Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(10)(2019) pp. 93-106

Weighted Simpson's type inequalities for GA-convex functions

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Received: 20 February, 2019 / Accepted: 26 June, 2016 / Published online: 01 September, 2019

Abstract. In this paper, some new weighted Simpson type integral inequalities are presented for the class of GA-convex functions.

AMS (MOS) Subject Classification Codes: 26A33; 41A55; 26D15; 26E60
Key Words: Simpson's inequality; convex functions; GA-convex function.

1. Introduction

The Simpson inequality states that if $\zeta : [\eta_1, \eta_2] \to \mathbb{R}$ is a four times continuously differentiable mapping on (η_1, η_2) and $\left\|\zeta^{(4)}\right\|_\infty = \sup_{y\in(\eta_1,\eta_2)} \left|\zeta^{(4)}(y)\right| < \infty$, then

$$\left|\int_{\eta_1}^{\eta_2} \zeta(\theta)\, d\theta - \frac{\eta_2-\eta_1}{3}\left[\frac{\zeta(\eta_1)+\zeta(\eta_2)}{2} + 2\zeta\left(\frac{\eta_1+\eta_2}{2}\right)\right]\right| \leq \frac{1}{2880}\left\|\zeta^{(4)}\right\|_\infty \cdot (\eta_2-\eta_1)^4 . \quad (1.\ 1)$$

There is a substantial literature on the generalizations of Simpson's inequality, Simpson type integral inequalities and Hermite-Hadamard type integral inequalities by using a variety of convexity conditions, see for example [1]-[46] and the references cited therein.

Recall that a function $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ is called convex function if the inequality

$$\zeta(y\delta + (1-y)\sigma) \leq y\zeta(\delta) + (1-y)\zeta(\sigma)$$

holds for all δ, $\sigma \in I$ and $y \in [0,1]$.

One of the generalizations of the convex functions, known as the GA-convex functions, is stated as follows:

Let $\zeta : I \subset (0,\infty) \to \mathbb{R}$. If the inequality

$$\zeta\left(\delta^{y}\sigma^{1-y}\right) \leq y\zeta\left(\delta\right) + \left(1-y\right)\zeta\left(\sigma\right)$$

holds for all δ, $\sigma \in I$ and $I \in [0,1]$, then is said to be GA-convex on I.

Example 1.1. *Consider the function $\zeta : (0,\infty) \to \mathbb{R}$ defined as $\zeta\left(x\right) = \ln x$, then this function is GA-convex function on $(0,\infty)$. Let δ, $\sigma \in (0,\infty)$ and $y \in [0,1]$, then*

$$\begin{aligned}\zeta\left(\delta^{y}\sigma^{1-y}\right) &= \ln\left(\delta^{y}\sigma^{1-y}\right) \leq y\ln\delta + \left(1-y\right)\ln\sigma \\ &\leq y\zeta\left(\delta\right) + \left(1-y\right)\zeta\left(\sigma\right).\end{aligned}$$

Thus the function $\zeta\left(x\right) = \ln x$ is GA-convex function on $(0,\infty)$.

The aim of this paper is to present some new weighted Simpson type integral inequalities for the class of GA-convex functions.

2. Weighted Simpson's type inequalities for GA-convex functions

In order to prove the results for this paper, we need the following lemma.

Lemma 2.1. *Let $\zeta : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, where η_1, $\eta_2 \in I^{\circ}$ with $\eta_1 < \eta_2$ and let $\phi : [\eta_1,\eta_2] \to [0,\infty)$ be a continuous positive mapping and geometrically symmetric with respect to $\sqrt{\eta_1\eta_2}$. If $\zeta^{'}, \phi \in L_1[\eta_1,\eta_2]$, then*

$$\begin{aligned}&\frac{1}{8\left(\ln\eta_2 - \ln\eta_1\right)}\left[\zeta\left(\eta_1\right) + 6\zeta\left(\sqrt{\eta_1\eta_2}\right) + \zeta\left(\eta_2\right)\right]\int_{\eta_1}^{\eta_2}\frac{\phi\left(z\right)}{z}dz \\ &\quad - \frac{1}{\ln\eta_2 - \ln\eta_1}\int_{\eta_1}^{\eta_2}\frac{\phi\left(z\right)\zeta\left(z\right)}{z}dz = \frac{\ln\eta_2 - \ln\eta_1}{4} \\ &\quad \times\left\{\int_0^1 p_1\left(y\right)\lambda_1\left(y\right)\zeta^{'}\left(\lambda_1\left(y\right)\right)dy - \int_0^1 p_2\left(y\right)\lambda_2\left(y\right)\zeta^{'}\left(\lambda_2\left(y\right)\right)dy\right\}, \quad (2.\ 2)\end{aligned}$$

where

$$p_1\left(y\right) = \frac{3}{4}\int_0^1\phi\left(\lambda_1\left(s\right)\right)ds - \int_0^y\phi\left(\lambda_1\left(s\right)\right)ds$$

and

$$p_2\left(y\right) = \frac{3}{4}\int_0^1\phi\left(\lambda_2\left(s\right)\right)ds - \int_0^y\phi\left(\lambda_2\left(s\right)\right)ds.$$

