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Two New Cubically Convergent Iteration Schemes for Resolution of Nonlinear Equations Based On Quadrature Rules

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Abstract. In this paper, we suggest and analyze two new third order iterative methods for approximation of zeros of nonlinear equations based on quadrature rules. Convergence analysis of these iteration schemes have been discussed and computational comparison of these iteration schemes have been made with some known third order iteration schemes.

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1. Introduction and Preliminaries

The design of iterative methods for the resolution of nonlinear equations has acheived much attention of scholars in the field of numerical analysis. Many iteration schemes have been introduced by using different techniques such as Taylor series, decomposition techniques, homotopy and modified homotopy techniques, geometric methods and quadrature rules. In this paper, we focus on quadrature rules. Weerkoon and Fernando $[1]$ have reformulated the classical Newton's method approximating the definite integral

$$f(x) = f(x_n) + \int_{x_n}^{x} f^{'}(t)dt. \tag{1.1}$$

by the Rectangular Rule. They have also established a third order iterative method, approximating the integral in (1.1) by the Trapezoidal Rule. Frontini and Sormani $[2, 3]$ have derived third order method approximating the definite integral in (1.1) by the Mid Point Rule. Hasanov et al $[4]$ established a new cubically convergent method, approximating the integral in (1.1) by the Simpon's $\frac{1}{3}$ Rule.

We shall use the following result to establish our new algorithms. Consider the integral

$$I = \int_a^b f(t)dt. \tag{1.2}$$

Approximation of integral in (1.2) by Rectangular rule, Mid point rule and Trapeziodal rule are given below;

Rectangular rule;

$$I_R = (b-a)f(a).$$

Mid point rule;

$$I_M = (b-a)f\left(\frac{a+b}{2}\right).$$

Trapeziodal rule;

$$I_T = (b-a)\frac{f(a)+f(b)}{2}.$$

2. Development of New Iteration schemes

Consider the nonlinear equation is of the form

$$f(x) = 0. \tag{2.1}$$

Rewritting equation (2.1) as

$$x = g(x). \tag{2.2}$$

By Newton's theorem, we have

$$g(x) = g(x_n) + \int_{x_n}^{x} g^{'}(t)dt. \tag{2.3}$$

Approximating the definite integral in (2.3) by the Rectangular Rule and from equation (2.2) and (2.3), we have the following iteration scheme;

Algorithm 2.1 For any initial value x_0, we approximate the solution x_{n+1}, by the iteration scheme.

$$x_{n+1} = \frac{g\left(x_n\right) - x_n g^{'}\left(x_n\right)}{1 - g^{'}\left(x_n\right)}.$$

This algorithm has been introduced by Shin $[5]$ and has second order convergence.

Approximating the definite integral in (2.3) by the Mid point rule and from equation (2.2) and (2.3), we have the following iteration scheme.

Algorithm 2.2 For any initial value x_0, we approximate the solution x_{n+1}, by the iteration scheme.

Predictor step:

$$y_n = \frac{g\left(x_n\right) - x_n g^{'}\left(x_n\right)}{1 - g^{'}\left(x_n\right)},$$

Corrector step:

$$x_{n+1} = \frac{g(x_n) - x_n g^{'}\left(\frac{x_n+y_n}{2}\right)}{1 - g^{'}\left(\frac{x_n+y_n}{2}\right)}.$$

Approximating the definite integral in (2.3) by the Trapeziodal Rule and from equation (2.2) and (2.3), we have the following iteration scheme;

Algorithm 2.3 For any initial value x_0, we approximate the solution x_{n+1}, by the iteration scheme.

Predictor step:

$$y_n = \frac{g\left(x_n\right) - x_n g^{'}\left(x_n\right)}{1 - g^{'}\left(x_n\right)},$$

Corrector step:

$$x_{n+1} = \frac{2g(x_n) - x_n\{g^{'}(x_n) + g^{'}(y_n)\}}{2 - g^{'}(x_n) - g^{'}(y_n)}.$$

3. CONVEGENCE ANALYSIS

We discuss the convergence analysis of Algorithm 2.2 and Algorithm 2.3.

Theorem 3.1 Let I be an open interval and $f : I \subseteq R \to R$ be a sufficiently differentiable function. Let $\alpha \in I$ be a simple zero of f. If x_0 is initial guess sufficiently close to α, then convergence order of algorithm (2.2) is at least three.

