Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol.46(2)(2014) pp. 63-68

On Weakly Hereditary Rings

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Abstract. We define a particular case of hereditary rings called weak hereditary rings. A sufficient condition for a weak hereditary be hereditary is given. We investigate the transfer of this notion in homomorphic image of rings, amalgamated duplication of a ring along a proper ideal, subring retracts, direct product of rings, and in trivial ring extensions. For the pullback constructions, we give an example showing that the transfer does not hold.

AMS (MOS) Subject Classification Codes: 13B02
Key Words: Dedekind domain rings, hereditary rings, weak hereditary rings, trivial ring extensions, subring retracts, pullback constructions, direct product of rings.

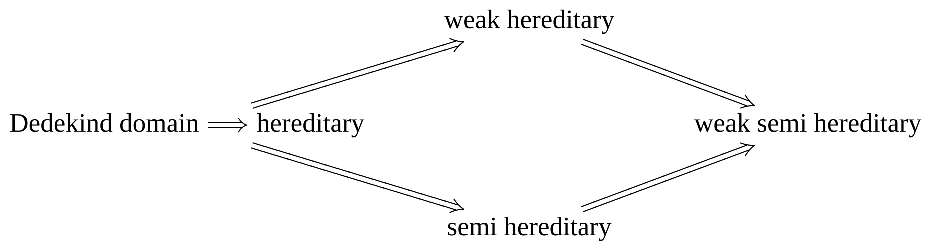
1. Introduction

If every ideal (resp., finitely generated ideal) of a ring A is projective, then A is called hereditary (resp., semi hereditary) [3]. A hereditary integral domain is called a Dedekind ring. In [1] the ring A is said weak semi hereditary if, for every two proper ideals of A I_1 and I_2 satisfying $I_1 \subseteq I_2$ with I_2 projective ideal and I_1 is finitely generated, then I_1 is projective. So, we present a particular case of hereditary, called weak hereditary.

Definition 1. If $I \subseteq J$ for every two ideals I and J of a ring A with J projective proper ideal implies that I is also projective, then A is called weak hereditary.

Remark 2. Every hereditary ring is a weak hereditary.

It is easy to see that the following diagram holds true:



Firstly in Proposition 3, a sufficient condition is given to have the converse. Secondly, it has been proven that A is weak hereditary provided A is local total of quotients.
Suppose that A is a ring and N is A-module, a ring $A \propto N$ whose underling group is $A \times N$ with pairwise addition and multiplication given by $(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1b_2 + a_2b_1)$ is said to be trivial ring extension of A by N [12, 10, 11].
Two Corollaries (5 and 6) and Theorem 7 have been used to show the transfer of a weak hereditary ring into trivial ring extensions.
The amalgamated duplication of a ring A along an ideal J was introduced by D'Anna and Fontana [8] as a subring of $A \times A$ with $(1_A, 1_A)$ unit element. It is defined by:

$$A \bowtie J = \{(a, a + j) | r \in A, j \in J\}.$$

However, in Proposition 8, we give condition so that the descent of the weak hereditary ring holds in extensions of ring. Namely, if A be a subring retract of a faithfully flat A-module R, then R is weak hereditary implies that A is weak hereditary. However, in Example 9, we show that the homomorphic image of weak hereditary is not necessarily in general weak hereditary.
And, in Proposition 10, we illustrate the transfer of weak hereditary from a ring A to a ring $A \bowtie J$.
In Example 12, we show that, in general, the transfer of weak hereditary notion does not hold in pullback constructions. In Theorem 13, we investigate the weak hereditary rings in direct product of rings. In Example 15 we show that, in general, the direct product of a weak hereditary ring is not weak hereditary.

2. Main Results

We start this section by the following proposition which gives a sufficient condition so that the converse of Remark 1.2 holds true.

Proposition 3. *Let A be a ring, then :*

(1) *If A contains a regular element, then A is weak hereditary if and only if A is hereditary.*
(2) *If A is a local total ring of quotients, then A is weak hereditary.*

Proof. (i) The "if part" of the assertion is evident. Conversely, suppose that J is proper ideal of A. Let a be a regular element of A, then $aJ \subseteq aA$. On the other hand, aA proper ideal and $aA \cong A$, then aA is free implies projective. Then aJ is projective ideal, since A is weak hereditary. But $aI(\cong J)$, so J is projective.
(ii) Let A be a local and total ring of quotients. We shall to display that A is weak hereditary. Deny. Suppose that there exist $I_1 \subseteq I_2 \subseteq M$, with M is a maximal ideal of A, I_1 is a non-projective ideal of A and I_2 is a proper projective ideal. Since A is local, then I_2 is free, so $I_2 = aA$, with a is a regular element from A. A contradiction, so A is weak hereditary. □

Example 4. Suppose that (T, N) is a local ring with $N^2 = 0$, where N is a maximal ideal of T. Since $T = Q(T)$, then T is weak hereditary by Proposition 3(ii).

