\kappa) \le 1$. □

Theorem 3.2. *Let $\varphi, \psi : \mathcal{W} = [\mu, \mu+\varrho(\nu,\mu)] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be two similarly ordered generalized preinvex functions. If $\varphi\psi \in \mathcal{L}[\mu, \mu+\varrho(\nu,\mu)]$, then*

$$\frac{1}{\varrho(\nu,\mu)} \int\limits_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)\psi(x)dx$$

$$\le \varphi(\mu)\psi(\mu)\int\limits_0^1 \hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa + \varphi(\nu)\psi(\nu)\int\limits_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa.$$

Proof. From inequality (3. 6), we have

$$\varphi(\mu+\kappa\varrho(\nu,\mu))\psi(\mu+\kappa\varrho(\nu,\mu))$$
$$\leq [\hbar_1(1-\kappa)\hbar_2(\kappa)]\varphi(\mu)\psi(\mu)+[\hbar_1(\kappa)\hbar_2(1-\kappa)]\varphi(\nu)\psi(\nu).$$

Integrating the above inequality on $[0,1]$, we obtain

$$\frac{1}{\varrho(\nu,\mu)}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi(x)\psi(x)dx$$
$$\leq \int_0^1\Big\{\hbar_1(1-\kappa)\hbar_2(\kappa)\varphi(\mu)\psi(\mu)+\hbar_1(\kappa)\hbar_2(1-\kappa)\varphi(\nu)\psi(\nu)\Big\}d\kappa$$
$$=\varphi(\mu)\psi(\mu)\int_0^1\hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa+\varphi(\nu)\psi(\nu)\int_0^1\hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa,$$

the required result. □

Some special cases of Theorem 3.2 are:

(IV). If $\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_2(1-\kappa)\hbar_1(\kappa)=1$, then Theorem 3.2 reduces to a new result

$$\frac{1}{\varrho(\nu,\mu)}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi(x)\psi(x)dx$$
$$\leq \varphi(\nu)\psi(\nu)+[\varphi(\mu)\psi(\mu)-\varphi(\nu)\psi(\nu)]\int_0^1\hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa.$$

(V). If $\hbar_1(\kappa)=\kappa^{s_1}$ and $\hbar_2(\kappa)=\kappa^{s_2}$, then Theorem 3.2 reduces to a new result

$$\frac{1}{\varrho(\nu,\mu)}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi(x)\psi(x)dx\leq[\varphi(\mu)\psi(\mu)+\varphi(\nu)\psi(\nu)]\mathbb{B}(1+s_1,1+s_2),$$

where $\mathbb{B}(x,y)$ is the beta function.

(VI). If $\hbar(\kappa_1)=\kappa^{-s_1}$ and $\hbar(\kappa_2)=\kappa^{-s_2}$, then Theorem 3.2, reduces to a new result.

$$\frac{1}{\varrho(\nu,\mu)}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi(x)\psi(x)dx$$
$$\leq\varphi(\mu)\psi(\mu)\int_0^1\hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa+\varphi(\nu)\psi(\nu)\int_0^1\hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa$$
$$\leq[\varphi(\mu)\psi(\mu)+\varphi(\nu)\psi(\nu)]\mathbb{B}(1-s_1,1-s_2).$$

We need the following result, which is essential in deriving the Fejér type inequality for generalized preinvex functions.

Lemma 3.3. *Let φ be a generalized preinvex function. Then*

$$\varphi(2\mu+\varrho(\nu,\mu)-x) \leq [\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_2(1-\kappa)\hbar_1(\kappa)][\varphi(\mu)+\varphi(\nu)]-\varphi(x).$$

Proof. Let $x=\mu+(1-\kappa)\varrho(\nu,\mu)\in\mathcal{W}$. Then

$$\begin{aligned}
& \varphi(2\mu+\varrho(\nu,\mu)-x) \\
= \;& \hbar_1(1-\kappa)\hbar_2(\kappa)\varphi(\mu)+\hbar_1(\kappa)\hbar_2(1-\kappa)\varphi(\nu). \\
\leq \;& [\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(\kappa)\hbar_2(1-\kappa)][\varphi(\mu)+\varphi(\nu)] \\
- \;& \hbar_1(1-\kappa)\hbar_2(\kappa)\varphi(\mu)-\hbar_1(\kappa)_2(1-\kappa)\varphi(\nu). \\
= \;& [\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(\kappa)\hbar_2(1-\kappa)][\varphi(\mu)+\varphi(\nu)]-\varphi(\mu+\kappa\varrho(\nu,\mu)) \\
= \;& [\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_1(\kappa)\hbar_2(1-\kappa)][\varphi(\mu)+\varphi(\nu)]-\varphi(x).
\end{aligned}$$

□

We now derive the Fejér type inequality, which is the main motivation of our next result.