Proof. Let α be simple zero of $f(x) = 0$ (equivalently $x = g(x)$). Let errors at n^{th} and $(n+1)^{th}$ iterations be e_n and e_{n+1} respectively. Expanding $g(x_n)$ and $g^{'}(x_n)$ by Taylor's expansion about α, we have

$$g(x_n) = \alpha + e_n g^{'}(\alpha) + \frac{1}{2}e_n^2 g^{''}(\alpha) + \frac{1}{6}e_n^3 g^{'''}(\alpha) + O(e_n^4) \qquad (3.1)$$

$$g^{'}(x_n) = g^{'}(\alpha) + e_n g^{''}(\alpha) + \frac{1}{2}e_n^2 g^{'''}(\alpha) + \frac{1}{6}e_n^3 g^{iv}(\alpha) + O(e_n^4) \qquad (3.2)$$

$$\begin{aligned} g(x_n) - x_n g^{'}(x_n) \quad &= \quad \alpha - \alpha g^{'}(\alpha) - \alpha g''(\alpha)e_n - \frac{1}{2}(g^{''}(\alpha) + \alpha g^{'''}(\alpha))e_n^2 - \\ &\quad \frac{1}{6}(2g^{'''}(\alpha) + \alpha g^{iv}(\alpha))e_n^3 + O(e_n^4)3.3 \qquad (3.\ 1) \end{aligned}$$

$$1 - g^{'}(x_n) = 1 - g^{'}(\alpha) - e_n g^{''}(\alpha) - \frac{1}{2}e_n^2 g^{'''}(\alpha) - \frac{1}{6}e_n^3 g^{iv}(\alpha) + O(e_n^4) \qquad (3.4)$$

On dividing Equation (3.3) by (3.4) and simplifying, we get

$$\begin{aligned} \frac{g\left(x_n\right) - x_n g^{'}\left(x_n\right)}{1 - g^{'}\left(x_n\right)} \quad &= \quad \alpha + \frac{g^{''}(\alpha)}{2(-1 + g^{'}(\alpha))}e_n^2 - \frac{1}{6(-1 + g^{'}(\alpha))^2}\{2g^{'''}(\alpha) - \\ &\quad 2g^{'''}(\alpha)g^{'}(\alpha) + 3g^{''^2}(\alpha)\}e_n^3 + O(e_n^4) \\ y_n \quad &= \quad \alpha + \frac{g^{''}(\alpha)}{2(-1 + g^{'}(\alpha))}e_n^2 - \frac{1}{6(-1 + g^{'}(\alpha))^2}\{2g^{'''}(\alpha) - \\ &\quad 2g^{'''}(\alpha)g^{'}(\alpha) + 3g^{''^2}(\alpha)\}e_n^3 + O(e_n^4)3.5 \qquad (3.\ 2) \end{aligned}$$

Expanding $g'\left(\frac{x_n+y_n}{2}\right)$ by Taylor's expansion about $\alpha,$ we have

$$\begin{aligned} g'\left(\frac{x_n+y_n}{2}\right) &= g^{'}(\alpha)+\frac{1}{2}g^{''}(\alpha)e_n+\frac{2g^{''2}(\alpha)-g^{'''}(\alpha)+g^{'''}(\alpha)g^{'}(\alpha)}{8(-1+g^{'}(\alpha))}e_n^2- \\ &\frac{1}{48(-1+g^{'}(\alpha))^2}\{14g^{''}(\alpha)g^{'''}(\alpha)-14g^{'}(\alpha)g^{''}(\alpha)g^{'''}(\alpha) \\ &+12g^{'''3}(\alpha)-g^{iv}(\alpha)+2g^{iv}(\alpha)g^{'}(\alpha)-g^{iv}(\alpha)g^{'2}(\alpha)\}e_n^3 \\ &+O(e_n^4)3.6 \qquad (3.\ 3) \end{aligned}$$