Proof. By integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_0^1 p_1(y)\,\lambda_1(y)\,\zeta^{'}(\lambda_1(y))\,dy \\
&= -\frac{2}{\ln\eta_2-\ln\eta_1}\int_0^1\left[\frac{3}{4}\int_0^1\phi(\lambda_1(s))\,ds-\int_0^y\phi(\lambda_1(s))\,ds\right]d\left[\zeta(\lambda_1(y))\right] \\
&= -\frac{2}{\ln\eta_2-\ln\eta_1}\left[\frac{3}{4}\int_0^1\phi(\lambda_1(s))\,ds-\int_0^y\phi(\lambda_1(s))\,ds\right]\zeta(\lambda_1(y))\Bigg|_0^1 \\
&-\frac{2}{\ln\eta_2-\ln\eta_1}\int_0^1\phi(\lambda_1(y))\,\zeta(\lambda_1(y))\,dy=\frac{2}{\ln\eta_2-\ln\eta_1}\left[\frac{\zeta(\eta_1)}{4}\int_0^1\phi(\lambda_1(y))\,dy\right] \\
&+\frac{2}{\ln\eta_2-\ln\eta_1}\left[\frac{3\zeta\left(\sqrt{\eta_1\eta_2}\right)}{4}\int_0^1\phi(\lambda_1(y))\,dy\right] \\
&-\frac{2}{\ln\eta_2-\ln\eta_1}\int_0^1\phi(\lambda_1(y))\,\zeta(\lambda_1(y))\,dy.
\end{aligned}
$$

By making the substitution $z=\lambda_1(y)$, we get

$$
\begin{aligned}
I_1 &= \frac{1}{(\ln\eta_2-\ln\eta_1)^2}\left[\zeta(\eta_1)+3\zeta\left(\sqrt{\eta_1\eta_2}\right)\right]\int_{\eta_1}^{\sqrt{\eta_1\eta_2}}\frac{\phi(z)}{z}dz \\
&-\frac{4}{(\ln\eta_2-\ln\eta_1)^2}\int_{\eta_1}^{\sqrt{\eta_1\eta_2}}\frac{\phi(z)\,\zeta(z)}{z}dz.
\end{aligned}
$$

Similarly, we can have

$$
\begin{aligned}
I_2 &= \int_0^1 p_2(y)\,\lambda_2(y)\,\zeta^{'}(\lambda_2(y))\,dy \\
&= -\frac{1}{(\ln\eta_2-\ln\eta_1)^2}\left[3\zeta\left(\sqrt{\eta_1\eta_2}\right)+\zeta(\eta_2)\right]\int_{\sqrt{\eta_1\eta_2}}^{\eta_2}\frac{\phi(z)}{z}dz \\
&+\frac{4}{(\ln\eta_2-\ln\eta_1)^2}\int_{\sqrt{\eta_1\eta_2}}^{\eta_2}\frac{\phi(z)\,\zeta(z)}{z}dz.
\end{aligned}
$$

Since $\phi(z)$ is geometrically symmetric with respect to $\sqrt{\eta_1\eta_2}$, we have

$$\int_{\eta_1}^{\sqrt{\eta_1\eta_2}} \frac{\phi(z)}{z} dz = \int_{\sqrt{\eta_1\eta_2}}^{\eta_2} \frac{\phi(z)}{z} dz = \frac{1}{2}\int_{\eta_1}^{\eta_2} \frac{\phi(z)}{z} dz.$$

Thus, we have

$$\begin{aligned}
&\frac{\ln\eta_2 - \ln\eta_1}{4}(I_1 - I_2) \\
&= \frac{1}{8(\ln\eta_2 - \ln\eta_1)}\left[\zeta(\eta_1) + 6\zeta\left(\sqrt{\eta_1\eta_2}\right) + \zeta(\eta_2)\right]\int_{\eta_1}^{\eta_2} \frac{\phi(z)}{z} dz \\
&- \frac{1}{\ln\eta_2 - \ln\eta_1}\int_{\eta_1}^{\eta_2} \frac{\phi(z)\zeta(z)}{z} dz.
\end{aligned}$$

□

Remark 2.2. *Throughout this manuscript we will use the following notation for the sake of convenience*

$$\begin{aligned}
\Psi(\eta_1, \eta_2; \zeta, \phi) &= \frac{1}{8(\ln\eta_2 - \ln\eta_1)}\left[\zeta(\eta_1) + 6\zeta\left(\sqrt{\eta_1\eta_2}\right) + \zeta(\eta_2)\right]\int_{\eta_1}^{\eta_2} \frac{\phi(z)}{z} dz \\
&- \frac{1}{\ln\eta_2 - \ln\eta_1}\int_{\eta_1}^{\eta_2} \frac{\phi(z)\zeta(z)}{z} dz
\end{aligned}$$

Corollary 2.3. *Under the assumptions of Lemma 2.1, the following inequality holds*

$$\begin{aligned}
&\Psi(\eta_1, \eta_2; \zeta, \phi) \le \frac{(\ln\eta_2 - \ln\eta_1)\|\phi\|_{[\eta_1,\eta_2],\infty}}{4} \\
&\times\left\{\int_0^1 \left(\frac{3}{4} - y\right)\lambda_1(y)\zeta'(\lambda_1(y))\,dy - \int_0^1 \left(\frac{3}{4} - y\right)\lambda_2(y)\zeta'(\lambda_2(y))\,dy\right\}, \quad (2.\ 3)
\end{aligned}$$

where $\|\phi\|_{[\eta_1,\eta_2],\infty} = \sup_{y\in[\eta_1,\eta_2]} |\phi(y)|$.