Theorem 3.4. *Let $\varphi:\mathcal{W}\longrightarrow\mathbb{R}$ be a generalized preinvex function with $\varrho(\nu,\mu)>0$, $\hbar_1(\frac{1}{2})\neq 0$, $\hbar_2(\frac{1}{2})\neq 0$. If $\psi:[\mu,\mu+\varrho(\nu,\mu)]\longrightarrow\mathbb{R}$ is a nonnegative, symmetric about $\mu+\frac{1}{2}\varrho(\nu,\mu)$ integrable function and condition* C *holds, then*

$$\begin{aligned}
&\tfrac{1}{2\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\varphi\left(\tfrac{2\mu+\varrho(\nu,\mu)}{2}\right)\int_{\mu}^{\mu+\varrho(\nu,\mu)}\psi(x)dx \leq \int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi(x)\psi(x)dx \\
&\leq \tfrac{\varphi(\mu)+\varphi(\nu)}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\Big[\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_2(1-\kappa)\hbar_1(\kappa)\Big]\psi(x)dx.
\end{aligned}$$

Proof. Let φ be a generalized preinvex function. Then, by virtue of Lemma 3.3, we have

$$\frac{1}{2\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\varphi\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)\int_{\mu}^{\mu+\varrho(\nu,\mu)}\psi(x)dx.$$

$$= \frac{1}{2\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)\psi(x)dx.$$

$$= \frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi\left(\frac{2\mu+\varrho(\nu,\mu)-x+x}{2}\right)\psi(x)dx$$

$$\leq \frac{1}{2\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\Big[\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})[\varphi(2\mu+\varrho(\nu,\mu)-x)+\varphi(x)]\Big]\psi(x)dx$$

$$= \frac{1}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi(2\mu+\varrho(\nu,\mu)-x)\psi(2\mu+\varrho(\nu,\mu)-x)dx+\frac{1}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi(x)\psi(x)dx.$$

$$= \int_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)\psi(x)dx$$

$$= \frac{1}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(2\mu+\varrho(\nu,\mu)-x)\psi(2\mu+\varrho(\nu,\mu)-x)dx + \frac{1}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)\psi(x)dx.$$

$$= \frac{1}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(2\mu+\varrho(\nu,\mu)-x)\psi(x)dx + \frac{1}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)\psi(x)dx$$

$$\leq \frac{1}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)} \Big[\Big(\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_2(1-\kappa)\hbar_1(\kappa)\Big)[\varphi(\mu)+\varphi(\nu)]-\varphi(x)\Big]\psi(x)dx$$

$$+ \frac{1}{2}\int_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)\psi(x)dx.$$

$$\leq \frac{\varphi(\mu)+\varphi(\nu)}{2}\Big[\hbar_1(1-\kappa)\hbar_2(\kappa)+\hbar_2(1-\kappa)\hbar_1(\kappa)\Big]\int_{\mu}^{\mu+\varrho(\nu,\mu)} \psi(x)dx.$$

This complete the proof. □

We derive a new Hermit-Hadamard inequality (HHI) for differentiable generalized preinvex functions.

Lemma 3.5. *Let $\varphi : \mathcal{W} = [\mu, \mu+\varrho(\nu,\mu)] \longrightarrow \mathbb{R}$ be a differentiable preinvex function on the interior $\mathcal{W}^\circ$ of $\mathcal{W}$ with $\varrho(\nu,\mu) > 0$. If $\varphi'' \in \mathcal{L}[\mu, \mu+\varrho(\nu,\mu)]$ and $\left|\varphi''\right|$ is generalized preinvex function on $[\mu, \mu+\varrho(\nu,\mu)]$, then*

$$\varphi\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) - \frac{1}{\varrho(\nu,\mu)}\int_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)dx$$

$$= -\frac{(\varrho(\nu,\mu))^2}{16}\Big[\int_0^1 \kappa^2\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+(1-\kappa)\mu\right)d\kappa$$

$$+\int_0^1 (\kappa-1)^2\varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+\kappa(\mu+\varrho(\nu,\mu))\right)d\kappa\Big].$$