$$\begin{aligned} g(x_n)-x_ng^{'}\left(\frac{x_n+y_n}{2}\right) &= \alpha-\alpha g^{'}(\alpha)-\frac{1}{2}\alpha g^{''}(\alpha)e_n-\frac{1}{8(-1+g^{'}(\alpha))}\{ \\ &\alpha(2g^{''2}(\alpha)-g^{'''}(\alpha)+g^{'''}(\alpha)g^{'}(\alpha))\}e_n^2 \\ &+\frac{1}{48(-1+g^{'}(\alpha))^2}\{14\alpha g^{''}(\alpha)g^{'''}(\alpha)- \\ &14\alpha g^{'}(\alpha)g^{''}(\alpha)g^{'''}(\alpha)+12\alpha g^{'''3}(\alpha)-\alpha g^{iv}(\alpha) \\ &+2\alpha g^{iv}(\alpha)g^{'}(\alpha)+12g^{''2}(\alpha)-12g^{''2}(\alpha)g^{'}(\alpha) \\ &+2g^{'''}(\alpha)-4g^{'''}(\alpha)g^{'}(\alpha)+2g^{''2}(\alpha)g^{'''}(\alpha)\}e_n^3 \\ &+O(e_n^4)3.7 \qquad (3.\ 4) \end{aligned}$$

$$\begin{aligned} 1-g^{'}\left(\frac{x_n+y_n}{2}\right) &= 1-g^{'}(\alpha)-\frac{1}{2}g^{''}(\alpha)e_n+\frac{1}{8(-1+g^{'}(\alpha))}\{2g^{''2}(\alpha) \\ &-g^{'''}(\alpha)+g^{'''}(\alpha)g^{'}(\alpha)\}e_n^2+\frac{1}{48(-1+g^{'}(\alpha))^2}\{ \\ &14g^{''}(\alpha)g^{'''}(\alpha)-14g^{'}(\alpha)g^{''}(\alpha)g^{'''}(\alpha)+12g^{'''3}(\alpha) \\ &-g^{iv}(\alpha)+2g^{iv}(\alpha)g^{'}(\alpha)-g^{iv}(\alpha)g^{'2}(\alpha)\}e_n^3+O(e_n^4)3.8 \qquad (3.\ 5) \end{aligned}$$

Dividing Eq. (3.7) by Eq. (3.8) and simplifying, we have

$$\frac{g(x_n)-x_ng^{'}\left(\frac{x_n+y_n}{2}\right)}{1-g^{'}\left(\frac{x_n+y_n}{2}\right)}=\alpha+\frac{g^{'''}(\alpha)-g^{'''}(\alpha)g^{'}(\alpha)+6g^{''2}(\alpha)}{24(-1+g^{'}(\alpha))^2}e_n^3+O(e_n^4)$$

$$x_{n+1}=\alpha+\frac{g^{'''}(\alpha)-g^{'''}(\alpha)g^{'}(\alpha)+6g^{''2}(\alpha)}{24(-1+g^{'}(\alpha))^2}e_n^3+O(e_n^4)$$

Thus algorithm 2.2 has at least third order convergence. □

Theorem 3.2 Let I be an open interval and $f: I\subseteq R\rightarrow R$ be a sufficiently differentiable function. Let $\alpha\in I$ be a simple zero of f. If x_0 is initial guess sufficiently close to α, then convergence order of algorithm (2.2) is three.

Proof. From Eq. (3.5), we have

$$\begin{aligned} y_n &= \alpha+\frac{g^{''}(\alpha)}{2(-1+g^{'}(\alpha))}e_n^2-\frac{2g^{'''}(\alpha)-2g^{'''}(\alpha)g^{'}(\alpha)+3g^{''^2}(\alpha)}{6(-1+g^{'}(\alpha))^2}e_n^3 \\ &+O(e_n^4) \end{aligned}$$

Expanding $g^{'}(y_n)$ by Taylor's expansion about α, we have

$$\begin{aligned} g^{'}(y_n) &= g^{'}(\alpha)+\frac{g^{''2}(\alpha)}{2(-1+g^{'}(\alpha))}e_n^2-\frac{1}{6(-1+g^{'}(\alpha))^2}\{2g^{'''}(\alpha)g^{''}(\alpha) \\ &\quad -2g^{'''}(\alpha)g^{''}(\alpha)g^{'}(\alpha)+3g^{''3}(\alpha)\}e_n^3+O(e_n^4)3.9 \end{aligned} \quad (3.\ 6)$$