Proof. Proof follows from the fact that

$$\|\phi\|_{\left[\eta_1,\sqrt{\eta_1\eta_2}\right],\infty} \le \|\phi\|_{[\eta_1,\eta_2],\infty}$$

and

$$\|\phi\|_{\left[\sqrt{\eta_1\eta_2},\eta_2\right],\infty} \le \|\phi\|_{[\eta_1,\eta_2],\infty}.$$

□

Theorem 2.4. *Let $\zeta : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, where η_1, $\eta_2 \in I^{\circ}$ with $\eta_1 < \eta_2$ and let $\phi : [\eta_1,\eta_2] \to [0,\infty)$ be a continuous positive mapping and geometrically symmetric with respect to $\sqrt{\eta_1\eta_2}$. If $\zeta^{'}, \phi \in L_1[\eta_1,\eta_2]$ and $\left|\zeta^{'}\right|^{\vartheta}$ is GA-convex on $[\eta_1,\eta_2]$ for $\vartheta \geq 1$, then*

$$\begin{aligned}
\left|\Psi\left(\eta_1,\eta_2;\zeta,\phi\right)\right| &\leq \frac{\eta_2\left(\ln\eta_2-\ln\eta_1\right)\|\phi\|_{[\eta_1,\eta_2],\infty}}{4}\left(\frac{5}{16}\right)^{1-\frac{1}{\vartheta}} \\
&\times\left\{\left[\frac{\theta\left(8\left(1-2\theta^{3/4}+\theta\right)-\left(1-14\theta^{3/4}+9\theta\right)\ln\theta+\left(2\theta-3\right)\left(\ln\theta\right)^2\right)}{8\left(\ln\theta\right)^3}\left|\zeta^{'}\left(\eta_1\right)\right|^{\vartheta}\right.\right. \\
&\left.+\frac{\theta\left(-8\left(1-2\theta^{3/4}+\theta\right)+\left(\theta+2\theta^{3/4}-7\right)\ln\theta-3\left(\ln\theta\right)^2\right)}{8\left(\ln\theta\right)^3}\left|\zeta^{'}\left(\eta_2\right)\right|^{\vartheta}\right]^{\frac{1}{\vartheta}} \\
&+\left[\frac{\left(-8\left(1-2\theta^{1/4}+\theta\right)-\left(9-14\theta^{1/4}+\theta\right)\ln\theta+\left(3\theta-3\right)\left(\ln\theta\right)^2\right)}{8\left(\ln\theta\right)^3}\left|\zeta^{'}\left(\eta_2\right)\right|^{\vartheta}\right. \\
&\left.\left.+\frac{\left(8\left(1-2\theta^{1/4}+\theta\right)+\left(1+2\theta^{1/4}-7\theta\right)\ln\theta+3\theta\left(\ln\theta\right)^2\right)}{8\left(\ln\theta\right)^3}\left|\zeta^{'}\left(\eta_1\right)\right|^{\vartheta}\right]^{\frac{1}{\vartheta}}\right\}, \quad (2.\ 4)
\end{aligned}$$

where $\theta = \left(\eta_1/\eta_2\right)^{\vartheta/2}$.

Proof. From (2. 3) and using the power-mean inequality, we have

$$\begin{aligned}
\left|\Psi\left(\eta_1,\eta_2;\zeta,\phi\right)\right| &\leq \frac{\left(\ln\eta_2-\ln\eta_1\right)\|\phi\|_{[\eta_1,\eta_2],\infty}}{4} \\
&\times\left\{\left(\int_0^1\left|\frac{3}{4}-y\right|dy\right)^{1-\frac{1}{\vartheta}}\left(\int_0^1\left|\frac{3}{4}-y\right|\lambda_1^{\vartheta}(y)\left|\zeta^{'}\left(\lambda_1(y)\right)\right|^{\vartheta}dy\right)^{\frac{1}{\vartheta}}\right. \\
&\left.+\left(\int_0^1\left|\frac{3}{4}-y\right|dy\right)^{1-\frac{1}{\vartheta}}\left(\int_0^1\left|\frac{3}{4}-y\right|\lambda_2^{\vartheta}(y)\left|\zeta^{'}\left(\lambda_2(y)\right)\right|^{\vartheta}dy\right)^{\frac{1}{\vartheta}}\right\}. \quad (2.\ 5)
\end{aligned}$$