Proof. Integrating by parts, we get

$$
\begin{aligned}
I_1 &= \int_0^1 \kappa^2 \varphi^{''}\Big(\kappa\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)+(1-\kappa)\mu\Big)d\kappa \\
&= \kappa^2 \frac{\varphi^{'}\Big(\kappa\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)+(1-\kappa)\mu\Big)}{\frac{\varrho(\nu,\mu)}{2}}\Big|_0^1 - \frac{4}{\varrho(\nu,\mu)}\int_0^1 \kappa f^{'}\Big(\kappa\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)+(1-\kappa)\mu\Big)d\kappa. \\
&= \frac{2}{\varrho(\nu,\mu)}\Big[\varphi^{'}\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)\Big] - \frac{4}{\varrho(\nu,\mu)}\Big[\kappa\frac{\varphi\Big(\kappa\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)+(1-\kappa)\mu\Big)}{\frac{\varrho(\nu,\mu)}{2}}\Big|_0^1 \\
&- \frac{2}{\varrho(\nu,\mu)}\int_0^1 \varphi\Big(\kappa\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)+(1-\kappa)\mu\Big)d\kappa\Big] \\
&= \frac{2}{\varrho(\nu,\mu)}\varphi^{'}\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) - \frac{8}{(\varrho(\nu,\mu))^2}\Bigg[\varphi\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) \\
&- \int_0^1 \varphi\Big(\kappa\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)+(1-\kappa)\mu\Big)d\kappa\Bigg] \\
&= \frac{2}{\varrho(\nu,\mu)}\varphi^{'}\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) - \frac{8}{(\varrho(\nu,\mu))^2}\varphi\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) + \frac{16}{(\varrho(\nu,\mu))^3}\int_\mu^{\frac{2\mu+\varrho(\nu,\mu)}{2}} \varphi(x)dx.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
I_2 &= \int_0^1 (\kappa-1)^2\varphi^{''}\ \ (1-\kappa)\ \ \frac{2\mu+\varrho(\nu,\mu)}{2}\ \ +\kappa(\mu+\varrho(\nu,\mu)\ \ d\kappa \\
&= (\kappa-1)^2\frac{\varphi^{'}\ \ (1-\kappa)\ \ \frac{2\mu+\varrho(\nu,\mu)}{2}\ \ +\kappa(\mu+\varrho(\nu,\mu))}{\frac{\varrho(\nu,\mu)}{2}}\ \ {}_0^1 \\
&- \frac{4}{\varrho(\nu,\mu)}\int_0^1 (\kappa-1)\varphi^{'}\ \ (1-\kappa)\ \ \frac{2\mu+\varrho(\nu,\mu)}{2}\ \ +\kappa(\mu+\varrho(\nu,\mu))\ \ d\kappa. \\
&= -\frac{2}{\varrho(\nu,\mu)}\Big[\varphi^{'}\ \ \frac{2\mu+\varrho(\nu,\mu)}{2}\ \ \Big] - \frac{4}{(\varrho(\nu,\mu))}\Big[(\kappa-1)\frac{\varphi\ \ (1-\kappa)\ \ \frac{2\mu+\varrho(\nu,\mu)}{2}\ \ +\kappa(\mu+\varrho(\nu,\mu))}{\frac{\varrho(\nu,\mu)}{2}}\ \ {}_0^1 \\
&- \frac{2}{\varrho(\nu,\mu)}\int_0^1 \varphi\ \ (1-\kappa)\ \ \frac{2\mu+\varrho(\nu,\mu)}{2}\ \ +\kappa(\mu+\varrho(\nu,\mu))\ \ d\kappa\Big]
\end{aligned}
$$

$$= -\frac{2}{\varrho(\nu,\mu)}\varphi'\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) - \frac{8}{(\varrho(\nu,\mu))^2}\varphi\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)$$
$$+ \frac{8}{(\varrho(\nu,\mu))^2}\int_0^1 \varphi\left((1-)\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+\kappa(\mu+\varrho(\nu,\mu))\right)d\kappa.$$
$$= -\frac{2}{\varrho(\nu,\mu)}\varphi'\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) - \frac{8}{(\varrho(\nu,\mu))^2}\varphi\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) + \frac{16}{(\varrho(\nu,\mu))^3}\int_{\frac{2\mu+\varrho(\nu,\mu)}{2}}^{\mu+\varrho(\nu,\mu)}\varphi(x)dx.$$