The transfer of the weak hereditary rings from a ring A to a ring $A \propto R$ will be studied in the following corollary.

Corollary 5. *Suppose that A is a domain and $R := Q(A)$, and let $S := A \propto R$. Then:*

(1) *S is not hereditary.*
(2) *If A is not a field, then S is not weak hereditary.*

(3) *If A is a field, then :*
  (a) *S is weak hereditary.*
  (b) *S is not hereditary.*

Proof. (i) From [13, Theorem 2.8].
(ii) Assume that A is not a field and $0 \neq a \in A$ where a is not invertible. Then $(a, 0)$ is a regular element of S, from Proposition 3(i) S is not weak hereditary.
(iii) Assume that A is a field, then :
  (a) Obvious from Proposition 3(ii).
  (b) from (i) S is not a hereditary ring.

□

This corollary gives an example of a ring which is weak hereditary but not hereditary.

Corollary 6. *Suppose that (R, N) is a local ring, with N is maximal ideal of R and E an R-module such that $NE = 0$.Then:*
(1) *$R \propto E$ is always weak hereditary.*
(2) *$R \propto E$ is never hereditary.*

Proof. (i) Obvious from Proposition 3(ii) and [13, Theorem 2.8].
(ii) From [13, Theorem 2.6]. □

Herein our aim is to show the stability of the transfer of weak hereditary from a ring R to a ring $R \propto N$.

Theorem 7. *Suppose that (R, N) is a local total ring of quotients with N is its maximal ideal. Suppose alos that the module retraction map ϕ establishes $ker(\phi) \subseteq Nil(T)$ and $Mker(\phi) := 0$. Then $T := R \propto N$ is a weak hereditary ring.*

Proof. Let (R, N) be a local and total ring of quotients, the map ϕ establishes $ker(\phi) \subseteq Nil(T)$ and $Nker(\phi) = 0$. We put $Y = Ker(\phi)$. We have to show that every element $n + y$ of T is zero-divisor or invertible element to prove that T is a total ring of quotients. Indeed:
If $n \in N$, so n is a non invertible element of R. Because R is total of quotients, hence n is zero-divisor in R. So there exists $(0 \neq b \in N)$ such that $nb = 0$. Then, $b(n + y) = 0$ as $NY = 0$, so $n + y$ is a zero-divisor element in T.
If $n \notin N$, so n is invertible in R and in T as well, then, $n + y$ is invertible in T as sum of a nilpotent and an invertible. As a result of that, T is weak hereditary from Proposition3(ii).
□

For two rings $B_1 \subseteq B_2$. The ring B_1 is a subring retract of the ring B_2 if there exists a homomorphism $\phi : B_2 \longrightarrow B_1$ such that $\phi|_{B_1} = id|_{B_1}$. If the map ϕ exists, then ϕ is called a module retraction map. Suitable background ([5, 11, 5, 13, 17]).

Proposition 8. *Let A be a subring retract of a faithfully flat A-module R, for each ideal I of A, $IR \neq R$. Then if R is weak hereditary, then A is weak hereditary.*

Proof. Let R be a weak hereditary ring and $I_1 \subseteq I_2$ two ideals of A where I_2 is proper projective. Since R is faithfully flat over A, thus $I_2 \bigotimes_A R = I_2R$ is a proper projective ideal of R. Hence, we have $I_1R \subseteq I_2R$, then I_1T is projective, since R is weak hereditary ring. We claim that I_1 is a projective ideal of A. Indeed, for any A-module M, and from

[6, p.118],

$$Ext_A\left(I_1, M\bigotimes_A R\right) \cong Ext_R\left(I_1\bigotimes_A R, M\bigotimes_A R\right) = 0.$$

On the other hand, M is a direct summand of $M\bigotimes_A R$ since A is a direct summand of R. Therefore, $Ext_A(I_1, M) = 0$ for every A-module M. Then that I_1 is a projective ideal of A. □

In general, the homomorphic image of weak hereditary is not necessarily weak hereditary, as it is shown in the following examples

Example 9. Suppose that (R, N) is a non-Dedekind domain ring and local, $0 \neq L$ is an $R-module$, such that $NL = 0$ and let $T = R \propto E$. Then:

(1) T is weak hereditary.
(2) $R(\cong T/0 \propto E)$ is not weak hereditary.