By using the GA-convexity of $\left|\zeta'\right|^{\vartheta}$ on $[\eta_1,\eta_2]$ for $\vartheta\geq 1$, we get

$$\begin{aligned}
&\int_0^1 \left|\frac{3}{4}-y\right| \lambda_1^{\vartheta}(y)\left|\zeta'(\lambda_1(y))\right|^{\vartheta} dy \\
&\leq \left|\zeta'(\eta_1)\right|^{\vartheta}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-y\right)\left(\frac{1+y}{2}\right)\lambda_1^{\vartheta}(y)\,dy+\int_{\frac{3}{4}}^1\left(y-\frac{3}{4}\right)\left(\frac{1+y}{2}\right)\lambda_1^{\vartheta}(y)\,dy\right] \\
&+\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-y\right)\left(\frac{1-y}{2}\right)\lambda_1^{\vartheta}(y)\,dy+\int_{\frac{3}{4}}^1\left(y-\frac{3}{4}\right)\left(\frac{1-y}{2}\right)\lambda_1^{\vartheta}(y)\,dy\right]\left|\zeta'(\eta_2)\right|^{\vartheta} \\
&=\frac{\eta_2^{\vartheta}\theta\left(8\left(1-2\theta^{3/4}+\theta\right)+\left(-1+14\theta^{3/4}-9\theta\right)\ln\theta+(-3+2\theta)(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta'(\eta_1)\right|^{\vartheta} \\
&+\frac{\eta_2^{\vartheta}\theta\left(-8\left(1-2\theta^{3/4}+\theta\right)+\left(-7+2\theta^{3/4}+\theta\right)\ln\theta-3(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta'(\eta_2)\right|^{\vartheta} \qquad (2.\ 6)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{3}{4}-y\right| \lambda_2^{\vartheta}(y)\left|\zeta'(\lambda_2(y))\right|^{\vartheta} dy \\
&\leq \left|\zeta'(\eta_1)\right|^{\vartheta}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-y\right)\left(\frac{1-y}{2}\right)\lambda_2^{\vartheta}(y)\,dy+\int_{\frac{3}{4}}^1\left(y-\frac{3}{4}\right)\left(\frac{1-y}{2}\right)\lambda_2^{\vartheta}(y)\,dy\right] \\
&+\left|\zeta'(\eta_2)\right|^{\vartheta}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-y\right)\left(\frac{1+y}{2}\right)\lambda_2^{\vartheta}(y)\,dy+\int_{\frac{3}{4}}^1\left(y-\frac{3}{4}\right)\left(\frac{1+y}{2}\right)\lambda_2^{\vartheta}(y)\,dy\right] \\
&=\frac{\eta_2^{\vartheta}\left(-8\left(1-2\theta^{1/4}+\theta\right)-\left(9-14\theta^{1/4}+\theta\right)\ln\theta+(-2+3\theta)(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta'(\eta_2)\right|^{\vartheta} \\
&+\frac{\eta_2^{\vartheta}\left(8\left(1-2\theta^{1/4}+\theta\right)+\left(1+2\theta^{1/4}-7\theta\right)\ln\theta+3\theta(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta'(\eta_1)\right|^{\vartheta}, \qquad (2.\ 7)
\end{aligned}$$

where $\theta=(\eta_1/\eta_2)^{\vartheta/2}$.
Using (2. 6) and (2. 7) in (2. 5) we get (2. 4). □

Theorem 2.5. *Let $\zeta : I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I°, where η_1, $\eta_2\in I^{\circ}$ with $\eta_1<\eta_2$ and let $\phi:[\eta_1,\eta_2]\to[0,\infty)$ be a continuous positive mapping and geometrically symmetric with respect to $\sqrt{\eta_1\eta_2}$. If $\zeta',\phi\in L_1[\eta_1,\eta_2]$ and $\left|\zeta'\right|^{\vartheta}$ is*

GA-convex on $[\eta_1, \eta_2]$ *for* $\vartheta > 1$*, then*

$$|\Psi(\eta_1, \eta_2; \zeta, \phi)| \leq \frac{(\ln \eta_2 - \ln \eta_1) \|\phi\|_{[\eta_1,\eta_2],\infty}}{4} \left[L\left(\eta_1^{\frac{\vartheta}{2(\vartheta-1)}}, \eta_2^{\frac{\vartheta}{2(\vartheta-1)}} \right) \right]^{1-\frac{1}{\vartheta}}$$
$$\times \left\{ \eta_1^{\frac{1}{2}} \left[\left(\frac{2^{-2\vartheta-5} \times 3^{\vartheta+1}(4\vartheta+11) + 2^{-2\vartheta-5}(8\vartheta+15)}{(\vartheta+1)(\vartheta+2)} \right) \left|\zeta'(\eta_1)\right|^{\vartheta} \right.\right.$$
$$\left. + \left(\frac{2^{-2\vartheta-5} + 2^{-2\vartheta-5} \times 3^{\vartheta+1}(4\vartheta+5)}{(\vartheta+1)(\vartheta+2)} \right) \left|\zeta'(\eta_2)\right|^{\vartheta} \right]^{\frac{1}{\vartheta}}$$
$$+ \eta_2^{\frac{1}{2}} \left[\left(\frac{2^{-2\vartheta-5} + 2^{-2\vartheta-5} \times 3^{\vartheta+1}(4\vartheta+5)}{(\vartheta+1)(\vartheta+2)} \right) \left|\zeta'(\eta_1)\right|^{\vartheta} \right.$$
$$\left.\left. + \left(\frac{2^{-2\vartheta-5} \times 3^{\vartheta+1}(4\vartheta+11) + 2^{-2\vartheta-5}(8\vartheta+15)}{(\vartheta+1)(\vartheta+2)} \right) \left|\zeta'(\eta_2)\right|^{\vartheta} \right]^{\frac{1}{\vartheta}} \right\}. \quad (2.\ 8)$$