Adding I_1 and I_2, we obtain

$$\varphi\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) - \frac{1}{\varrho(\nu,\mu)}\int_\mu^{\mu+\varrho(\nu,\mu)}\varphi(x)dx$$
$$= -\frac{(\varrho(\nu,\mu))^2}{16}\Big[\int_0^1 \kappa^2\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+(1-\kappa)\mu\right)d\kappa$$
$$+ \int_0^1 (\kappa-1)^2\varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+\kappa(\mu+\varrho(\nu,\mu)\right)d\kappa\Big],$$

the required result. □

Theorem 3.6. *Let $\varphi : \mathcal{W} = [\mu, \mu+\varrho(\nu,\mu)] \longrightarrow \mathbb{R}$ be a differentiable preinvex function on the interior $\mathcal{W}^\circ$ of $\mathcal{W}$ with $\varrho(\nu,\mu) > 0$. If $\varphi'' \in \mathcal{L}[\mu, \mu+\varrho(\nu,\mu)]$ and $\left|\varphi''\right|^q$ is generalized preinvex function, where $q \geq 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|\varphi\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) - \frac{1}{\varrho(\nu,\mu)}\int_\mu^{\mu+\varrho(\nu,\mu)}\varphi(x)dx\right|$$
$$\leq \frac{(\varrho(\nu,\mu))^2}{16}\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\left[\Big[H_1(\mu,\nu,\hbar_1,\hbar_2)\left|\varphi''(\mu)\right|^q + H_2(\mu,\nu,\hbar_1,\hbar_2)\left|\varphi''\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)\right|^q\Big]^{\frac{1}{q}}\right.$$
$$\left.+\Big[H_3(\mu,\nu,\hbar_1,\hbar_2)\left|\varphi''\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)\right|^q + H_4(\mu,\nu,\hbar_1,\hbar_2)\left|\varphi''(\nu)\right|^q\Big]^{\frac{1}{q}}\right],$$

where

$$H_1(\mu,\nu,\hbar_1,\hbar_2) = H_3(\mu,\nu,\hbar_1,\hbar_2) = \int_0^1 \hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa$$
$$H_2(\mu,\nu,\hbar_1,\hbar_2) = H_4(\mu,\nu,\hbar_1,\hbar_2) = \int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa$$

Proof. Let $|\varphi''|^q$ be a generalized preinvex function. Then, using Lemma 3.5, we obtain

$$\varphi\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) - \frac{1}{\varrho(\nu,\mu)}\int_{\mu}^{\mu+\varrho(\nu,\mu)}\varphi(x)dx$$
$$= \frac{(\varrho(\nu,\mu))^2}{16}\Big[\int_0^1 \kappa^2\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+(1-\kappa)\mu\right)d\kappa$$
$$+\int_0^1(\kappa-1)^2\varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+\kappa(\mu+\varrho(\nu,\mu)\right)d\kappa\Big].$$
$$\le \frac{(\varrho(\nu,\mu))^2}{16}\Big[\int_0^1 \left|\kappa^2\right|\left|\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+(1-\kappa)\mu\right)\right|d\kappa$$
$$+\int_0^1\left|(\kappa-1)^2\right|\left|\varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+\kappa(\mu+\varrho(\nu,\mu)\right)\right|d\kappa\Big].$$

$$\le \frac{(\varrho(\nu,\mu))^2}{16}\Bigg[\left(\int_0^1 \left|\kappa^2\right|^p d\kappa\right)^{\frac{1}{p}}\left(\left|\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+(1-\kappa)\mu\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|(\kappa-1)^2\right|^p d\kappa\right)^{\frac{1}{p}}\left(\left|\varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+\kappa(\mu+\varrho(\nu,\mu)\right)\right|^q d\kappa\right)^{\frac{1}{q}}\Bigg]. \tag{3. 7}$$

Since $|\varphi''|^q$ is generalized preinvex function, so

$$\int_0^1\left|\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+(1-\kappa)\mu\right)\right|^q d\kappa$$
$$\le\int_0^1\Big[\hbar_1(1-\kappa)\hbar_2(\kappa)\left|\varphi''(\mu)\right|^q+\hbar_1(\kappa)\hbar_2(1-\kappa)\left|\varphi''\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)\right|^q\Big]d\kappa. \tag{3. 8}$$

$$\int_0^1\left|\varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)+\kappa(\mu+\varrho(\nu,\mu))\right)\right|^q d\kappa$$
$$\le\int_0^1\Big[\hbar_1(1-\kappa)\hbar_2(\kappa)\left|\varphi''\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)\right|^q+\hbar_1(\kappa)\hbar_2(1-\kappa)\left|\varphi''(\nu)\right|^q\Big]d\kappa. \tag{3. 9}$$

and

$$\int_0^1|\kappa^2|^p d\kappa=\int_0^1|(\kappa-1)^2|^p d\kappa=\frac{1}{2p+1}. \tag{3. 10}$$