Adding Eq. (3.2) and Eq. (3.9), we get

$$\begin{aligned} g^{'}(x_n)+g^{'}(y_n) &= 2g^{'}(\alpha)+g^{''}(\alpha)e_n+\frac{1}{2(-1+g^{'}(\alpha))}\{-g^{'''}(\alpha)+ \\ &\quad g^{'''}(\alpha)g^{'}(\alpha)+g^{''^2}(\alpha)\}e_n^2+\frac{1}{6(-1+g^{'}(\alpha))^2}\{-g^{^{iv}}(\alpha) \\ &\quad +g^{^{iv}}(\alpha)g^{'}(\alpha)-g^{^{iv}}(\alpha)g^{'2}(\alpha)+2g^{''}(\alpha)g^{'''}(\alpha) \\ &\quad -2g^{'}(\alpha)g^{''}(\alpha)g^{'''}(\alpha)+3g^{''3}(\alpha)\}e_n^3+O(e_n^4)3.10 \end{aligned} \quad (3.\ 7)$$

$$\begin{aligned} 2g(x_n)-x_n(g^{'}(x_n)+g^{'}(y_n)) &= 2\alpha-2\alpha g^{'}(\alpha)-\alpha g^{''}(\alpha)e_n-\frac{\alpha}{2(-1+g^{'}(\alpha))}\{ \\ &\quad -g^{'''}(\alpha)+g^{'''}(\alpha)g^{'}(\alpha)g^{''^2}(\alpha)\}e_n^2+ \\ &\quad \frac{1}{6(-1+g^{'}(\alpha))^2}\{-\alpha g^{^{iv}}(\alpha)+\alpha g^{^{iv}}(\alpha)g^{'}(\alpha) \\ &\quad -\alpha g^{^{iv}}(\alpha)g^{'2}(\alpha)+2\alpha g^{''}(\alpha)g^{'''}(\alpha)+3\alpha g^{''3}(\alpha) \\ &\quad -g^{'''}(\alpha)+2g^{'''}(\alpha)g^{'}(\alpha)-g^{'''}(\alpha)g^{'2}(\alpha) \\ &\quad +3g^{''2}(\alpha)-3g^{''2}(\alpha)g^{'}(\alpha)\}e_n^3+O(e_n^4)3.11 \end{aligned} \quad (3.\ 8)$$

$$\begin{aligned} 2-g^{'}(x_n)-g^{'}(y_n) &= 2-2g^{'}(\alpha)-g^{''}(\alpha)e_n-\frac{1}{2(-1+g^{'}(\alpha))}\{-g^{'''}(\alpha)+ \\ &\quad g^{'''}(\alpha)g^{'}(\alpha)+g^{''^2}(\alpha)\}e_n^2-\frac{1}{6(-1+g^{'}(\alpha))^2}\{-g^{^{iv}}(\alpha) \\ &\quad +g^{^{iv}}(\alpha)g^{'}(\alpha)-g^{^{iv}}(\alpha)g^{'2}(\alpha)+2g^{''}(\alpha)g^{'''}(\alpha) \\ &\quad -2g^{'}(\alpha)g^{''}(\alpha)g^{'''}(\alpha)-3g^{''3}(\alpha)\}e_n^3+O(e_n^4)3.12 \end{aligned} \quad (3.\ 9)$$

On dividing Eq. (3.11) by Eq. (3.12) and after simplification. we have

$$\frac{2g(x_n)-x_n(g^{'}(x_n)+g^{'}(y_n))}{2-g^{'}(x_n)-g^{'}(y_n)}=\alpha+\frac{-g^{'''}(\alpha)+g^{'''}(\alpha)g^{'}(\alpha)+3g^{''2}(\alpha)}{12(-1+g^{'}(\alpha))^2}e_n^3+O(e_n^4)$$

$$x_{n+1}=\alpha+\frac{-g^{'''}(\alpha)+g^{'''}(\alpha)g^{'}(\alpha)+3g^{''2}(\alpha)}{12(-1+g^{'}(\alpha))^2}e_n^3+O(e_n^4)$$

□

Hence algorithm 2.3 has third order convergence.