Proof. From (2. 3) and using the Hölder integral inequality, we have

$$|\Psi(\eta_1, \eta_2; \zeta, \phi)| \leq \frac{(\ln \eta_2 - \ln \eta_1) \|\phi\|_{[\eta_1,\eta_2],\infty}}{4}$$
$$\times \left\{ \left(\int_0^1 \lambda_1^{\frac{\vartheta}{\vartheta-1}}(y)\, dy \right)^{1-\frac{1}{\vartheta}} \left(\int_0^1 \left|\frac{3}{4} - y\right|^{\vartheta} \left|\zeta'(\lambda_1(y))\right|^{\vartheta} dy \right)^{\frac{1}{\vartheta}} \right.$$
$$\left. + \left(\int_0^1 \lambda_2^{\frac{\vartheta}{\vartheta-1}}(y)\, dy \right)^{1-\frac{1}{\vartheta}} \left(\int_0^1 \left|\frac{3}{4} - y\right|^{\vartheta} \left|\zeta'(\lambda_2(y))\right|^{\vartheta} dy \right)^{\frac{1}{\vartheta}} \right\}. \quad (2.\ 9)$$

By using the GA-convexity of $\left|\zeta'\right|^{\vartheta}$ on $[\eta_1, \eta_2]$ for $\vartheta \geq 1$, we get

$$\int_0^1 \left|\frac{3}{4} - y\right|^{\vartheta} \left|\zeta'(\lambda_1(y))\right|^{\vartheta} dy \leq \left|\zeta'(\eta_1)\right|^{\vartheta} \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - y\right)^{\vartheta} \left(\frac{1+y}{2}\right) dy \right.$$
$$\left. + \int_{\frac{3}{4}}^1 \left(y - \frac{3}{4}\right)^{\vartheta} \left(\frac{1+y}{2}\right) dy \right] + \left|\zeta'(\eta_2)\right|^{\vartheta} \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - y\right)^{\vartheta} \left(\frac{1-y}{2}\right) dy \right.$$
$$\left. + \int_{\frac{3}{4}}^1 \left(y - \frac{3}{4}\right)^{\vartheta} \left(\frac{1-y}{2}\right) dy \right] = \left(\frac{2^{-2\vartheta-5} + 2^{-2\vartheta-5} \times 3^{\vartheta+1}(4\vartheta+5)}{(\vartheta+1)(\vartheta+2)} \right) \left|\zeta'(\eta_2)\right|^{\vartheta}$$
$$+ \left(\frac{2^{-2\vartheta-5} \times 3^{\vartheta+1}(4\vartheta+11) + 2^{-2\vartheta-5}(8\vartheta+15)}{(\vartheta+1)(\vartheta+2)} \right) \left|\zeta'(\eta_1)\right|^{\vartheta} \quad (2.\ 10)$$

and

$$\int_0^1 \left|\frac{3}{4}-y\right|^{\vartheta}\left|\zeta^{'}\left(\lambda_2\left(y\right)\right)\right|^{\vartheta}dy \leq \left|\zeta^{'}\left(\eta_1\right)\right|^{\vartheta}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-y\right)^{\vartheta}\left(\frac{1-y}{2}\right)dy\right.$$

$$\left.+\int_{\frac{3}{4}}^1\left(y-\frac{3}{4}\right)^{\vartheta}\left(\frac{1-y}{2}\right)dy\right]+\left|\zeta^{'}\left(\eta_2\right)\right|^{\vartheta}\left[\int_0^{\frac{3}{4}}\left(\frac{3}{4}-y\right)^{\vartheta}\left(\frac{1+y}{2}\right)dy\right.$$

$$\left.+\int_{\frac{3}{4}}^1\left(y-\frac{3}{4}\right)^{\vartheta}\left(\frac{1+y}{2}\right)dy\right]=\left(\frac{2^{-2\vartheta-5}+2^{-2\vartheta-5}\times 3^{\vartheta+1}\left(4\vartheta+5\right)}{\left(\vartheta+1\right)\left(\vartheta+2\right)}\right)\left|\zeta^{'}\left(\eta_1\right)\right|^{\vartheta}$$

$$+\left(\frac{2^{-2\vartheta-5}\times 3^{\vartheta+1}\left(4\vartheta+11\right)+2^{-2\vartheta-5}\left(8\vartheta+15\right)}{\left(\vartheta+1\right)\left(\vartheta+2\right)}\right)\left|\zeta^{'}\left(\eta_2\right)\right|^{\vartheta}. \quad (2.\ 11)$$

We also observe that

$$\int_0^1 \lambda_1^{\frac{\vartheta}{\vartheta-1}}\left(y\right)dy=\eta_1^{\frac{\vartheta}{2(\vartheta-1)}}L\left(\eta_1^{\frac{\vartheta}{2(\vartheta-1)}},\eta_2^{\frac{\vartheta}{2(\vartheta-1)}}\right) \quad (2.\ 12)$$

and

$$\int_0^1 \lambda_2^{\frac{\vartheta}{\vartheta-1}}\left(y\right)dy=\eta_2^{\frac{\vartheta}{2(\vartheta-1)}}L\left(\eta_1^{\frac{\vartheta}{2(\vartheta-1)}},\eta_2^{\frac{\vartheta}{2(\vartheta-1)}}\right). \quad (2.\ 13)$$