Substituting (3. 8)-(3. 10) into (3. 7), yields

$$\varphi \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad - \frac{1}{\varrho(\nu,\mu)} \int\limits_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)dx\|$$

$$\leq \frac{(\varrho(\nu,\mu))^2}{16} \quad \frac{1}{2p+1} \quad ^{\frac{1}{p}} \Bigg[\Big[\int\limits_0^1 \Big[\hbar_1(1-\kappa)\hbar_2(\kappa) \ \varphi^{''}(\mu) \ ^q + \hbar_1(\kappa)\hbar_2(1-\kappa) \ \varphi^{''} \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad ^q\Big]d\kappa\Big]^{\frac{1}{q}}$$

$$+ \Big[\int\limits_0^1 \Big[\hbar_1(1-\kappa)\hbar_2(\kappa) \ \varphi^{''} \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad ^q + \hbar_1(\kappa)\hbar_2(1-\kappa) \ \varphi^{''}(\nu) \ ^q\Big]d\kappa\Big]^{\frac{1}{q}}\Bigg].$$

$$= \frac{(\varrho(\nu,\mu))^2}{16} \quad \frac{1}{2p+1} \quad ^{\frac{1}{p}} \Bigg[\Big[H_1(\mu,\nu,\hbar_1,\hbar_2) \ \varphi^{''}(\mu) \ ^q + H_2(\mu,\nu,\hbar_1,\hbar_2) \ \varphi^{''} \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad ^q\Big]^{\frac{1}{q}}$$

$$+ \Big[H_3(\mu,\nu,\hbar_1,\hbar_2) \ \varphi^{''} \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad ^q + H_4(\mu,\nu,\hbar_1,\hbar_2) \ \varphi^{''}(\nu) \ ^q\Big]^{\frac{1}{q}}\Bigg],$$

which completes the proof. □

Theorem 3.7. *Let $\varphi : \mathcal{W} = [\mu, \mu+\varrho(\nu,\mu)] \longrightarrow \mathbb{R}$ be a differentiable mapping on $\mathcal{W}^\circ$, where $\mathcal{W}^\circ$ is the interior of $\mathcal{W}$ with $\varrho(\nu,\mu) > 0$. If $\varphi^{''} \in \mathcal{L}[\mu, \mu+\varrho(\nu,\mu)]$ and $\Big|\varphi^{''}\Big|^q$ is generalized preinvex function, then*

$$\Bigg|\varphi\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) - \frac{1}{\varrho(\nu,\mu)} \int\limits_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)dx\Bigg|$$

$$\leq \frac{(\varrho(\nu,\mu))^2}{16}\Big(\frac{1}{3}\Big)^{1-\frac{1}{q}} \Bigg[\Big[H_5(\mu,\nu,\hbar_1,\hbar_2)\Big|\varphi^{''}(\mu)\Big|^q + H_6(\mu,\nu,\hbar_1,\hbar_2)\Big|\varphi^{''}\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)\Big|^q\Big]^{\frac{1}{q}}$$

$$+ \Big[H_7(\mu,\nu,\hbar_1,\hbar_2)\Big|\varphi^{''}\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)\Big|^q + H_8(\mu,\nu,\hbar_1,\hbar_2)\Big|\varphi^{''}(\nu)\Big|^q\Big]^{\frac{1}{q}}\Bigg],$$

where

$$H_5(\mu,\nu,\hbar_1,\hbar_2) = \int\limits_0^1 |\kappa^2|\hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa, \tag{3. 11}$$

$$H_6(\mu,\nu,\hbar_1,\hbar_2) = \int\limits_0^1 |\kappa^2|\hbar_2(1-\kappa)\hbar_1(\kappa)d\kappa, \tag{3. 12}$$

$$H_7(\mu,\nu,\hbar_1,\hbar_2) = \int\limits_0^1 |\kappa-1|^2\hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa, \tag{3. 13}$$

$$H_8(\mu,\nu,\hbar_1,\hbar_2) = \int_0^1 |\kappa-1|^2 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa. \tag{3. 14}$$