4. Applications

We consider some nonlinear equations to make the comparision of our newly established iteration schemes with classical Newton method(NM), Householder method (HHM), Chun method (CM) [7], Noor method(NR) [8], Abbassbandy method(AM) [9], Weerakon and Fernando method (WAF) [1]. The number of iterations to approximate the zero (IT), the absolute value of function ($|f(x_n)|$) and the computational order of convergence (COC) are also shown in comparison table given below. Here, COC is defined by

$$\rho \approx \frac{\ln\left|\frac{(x_{n+1}-x_n)}{(x_n-x_{n-1})}\right|}{\ln\left|\frac{(x_n-x_{n-1})}{(x_{n-1}-x_{n-2})}\right|}$$

All the computations are performed on Core i5, 2.40 GHz by using Maple 13. We use $\in = 10^{-30}$. The following stopping criteria is used for estimating the zero:

$$\begin{array}{lrcl} (i) & |x_n - x_{n-1}| & < & \in \\ (ii) & |f(x_n)| & < & \in \end{array}$$

The following nonlinear equations are considered to illustarate the performance of our newly introduced iteration schemes.

$$\begin{aligned}
f_1(x) &= \sin^2 x - x^2 + 1,\ g(x) = \sin x + \frac{1}{x + \sin x} \\
f_2(x) &= x^2 - e^x - 3x + 2,\ g(x) = \frac{x^2 - e^x + 2}{3} \\
f_3(x) &= \cos x - x,\ g(x) = \cos x \\
f_4(x) &= (x-1)^3 - 1,\ g(x) = 1 + \sqrt{\frac{1}{x-1}} \\
f_5(x) &= x^3 - 10,\ g(x) = \sqrt{\frac{10}{x}} \\
f_6(x) &= e^{x^2+7x-30} - 1,\ g(x) = \frac{1}{7}(30 - x^2)
\end{aligned}$$

Table 1. Comparison of NM, AM, HHM, CM, NR, WAF, Alg. 2.2 and Alg. 2.3 ($f_1(x) = \sin^2 x - x^2 + 1, g(x) = \sin x + \frac{1}{x+\sin x}, x_0 = -1$)

Method	IT	x_n	$\lvert f(x_n)\rvert$	COC
NM	7	1.404491648315341226350868177	$1.04e^{-50}$	2
AM	5	1.404491648315341226350868177	$5.80e^{-55}$	2.8643
HHM	6	1.404491648315341226350868178	$1.45e^{-82}$	2.9943
CM	5	1.404491648315341226350868178	$2.01e^{-62}$	2.9969
NR	5	1.404491648315341226350868176	$2.49e^{-86}$	3.0165
WAF	4	1.404491648215341226035086817	$4.91e^{-30}$	3.0453
Alg.2.2	4	1.404491648215341226035086820	$1.51e^{-81}$	3.0008
Alg.2.3	4	1.404491648215341226029277224	$7.90e^{-62}$	3.001

Table 2. Comparison of NM, AM, HHM, CM, NR, WAF, Alg. 2.2 and Alg. 2.3 ($f_2(x) = x^2 - e^x - 3x + 2, g(x) = \frac{x^2-e^x+2}{3}, x_0 = 2$)

$Method$	IT	x_n	$\lvert f(x_n)\rvert$	COC
NM	6	0.257530285439860760455367303	$2.92e^{-55}$	2.0005
AM	5	0.257530285439860760455367304	$1.89e^{-35}$	3.0003
HHM	4	0.257530285439860760455367305	$5.33e^{-63}$	3.0050
CM	4	0.257530285439860760455367304	$1.01e^{-62}$	3.0005
NR	4	0.257530285439860760455367306	$1.91e^{-72}$	3.0001
WAF	4	0.257530285439860760455367305	$6.10e^{-34}$	3.0010
Alg.2.2	4	0.257530285439860763223411636	$2.77e^{-54}$	3.0083
Alg.2.3	4	0.257530285439860760455367305	$2.67e^{-34}$	3.0100

Table 3. Comparison of NM, AM, HHM, CM, NR, WAF, Alg. 2.2 and Alg. 2.3
$(f_3(x) = \cos x - x,\, g(x) = \cos x,\, x_0 = 1.7)$