Applying (2. 10)-(2. 13) in (2. 9), we get (2. 8). □

Theorem 2.6. *Let $\zeta : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, where η_1, $\eta_2 \in I^{\circ}$ with $\eta_1 < \eta_2$ and let $\phi : [\eta_1,\eta_2] \to [0,\infty)$ be a continuous positive mapping and geometrically symmetric with respect to $\sqrt{\eta_1\eta_2}$. If $\zeta^{'},\phi \in L_1[\eta_1,\eta_2]$ and $\left|\zeta^{'}\right|^{\vartheta}$ is GA-convex on $[\eta_1,\eta_2]$ for $\vartheta > 1$, then*

$$\left|\Psi\left(\eta_1,\eta_2;\zeta,\phi\right)\right| \leq \frac{\left(\ln\eta_2-\ln\eta_1\right)\left\|\phi\right\|_{[\eta_1,\eta_2],\infty}}{4}\left(\frac{\vartheta-1}{2\vartheta-1}\right)^{1-\frac{1}{\vartheta}}$$

$$\times\left[4^{-\frac{2\vartheta-1}{\vartheta-1}}\left(3^{\frac{2\vartheta-1}{\vartheta-1}}+1\right)\right]^{1-\frac{1}{\vartheta}}\left\{\eta_1^{\frac{1}{2}}\left[\left(\frac{2\eta_1^{\frac{\vartheta}{2}}-\eta_2^{\frac{\vartheta}{2}}-L\left(\eta_1^{\frac{\vartheta}{2}},\eta_2^{\frac{\vartheta}{2}}\right)}{\vartheta\left(\ln\eta_1-\ln\eta_2\right)}\right)\left|\zeta^{'}\left(\eta_1\right)\right|^{\vartheta}\right.\right.$$

$$\left.+\left(\frac{L\left(\eta_1^{\frac{\vartheta}{2}},\eta_2^{\frac{\vartheta}{2}}\right)-\eta_2^{\frac{\vartheta}{2}}}{\vartheta\left(\ln\eta_1-\ln\eta_2\right)}\right)\left|\zeta^{'}\left(\eta_2\right)\right|^{\vartheta}\right]^{\frac{1}{\vartheta}}+\eta_2^{\frac{1}{2}}\left[\left(\frac{\eta_2^{\frac{\vartheta}{2}}-L\left(\eta_1^{\frac{\vartheta}{2}},\eta_2^{\frac{\vartheta}{2}}\right)}{\vartheta\left(\ln\eta_1-\ln\eta_2\right)}\right)\left|\zeta^{'}\left(\eta_1\right)\right|^{\vartheta}\right.$$

$$\left.\left.+\left(\frac{L\left(\eta_1^{\frac{\vartheta}{2}},\eta_2^{\frac{\vartheta}{2}}\right)+\eta_1^{\frac{\vartheta}{2}}-2\eta_2^{\frac{\vartheta}{2}}}{\vartheta\left(\ln\eta_1-\ln\eta_2\right)}\right)\left|\zeta^{'}\left(\eta_2\right)\right|^{\vartheta}\right]^{\frac{1}{\vartheta}}\right\}. \quad (2.\ 14)$$

Proof. From (2. 3) and using the Hölder integral inequality, we have

$$|\Psi(\eta_1,\eta_2;\zeta,\phi)| \le \frac{(\ln\eta_2-\ln\eta_1)\|\phi\|_{[\eta_1,\eta_2],\infty}}{4} \times \left\{ \left(\int_0^1 \left|\frac{3}{4}-y\right|^{\frac{\vartheta}{\vartheta-1}} dy\right)^{1-\frac{1}{\vartheta}} \left(\int_0^1 \lambda_1^{\vartheta}(y)\left|\zeta'(\lambda_1(y))\right|^{\vartheta} dy\right)^{\frac{1}{\vartheta}} + \left(\int_0^1 \left|\frac{3}{4}-y\right|^{\frac{\vartheta}{\vartheta-1}} dy\right)^{1-\frac{1}{\vartheta}} \left(\int_0^1 \lambda_2^{\vartheta}(y)\left|\zeta'(\lambda_2(y))\right|^{\vartheta} dy\right)^{\frac{1}{\vartheta}} \right\}. \quad (2.\ 15)$$

Since $\left|\zeta'\right|^{\vartheta}$ is GA-convex on $[\eta_1,\eta_2]$ for $\vartheta \ge 1$, we get

$$\int_0^1 \lambda_1^{\vartheta}(y)\left|\zeta'(\lambda_1(y))\right|^{\vartheta} dy \le \left|\zeta'(\eta_1)\right|^{\vartheta}\int_0^1\left(\frac{1+y}{2}\right)\lambda_1^{\vartheta}(y)\,dy + \left|\zeta'(\eta_2)\right|^{\vartheta}\int_0^1\left(\frac{1-y}{2}\right)\lambda_1^{\vartheta}(y)\,dy$$
$$= \eta_1^{\frac{\vartheta}{2}}\left(\frac{2\eta_1^{\frac{\vartheta}{2}}-\eta_2^{\frac{\vartheta}{2}}-L\left(\eta_1^{\frac{\vartheta}{2}},\eta_2^{\frac{\vartheta}{2}}\right)}{\vartheta(\ln\eta_1-\ln\eta_2)}\right)\left|\zeta'(\eta_1)\right|^{\vartheta} + \eta_1^{\frac{\vartheta}{2}}\left(\frac{L\left(\eta_1^{\frac{\vartheta}{2}},\eta_2^{\frac{\vartheta}{2}}\right)-\eta_2^{\frac{\vartheta}{2}}}{\vartheta(\ln\eta_1-\ln\eta_2)}\right)\left|\zeta'(\eta_2)\right|^{\vartheta} \quad (2.\ 16)$$