Proof. Using Lemma 3.5, it follows that

$$\varphi\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) - \frac{1}{\varrho(\nu,\mu)} \int_\mu^{\mu+\varrho(\nu,\mu)} \varphi(x)dx$$

$$= \frac{(\varrho(\nu,\mu))^2}{16}\Big[\int_0^1 \kappa^2 \varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right)\right.$$

$$\left. + (1-\kappa)\mu\right) d\kappa + \int_0^1 (\kappa-1)^2 \varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) + \kappa(\mu+\varrho(\nu,\mu)\right) d\kappa\Big].$$

$$\begin{aligned} &\leq \frac{(\varrho(\nu,\mu))^2}{16}\Big[\int_0^1 \kappa^2 \left|\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) + (1-\kappa)\mu\right)\right| d\kappa \\ &+ \int_0^1 (\kappa-1)^2 \left|\varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) + \kappa(\mu+\varrho(\nu,\mu)\right)\right| d\kappa\Big]. \end{aligned} \tag{3. 15}$$

Using the power mean inequality, we have

$$\begin{aligned} &\int_0^1 |\kappa^2| \varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) + (1-\kappa)\mu\right) d\kappa \\ &\leq \left(\int_0^1 |\kappa^2| d\kappa\right)^{1-\frac{1}{q}} \left(\int_0^1 |\kappa^2| \left|\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) + (1-\kappa)\mu\right)\right|^q d\kappa\right)^{\frac{1}{q}}. \end{aligned} \tag{3. 16}$$

$$\begin{aligned} &\int_0^1 |\kappa-1|^2 \varphi''\left((1-\kappa)\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) + \kappa(\mu+\varrho(\nu,\mu))\right) d\kappa \\ &\leq \left(\int_0^1 |\kappa-1|^2 d\kappa\right)^{1-\frac{1}{q}} \left(\int_0^1 |(\kappa-1)^2| \left|\varphi''\left(\kappa\left(\frac{2\mu+\varrho(\nu,\mu)}{2}\right) + \kappa(\mu+\varrho(\nu,\mu))\right)\right|^q d\kappa\right)^{\frac{1}{q}}. \end{aligned} \tag{3. 17}$$

and

$$\int_0^1 |\kappa^2| d\kappa = \int_0^1 |(\kappa-1)^2| d\kappa = \frac{1}{3}. \tag{3. 18}$$

Substituting (3. 16)-(3. 18) in (3. 15), yields

$$\varphi \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad - \frac{1}{\varrho(\nu,\mu)} \int\limits_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)dx$$

$$\leq \frac{(\varrho(\nu,\mu))^2}{16}(\frac{1}{3})^{1-\frac{1}{q}}\Bigg[\Big[\quad \varphi^{''}(\mu) \quad ^q \int\limits_0^1 |\kappa^2|\hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa \quad + \quad \varphi^{''} \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad ^q$$

$$\int\limits_0^1 |\kappa^2|\hbar_2(1-\kappa)\hbar_1(\kappa)d\kappa \quad \Big]^{\frac{1}{q}} + \Big[\quad \varphi^{''} \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad ^q \int\limits_0^1 |(\kappa-1)^2|\hbar_1(1-\kappa)\hbar_2(\kappa)d\kappa$$

$$+ \quad \varphi^{''}(\nu) \quad ^q \int\limits_0^1 |(\kappa-1)^2|\hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa \quad \Big]^{\frac{1}{q}}\Bigg].$$

$$= \frac{(\varrho(\nu,\mu))^2}{16}(\frac{1}{3})^{1-\frac{1}{q}}\Bigg[\Big[H_5(\mu,\nu,\hbar_1,\hbar_2) \quad \varphi^{''}(\mu) \quad ^q + H_6(\mu,\nu,\hbar_1,\hbar_2) \quad \varphi^{''} \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad ^q\Big]^{\frac{1}{q}}$$

$$+ \Big[H_7(\mu,\nu,\hbar_1,\hbar_2) \quad \varphi^{''} \quad \frac{2\mu+\varrho(\nu,\mu)}{2} \quad ^q + H_8(\mu,\nu,\hbar_1,\hbar_2) \quad \varphi^{''}(\nu) \quad ^q\Big]^{\frac{1}{q}}\Bigg],$$