$Method$	IT	x_n	$\lvert f(x_n)\rvert$	COC
NM	5	0.739085133215160641655372084	$2.03e^{-32}$	2
AM	4	0.739085133215160641655372085	$7.14e^{-47}$	3.0014
HHM	4	0.739085133215160641655372086	$3.08e^{-42}$	2.9877
CM	4	0.739085133215160641655372087	0	2.9923
NR	4	0.739085133215160641655372089	$6.76e^{-47}$	3.0105
WAF	4	0.739085133215160641655312088	$2.84e^{-65}$	3.0017
Alg.2.2	4	0.739085133215160641669785117	$3.32e^{-61}$	2.9976
Alg.2.3	4	0.739085133215160641654275011	$2.84e^{-65}$	3.0117

Table 4. Comparison of NM, AM, HHM, CM, NR, WAF, Alg. 2.2 and Alg. 2.3
$(f_4(x) = (x-1)^3 - 1,\, g(x) = 1 + \sqrt{\frac{1}{x-1}},\, x_0 = 3.5)$

$Method$	IT	x_n	$\lvert f(x_n)\rvert$	COC
NM	8	2.000000000000000000000000023	$2.06e^{-42}$	2
AM	5	2	$7.98e^{-64}$	2.9901
HHM	5	2	$3.67e^{-31}$	2.9706
CM	5	2	0	2.9864
NR	5	2.000000000000000000000000000	$4.10e^{-71}$	2.9959
WAF	5	2.000000000000000000000000000	$9.83e^{-37}$	2.9864
Alg 2.2	4	2.000000000000000000000000000	$8.92e^{-74}$	3.0071
Alg 2.3	4	2.000000000000000000000000000	$4.35e^{-33}$	2.9979

Table 5. Comparison of NM, AM, HHM, CM, NR, WAF, Alg. 2.2 and Alg. 2.3
$(f_5(x) = x^3 - 10,\, g(x) = \sqrt{\frac{10}{x}},\, x_0 = 1.5)$

$Method$	IT	x_n	$\lvert f(x_n)\rvert$	COC
NM	7	2.154434690031883721759235664	$2.06e^{-54}$	1.9950
AM	5	2.154434690031883721759293566	$1.64e^{-75}$	2.9970
HHM	5	2.154434690031883721759235667	$3.27e^{-52}$	3.0029
CM	5	2.154434690031883721759235667	$5.0e^{-63}$	3.0001
NR	4	2.154434690031883721759235663	$6.54e^{-42}$	2.6530
WAF	4	2.154434690031883721759293566	$7.06e^{-31}$	3.0267
Alg 2.2	3	2.154434690031883721759293566	$1.69e^{-33}$	3.2451
Alg 2.3	4	2.154434690031883721759293566	$3.87e^{-67}$	3.0080

Table 6. Comparison of NM, AM, HHM, CM, NR, WAF, Alg. 2.2 and Alg. 2.3
$(f_6(x) = e^{x^2+7x-30} - 1,\, g(x) = \frac{1}{7}(30 - x^2),\, x_0 = 3.5)$

$Method$	IT	x_n	$\lvert f(x_n)\rvert$	COC
NM	13	3.0000000000000000000000000000	$1.52e^{-47}$	2.002
AM	7	3.0000000000000000000000000000	$4.33e^{-48}$	2.9840
HHM	9	3.0000000000000000000000000000	$1.50e^{-81}$	3.0661
CM	8	3.0000000000000000000000000000	$2.0e^{-62}$	2.9884
NR	7	3.0000000000000000000000000000	$1.71e^{-87}$	2.3856
WAF	9	3.0000000000000000000000000000	$5.07e^{-71}$	2.9998
Alg 2.2	3	3.0000000000000000000000000000	$3.81e^{-37}$	2.9835
Alg 2.3	3	3.0000000000000000000000000000	$3.81e^{-37}$	2.9999

4.1. **Efficiency Index.** Efficiency index of an iteration scheme is defined as $p^{\frac{1}{E}}$, where p is order of convergence and E is number of function and its derivative evaluations per iteration. The efficiency index of algorithm 2.2 and 2.3 developed in this paper is $3^{\frac{1}{3}} = 1.442\,2$, which is better than Newton method.

5. Conclusion

In this paper, we have introduced two new third order iterative methods for nonlinear equations based on quadrature rules which are free of second derivatives. Convergence analysis of these iterative methods have been discussed. Some nonlinear problems have been solved to check the performance and efficiency of our newly introduced iterative methods. We have made comparison of our new iterative methods with some well known iterative methods existing in literature. From numerical results, we conclude that our newly introduced iterative methods perform well as compared to other methods.

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