Proof. (i) Obvious from Corollary 6(i).
(ii) We have $R \cong T/0 \propto E$ with $0 \propto E$ is an ideal of T. We shall prove that R is not a weak hereditary ring. Deny. Suppose that R is hereditary, then R is Dedekind, since R is a domain. A contradiction, then $T/(0 \propto E)$ is not weak hereditary. □

Now, the transfer of weak hereditary properties from a ring A to a ring $A \bowtie J$ will be discussed.

Proposition 10. *Suppose that A is a ring and J is a proper ideal of R. If (A, M) is local total of quotients, then :*

(1) *$A \bowtie J$ is weak hereditary.*
(2) *$A \bowtie J$ is not hereditary.*

Proof. (i) $A \bowtie J$ is a local ring, since A is a local ring. For show that $A \bowtie J$ is a total ring of quotients, we must show that every element $(a, a + j)$ of $A \bowtie J$ is invertible or zero-divisor element. There are two possible cases:
Case 1. $a \notin M$, a is invertible in A and then $(a, a + j) \notin M \bowtie J$. Since $A \bowtie J$ is a local ring, where $M \bowtie J$ is a maximal ideal of the local ring $A \bowtie J$, then, $(a, a + j)$ is invertible in $A \bowtie J$.
Case 2. $a \in M$, since A is a total ring of quotients in which every element is either a zero divisor or an invertible, then a is a zero-divisor element of A, hence from [7, Proposition 2.2] we have $(a, a + j)$ is a zero-divisor element of $A \bowtie I$. And so $A \bowtie J$ is local total of quotients. Thus $A \bowtie J$ is weak hereditary from Proposition 3(ii).
(ii) It is obvious that $O_1 = \{(0, j), j \in J\}$ and $O_2 = \{(j, 0), j \in J\}$ are ideals of $A \bowtie J$. We shall to show that O_1 is not projective. Deny. O_1 is projective. $A \bowtie J$ is local (from [9, Theorem 3.5]), since A is local, so , then O_1 is free. A contradiction since $O_1O_2 = 0$. Therefore, O_1 is not projective and so $A \bowtie J$ cannot be a hereditary ring. □

Example 11. Let $R := \mathbf{Z}/(2^i\mathbf{Z})$, with $i \geq 2$ be an integer, and let $T = R \propto R$, and suppose that $J(\neq 0)$ is a proper ideal of T. Then:
(i) R is weak hereditary.
(ii) T is weak hereditary.
(iii) $T \bowtie J$ is a weak hereditary ring.

The following example generally proves that the transfer of a weak hereditary ring notion does not hold in Pullback constructions. We adopt the following riding notations: T_0 is a domain of the form $T_0 = K_0 + M_0$, with K_0 is a field, D_0 is a subring of K_0 such that $qf(D_0) = K_0$, M_0 is a non-zero maximal ideal of T_0; and $R_0 = D_0 + M_0$. For more details, see [9, 10, 11].

Example 12. Suppose that D_0 is a Dedekind which is not a field, $K_0 = qf(D_0)$. Suppose also the following Pullback:

$$\begin{array}{ccc} A_0 = D_0 \propto K_0 & \longrightarrow & T_0 = K_0 \propto K_0 \\ \downarrow & & \downarrow \\ D_0 \cong A_0/0 \propto K_0 & \longrightarrow & K_0 \end{array}$$

A_0 is not weak hereditary from Corollary 2.3(ii), however both D_0 and T_0 are weak hereditary rings.
We study now the transfer of weak hereditary in rings direct product.

Theorem 13. *Suppose that $(A_i)_{i=1,2,...,n}$ is rings and $A := \prod_{i=1}^{n} A_i$. Then: If A is a weak hereditary ring, then A_i for each $i = 1, \ldots, n$.*

The proof of this theorem has been relies mainly on the following Lemma.

Lemma 14. *(*[15, Lemma 2.5]*)*
Suppose that $(A_i)_{i=1,2}$ is a family of rings and M_i is an $A_i - module$ for $i = 1, 2$, then, $M_1 \prod M_2$ is a projective $A_1 \prod A_2 - module$ if and only if M_i is a projective $A_i - module$ for i = 1, 2.

Proof of Theorem 13. The result for $i = 1, 2$ will be verified. Then, it will be verified by induction on n.
Suppose that $\prod_{i=1}^{2} A_i$ is a weak hereditary ring. To prove that A_1 and A_2 are weak hereditary rings. Let I_1 and J_1 two ideals of A_1 and suppose that $I_1 \subseteq J_1$ where J_1 is projective proper, then $I_1 \times A_2$ is an ideal of $\prod_{i=1}^{2} A_i$ and $J_1 \times A_2$ is projective proper of $\prod_{i=1}^{2} A_i$. Since, $I_1 \times A_2 \subseteq J_1 \times A_2$ and $\prod_{i=1}^{2} A_i$ is a weak hereditary ring, so, $I_1 \times A_2$ is a projective. Then I_1 is a projective ideal. □

We know that the direct product of hereditary ring is hereditary. But the next example has been proved that the direct product of weak hereditary is not in general weak hereditary.