which completes the proof. □

Theorem 3.8. *Let $\varphi : \mathcal{W} = [\mu, \mu+\varrho(\nu,\mu)] \longrightarrow \mathbb{R}$ be a differentiable preinvex function on the interior $\mathcal{W}^{\circ}$ of $\mathcal{W}$ with $\varrho(\nu,\mu) > 0$. If $\varphi^{''} \in \mathcal{L}[\mu, \mu+\varrho(\nu,\mu)]$ and $\left|\varphi^{''}\right|$ is generalized preinvex function, then*

$$\left|\varphi\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big) - \frac{1}{\varrho(\nu,\mu)} \int\limits_{\mu}^{\mu+\varrho(\nu,\mu)} \varphi(x)dx\right|$$

$$\leq \frac{(\varrho(\nu,\mu))^2}{16}\Big[H_5(\mu,\nu,\hbar_1,\hbar_2)\left|\varphi^{''}(\mu)\right| + H_6(\mu,\nu,\hbar_1,\hbar_2)\left|\varphi^{''}\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)\right|$$

$$+ H_7(\mu,\nu,\hbar_1,\hbar_2)\left|\varphi^{''}\Big(\frac{2\mu+\varrho(\nu,\mu)}{2}\Big)\right| + H_8(\mu,\nu,\hbar_1,\hbar_2)\left|\varphi^{''}(\nu)\right|\Big],$$

where $H_5(\mu,\nu,\hbar_1,\hbar_2)$, $H_6(\mu,\nu,\hbar_1,\hbar_2)$, $H_7(\mu,\nu,\hbar_1,\hbar_2)$ and $H_8(\mu,\nu,\hbar_1,\hbar_2)$ are given in (3. 11)-(3. 14).

Proof. Its proof follows from Theorem 3.7 for $q = 1$. □

4. CONCLUSION:

We now mention some new particular cases, which can be obtained from our main results

(1) If $\varrho(\nu,\mu) = \nu - \mu$ and $\hbar_1(\kappa) = \hbar_2(\kappa) = \kappa$, $\hbar_1(\kappa) = \hbar_2(\kappa) = \kappa^s$, $\hbar_1(\kappa) = \hbar_2(\kappa) = \kappa^{-1}$ and $\hbar_1(\kappa) = \hbar_2(\kappa) = 1$ under this assumption, we will get the results for convexity, like s, Godunova levin and p-convexity, respectively.

(2) If $\hbar_1(\kappa) = \hbar_2(\kappa) = \kappa$
under this assumption, we get the results for preinvexity.
(3) If $\varrho(\nu, \mu) = \nu - \mu$ and $\hbar_1(\kappa) = \hbar_2(\kappa) = \kappa$, $\hbar_1(\kappa) = \hbar_2(\kappa) = \kappa^s$, $\hbar_1(\kappa) = \hbar_2(\kappa) = \kappa^{-1}$ and $\hbar_1(\kappa) = \hbar_2(\kappa) = 1$, then, we will get the results for preinvexity, like s, Godunova Levin, and p-preinvexity, respectively.

We have derived new upper and lower bounds for the functions of generalized preinvex functions. We have also developed some interesting results for this new class, which may have significant and important applications in various fields of pure and applied sciences.

5. Authors contributions

All the authors worked jointly and contributed equaly. They all read and approved the final manuscript.

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7. Conflicts of interest

All the authors declare no conflict of interest.

References

[1] A. Barani, A. Ghazanfari and S. S. Dragomir, *On Hermite-Hadamard inequality for function whose derivatives absolute values are preinvex*, J. Inequal. Appl, (2012) 2012:247.

[2] A. Ben-Israel and B. Mond, *What is invexity?* J. Austral. Math. Soc. Ser. B**28**, No 1 (1986) 19.

[3] M. Bombardelli, and S. Varosanec, *On properities of h-convex functions related to the Hermite-Hadamard -Fejér inequalities*, Compt. Math. Appl. **58**, (2009) 1869-1877.

[4] W. W. Breckner, *Stetigkeitsaussagen fiir eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen,* Publ. Inst. Math, **23**, (1978) 13-20.

[5] C. Hermite, *Sur deux limites d'une integrale definie*, Mathesis, **3**, (1883) 82.

[6] M. R. Delavar, S.S. Dragomir and M. De La Sen, *On estimation type results for Fejér inequality with applications*, J. Inequal. Appl. (2018), 2018: :85

[7] S. S. Dragomir, *Inequalities of Jensen type for φ-convex functions,* Fasciculi Math. **55**, (2015) 35-52.

[8] L. Fejér, *Uber die fourierreihen, II, Math Naturwise.* Anz Ungar. Akad. Wiss. **24**, (1906) 369-390.