and

$$\int_0^1 \lambda_2^{\vartheta}(y)\left|\zeta'(\lambda_2(y))\right|^{\vartheta} dy \le \left|\zeta'(\eta_1)\right|^{\vartheta}\int_0^1\left(\frac{1-y}{2}\right)\lambda_2^{\vartheta}(y)\,dy + \left|\zeta'(\eta_2)\right|^{\vartheta}\int_0^1\left(\frac{1+y}{2}\right)\lambda_2^{\vartheta}(y)\,dy$$
$$= \eta_2^{\frac{\vartheta}{2}}\left(\frac{\eta_2^{\frac{\vartheta}{2}}-L\left(\eta_1^{\frac{\vartheta}{2}},\eta_2^{\frac{\vartheta}{2}}\right)}{\vartheta(\ln\eta_1-\ln\eta_2)}\right)\left|\zeta'(\eta_1)\right|^{\vartheta} + \eta_2^{\frac{\vartheta}{2}}\left(\frac{L\left(\eta_1^{\frac{\vartheta}{2}},\eta_2^{\frac{\vartheta}{2}}\right)+\eta_1^{\frac{\vartheta}{2}}-2\eta_2^{\frac{\vartheta}{2}}}{\vartheta(\ln\eta_1-\ln\eta_2)}\right)\left|\zeta'(\eta_2)\right|^{\vartheta}. \quad (2.\ 17)$$

We notice that

$$\int_0^1 \left|\frac{3}{4}-y\right|^{\frac{\vartheta}{\vartheta-1}} dy = 4^{-\frac{2\vartheta-1}{\vartheta-1}}\left(\frac{\vartheta-1}{2\vartheta-1}\right)\left(3^{\frac{2\vartheta-1}{\vartheta-1}}+1\right). \quad (2.\ 18)$$

Applying (2. 16)-(2. 18) in (2. 15), we get (2. 14). □

Theorem 2.7. *Let $\zeta : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, where $\eta_1, \eta_2 \in I^{\circ}$ with $\eta_1 < \eta_2$ and let $\phi : [\eta_1,\eta_2] \to [0,\infty)$ be a continuous positive mapping and geometrically symmetric with respect to $\sqrt{\eta_1\eta_2}$. If $\zeta', \phi \in L_1[\eta_1,\eta_2]$ and $\left|\zeta'\right|^{\vartheta}$ is GA-convex on $[\eta_1,\eta_2]$ for $\vartheta \geq 1$, then*

$$
\begin{aligned}
|\Psi(\eta_1,\eta_2;\zeta,\phi)| \leq{} & \frac{\eta_2(\ln\eta_2-\ln\eta_1)\|\phi\|_{[\eta_1,\eta_2],\infty}}{4} \\
& \times\left\{\left[\frac{\theta\left((\theta-3)\ln\theta-4\left(\theta-2\theta^{3/4}+1\right)\right)}{4(\ln\theta)^2}\right]^{1-\frac{1}{\vartheta}}\right. \\
& \left[\frac{\theta\left(8\left(\theta-2\theta^{3/4}+1\right)-\left(9\theta-14\theta^{3/4}-1\right)(\ln\theta)+(2\theta-3)(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta'(\eta_1)\right|^{\vartheta}\right. \\
& \left.+\frac{\theta\left(-8\left(\theta-2\theta^{3/4}+1\right)+\left(\theta+2\theta^{3/4}-7\right)(\ln\theta)-3(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta'(\eta_2)\right|^{\vartheta}\right]^{\frac{1}{\vartheta}} \\
& +\left[\frac{(3\theta-1)\ln\theta-4\left(\theta-2\theta^{1/4}+1\right)}{4(\ln\theta)^2}\right]^{1-\frac{1}{\vartheta}} \\
& \times\left[\frac{\left(-8\left(\theta-2\theta^{1/4}+1\right)-\left(\theta-14\theta^{1/4}+9\right)(\ln\theta)+(2\theta-3)(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta'(\eta_2)\right|^{\vartheta}\right. \\
& \left.\left.+\frac{\left(8\left(\theta-2\theta^{1/4}+1\right)-\left(1+\theta^{1/4}-7\theta\right)(\ln\theta)+3\theta(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta'(\eta_1)\right|^{\vartheta}\right]^{\frac{1}{\vartheta}}\right\}, \quad (2.19)
\end{aligned}
$$

where $\theta = (\eta_1/\eta_2)^{1/2}$.