Example 15. Suppose that $R_1 = \mathbf{Z}$ and $R_2 = R \bowtie I$ are two rings with $R = K \propto K$ and K is a field, then $R_1 \times R_2$ is not weak hereditary.

Proof. $R_1 = \mathbf{Z}$ is a hereditary ring. Then $R_2 = R \bowtie I$ is a weak hereditary but not hereditary ring from Theorem 10. On the other hand, $p\mathbf{Z} \times O_1 = \{(0, i), i \in I\} \subseteq p\mathbf{Z} \times R_2$ with $p\mathbf{Z} \times O_1$ is a ideal of $R_1 \times R_2$ and $p\mathbf{Z} \times R_2$ is projective proper ideal of $R_1 \times R_2$ from Lemma 14, but $p\mathbf{Z} \times O_1$ is not projective of $R_1 \times R_2$ since O_1 is not a projective ideal of R_2 (since $O_1 O_2 = 0$ and R_2 is local), then $R_1 \times R_2$ is not weak hereditary. □

3. Conclusion

From this piece of work, it has been concluded that:

Proposition 16. *Suppose that R is a ring. Then:*

(1) *If A contains a regular element, then A is weak hereditary if and only if A is hereditary.*

(2) *A is weak hereditary provided A is local total of quotients.*

Theorem 17. *Suppose that (R, N) is a local total ring of quotients with N is its maximal ideal. Suppose alos that the module retraction map ϕ establishes $ker(\phi) \subseteq Nil(T)$ and $Mker(\phi) := 0$. Then $T := R \propto N$ is a weak hereditary ring.*

Proposition 18. *Suppose that A is a ring and J is a proper ideal of R. If (A, M) is local total of quotients, then:*

(1) *$A \bowtie J$ is weak hereditary.*
(2) *$A \bowtie J$ is not hereditary.*

Example 19. Let $R_1 = \mathbf{Z}$ and let $R_2 = R \bowtie I$ be two rings with $R = K \propto K$ and K is a field, then $R_1 \times R_2$ is not weak hereditary.

Acknowledgement

The author would like to thank the anonymous referees for their valuable comments and suggestions which improved the presentation of this paper.

References

[1] F. Almahdi and N. mahdou, *On Weakly semihereditary rings*, Int. J. Open Probl. Comput. Sci. Math. **4**, No. 2 (2011) 38-49.

[2] F. Almahdi and N. mahdou, *On P-Weakly semihereditary rings*, J. Adv. Res. Pure Math., **5**, No. 2 (2013) 95-104.

[3] C. Bakkari, *On P-hereditary and P-semihereditary rings*, Int. J. Algebra, **3**, No. 7 (2009) 309-314.

[4] C. Bakkari, S. Kabbaj and N. Mahdou, *Trivial extensions defined by Prüfer conditions*, J. Pure Appl. Algebra, **214**, No. 1 (2010) 53-60.

[5] C. Bakkari, N. Mahdou and H. Mouanis, *Prüfer-Like Conditions in Subrings Retract and Applications*, Comm. Algebra, **37**, No. 1 (2009) 47-55.

[6] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.

[7] H.R. Maimani and S. Yassemi, *Zero-divisor graphs of amalgamated duplication of a ring along an ideal*, J. Pure Appl. Algebra, **212**, No. 1 (2008) 168-174.

[8] M.D 'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat., **45**, No. 2 (2007) 241-252.

[9] M.D 'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl., **6**, No. 3 (2007) 443–459.

[10] S. Gabelli and E. Houston, *Coherentlike conditions in pullbacks*, Michigan Math. J., **44**, (1997) 99-123.

[11] S. Glaz, *Commutative coherent rings*, Springer-Verlag, Berlin, 1989.

[12] J.A. Huckaba, *Commutative Rings with Zero-Divisors*, Marcel Dekker, New York, 1988.

[13] S. Kabbaj and N. Mahdou, *Trivial extension defined by coherent-like conditions*, Comm. Algebra, **32**, No. 10 (2004) 3937-3953.

[14] N. Mahdou, *On weakly finite conductor rings, Comm. Algebra*, **10**, (2004) 4027-4036.

[15] N. Mahdou, *On Costa's conjecture, Comm. Algebra*, **29**, (2001) 2775-2785.

[16] J.J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.

[17] W.V. Vasconcelas, *Finiteness in projective ideals*, J. Algebra, **25**, (1973) 269-278.