[9] J. Hadamard, *Etude sur les proprietes des fonctionsentieres e. z en particulier dune fonction consideree par Riemann*, J. Math. Pure. Appl. **58**, (1983) 171-215.

[10] I. Iscan, *On Hermite-Hadamard and Simpson like inequalities for differentiable harmonically convex functions,* J. Math. **10**, (2014), 346305.

[11] M. Matloka, *On some Hadamard-type inequalities (h_1, h_2)-preinvex functions on the coordinates,* J. Inequal. Appl. **2013**, 2013:227.

[12] S. K. Mishra and M. A. Noor, *On vector variational-like inequality problems,* J. Math. Anal. Appl. **311**, (2005) 69-75.

[13] S. R. Mohan and S. K. Neogy, *On invex set and preinvex functions,* J. Math. Anal. Appl, **189**, (1995) 901-908.

[14] C. P. Niculescu and L. E. Persson, *Convex Functions and Their Applications.* Springer-Verlag, New York, (2018).

[15] M. A. Noor. *Variational inequalities and approximation,* Punjab Univer. J. Math, **8**, (2018) 25-40.

[16] M. A. Noor, *Hermite Hadamard integral inequalities for log-preinvex functions,* J. Math. Anal. Approx. Theory, **2**, (2007) 126-131.
[17] M. A. Noor, K. I. Noor, M. U. Awan and J. Li, *On Hermite-Hadamard inequalities for h preinvex functions,* Filomat. **28**, No. (7) (2018) 1463-1474.
[18] M. A. Noor, *Variational like inequalities,* Optimization, **30**, (1994) 323-330.
[19] M. A. Noor, K. I. Noor, S. Rashid, *Some new classes of preinvex functions and inequalities,* mathematics. **7**, No. 1 (2019).
[20] M. A. Noor, K. I Noor and S. Iftikhar, *On integral inequalities for differentiable relative harmonic preinvex functions, (survey),* TWMSJ. Pure. Appl. Math. **7**, (2016) 3-19.
[21] M. A. Noor, *Some developments in general variational inequalites,* Appl. Math. Comput. **151**, (2004) 199-277.
[22] M. A. Noor, *On hermite-Hadamard integral inequaliites for involving two log peinvex functions,* J. Inequal. Pure. Appl. Math. **3**, (2007) 75-81.
[23] M. A. Noor, *Hadamrd integral inequalites for product of two preinvex functions,* Nonlinear Anal. Forum, **14**, (2009) 167-173.
[24] M. A. Noor, K. I. Noor and S. Iftikhar, *Nonconvex Functions and Integral Inequalities,* Punjab Univ. J. Math, **47**, No. 2 (2015) 19-27
[25] M. E. Ozdemir and C. Yildiz, A. C. Akdemir and E. Set, *On some inequalities for s-convex functions and applications*, J. Inequal. Appl. (2013) 2013:333.
[26] J. Pecaric, F. Proschan and Y. L. Yang, *Convex Functions, Partial Orderings and Statistical applications,* Academic Press , New York, (1992).
[27] L. -E. Persson, M. A. Ragusa, N. Samko and P. Wall, *Commutators of Hardy operators in vanishing Morrey spaces,* Amer. Instit. Phy, Conference Proceedings, **1493**, (2012) 859-866.
[28] M. Z. Sarikaya, A. Saglam and H. Yildrim, *On some Hadamard- type inequalities for h- convex functions,* J. Math. Inequal. **2**, No. 3 (2008) 335- 341.
[29] W. Sun and Q. Liu, *New Hermite-Hadamard type inequalities for (α, m)-convex functions and applications to special means,* J. Math. Inequal. **11**, No. 2(2017) 383-397.
[30] G. Toader, *Some generalizations of the convexity,* Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca. (1985), 329-338.
[31] K. -L. Tseng, G.-S, Yang and K. -C. Hsu, *Some inequaliites for differentiable mappings and applications to Fejér inequality and weighted trapezoidal formula,* Taiwan. J. Math. **15**, (2011) 1737-1747.
[32] M. Tunc, E. Gov and U. Sanal, *On tgs- convex function and their inequalities,* Facta Universit(NIS). Math. Inform. **30**, (2015) 679- 691.
[33] S. Verosanec, *On h- convexity,* J. Math. Anal. Appl, **326,** (2007) 303-311.
[34] Z, Weir and B. Mond, *Preinvex functions in multiobjective optimization,* J. Math. Anal. Appl, **136**, (1988) 29-38.