Proof. From (2. 3) and using the power-mean inequality, we have

$$
\begin{aligned}
|\Psi(\eta_1,\eta_2;\zeta,\phi)| \leq{} & \frac{(\ln\eta_2-\ln\eta_1)\|\phi\|_{[\eta_1,\eta_2],\infty}}{4} \\
& \times\left\{\left(\int_0^1\left|\frac{3}{4}-y\right|\lambda_1(y)\,dy\right)^{1-\frac{1}{\vartheta}}\left(\int_0^1\left|\frac{3}{4}-y\right|\lambda_1(y)\left|\zeta'(\lambda_1(y))\right|^{\vartheta}dy\right)^{\frac{1}{\vartheta}}\right. \\
& \left.+\left(\int_0^1\left|\frac{3}{4}-y\right|\lambda_2(y)\,dy\right)^{1-\frac{1}{\vartheta}}\left(\int_0^1\left|\frac{3}{4}-y\right|\lambda_2(y)\left|\zeta'(\lambda_2(y))\right|^{\vartheta}dy\right)^{\frac{1}{\vartheta}}\right\}. \quad (2.20)
\end{aligned}
$$

By using the GA-convexity of $\left|\zeta^{'}\right|^{\vartheta}$ on $[\eta_1,\eta_2]$ for $\vartheta \geq 1$, we get

$$\begin{aligned}
&\int_0^1 \left|\frac{3}{4}-y\right| \lambda_1(y) \left|\zeta^{'}(\lambda_1(y))\right|^{\vartheta} dy \\
&\leq \left|\zeta^{'}(\eta_1)\right|^{\vartheta} \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-y\right)\left(\frac{1+y}{2}\right)\lambda_1(y)\,dy + \int_{\frac{3}{4}}^1 \left(y-\frac{3}{4}\right)\left(\frac{1+y}{2}\right)\lambda_1(y)\,dy\right] \\
&+ \left|\zeta^{'}(\eta_2)\right|^{\vartheta} \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-y\right)\left(\frac{1-y}{2}\right)\lambda_1(y)\,dy + \int_{\frac{3}{4}}^1 \left(y-\frac{3}{4}\right)\left(\frac{1-y}{2}\right)\lambda_1(y)\,dy\right] \\
&= \frac{\eta_2\theta\left(8\left(\theta-2\theta^{3/4}+1\right)-\left(9\theta-14\theta^{3/4}-1\right)(\ln\theta)+(2\theta-3)(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta^{'}(\eta_1)\right|^{\vartheta} \\
&+ \frac{\eta_2\theta\left(-8\left(\theta-2\theta^{3/4}+1\right)+\left(\theta+2\theta^{3/4}-7\right)(\ln\theta)-3(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta^{'}(\eta_2)\right|^{\vartheta} \qquad (2.\ 21)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{3}{4}-y\right| \lambda_2(y) \left|\zeta^{'}(\lambda_2(y))\right|^{\vartheta} dy \\
&\leq \left|\zeta^{'}(\eta_1)\right|^{\vartheta} \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-y\right)\left(\frac{1-y}{2}\right)\lambda_2(y)\,dy + \int_{\frac{3}{4}}^1 \left(y-\frac{3}{4}\right)\left(\frac{1-y}{2}\right)\lambda_2(y)\,dy\right] \\
&+ \left|\zeta^{'}(\eta_2)\right|^{\vartheta} \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4}-y\right)\left(\frac{1+y}{2}\right)\lambda_2(y)\,dy + \int_{\frac{3}{4}}^1 \left(y-\frac{3}{4}\right)\left(\frac{1+y}{2}\right)\lambda_2(y)\,dy\right] \\
&= \frac{\eta_2\left(-8\left(\theta-2\theta^{1/4}+1\right)-\left(\theta-14\theta^{1/4}+9\right)(\ln\theta)+(2\theta-3)(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta^{'}(\eta_2)\right|^{\vartheta} \\
&+ \frac{\eta_2\left(8\left(\theta-2\theta^{1/4}+1\right)-\left(1+\theta^{1/4}-7\theta\right)(\ln\theta)+3\theta(\ln\theta)^2\right)}{8(\ln\theta)^3}\left|\zeta^{'}(\eta_1)\right|^{\vartheta}. \qquad (2.\ 22)
\end{aligned}$$

We also have

$$\int_0^1 \left|\frac{3}{4}-y\right| \lambda_1(y)\,dy = \int_0^{\frac{3}{4}} \left(\frac{3}{4}-y\right) \lambda_1(y)\,dy + \int_{\frac{3}{4}}^1 \left(y-\frac{3}{4}\right) \lambda_1(y)\,dy$$
$$= \frac{\eta_2\theta\left((\theta-3)\ln\theta - 4\left(\theta - 2\theta^{3/4}+1\right)\right)}{4(\ln\theta)^2} \quad (2.\ 23)$$

and

$$\int_0^1 \left|\frac{3}{4}-y\right| \lambda_2(y)\,dy = \int_0^{\frac{3}{4}} \left(\frac{3}{4}-y\right) \lambda_2(y)\,dy + \int_{\frac{3}{4}}^1 \left(y-\frac{3}{4}\right) \lambda_2(y)\,dy$$
$$= \frac{\eta_2\left((3\theta-1)\ln\theta - 4\left(\theta - 2\theta^{1/4}+1\right)\right)}{4(\ln\theta)^2}, \quad (2.\ 24)$$

where $\theta = (\eta_1/\eta_2)^{1/2}$.
Using (2. 21) and (2. 24) in (2. 20) we get (2. 19). □

3. Acknowledgment

The authors are very thankful to the anonymous referees' for the useful commenents/suggestion for the improvement of the manuscript before its publication.

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