Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(3)(2019) pp. 113-131

Certain Properties of Bipolar Fuzzy Soft Topology Via Q-Neighborhood

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Received: July 05, 2018 / Accepted: September 25, 2017 / Published online: 28 December, 2018

Abstract. In the present work, we bring out some properties of bipolar fuzzy soft topology (BFS-topology) by using the concept of Q-neighborhood. Firstly, we define the concept of quasi-coincident and Q-neighborhood for BFS-point and BFS-set, then we discuss certain properties of BFS-topology including, BFS-accumulation point, BFS-dense subset, BFS-countability axioms and BFS-separable space by utilizing BFS Q-neighborhood. We also study the concept of BFS-Lindelöf space. Furthermore, we apply the concept of BFS quasi-coincident in decision-making of a real world problem by using BFS-AND, BFS-OR operations.

AMS (MOS) Subject Classification Codes: 54A05; 54A40; 90B50; 11B05; 03E72;
Key Words: BFS quasi-coincident, BFS Q-neighborhood, BFS-accumulation point, BFS-countability axioms, BFS decision making.

1. Introduction

Zadeh [58] brought out the idea of fuzzy sets (1965). The theory of fuzzy set is universality of classical set theory. Fuzzy sets have been used in many fields of life including, commercial appliances (air conditioner, washing machine and heating ventilation etc), forecasting system of weather, system of traffic monitoring (in Japan fuzzy controller use to run the train all day). In short, fuzzy set theory has been auspiciously adapted in fields of computer sciences, physics, chemistry and medical sciences. In (1968), Chang [13] interpreted fuzzy topology on fuzzy set. Kelava and Seikkala [21] presented the idea of fuzzy metric spaces. Pao-Ming and Ying-Ming [33, 34] introduced the structure of neighborhood of fuzzy-point. They provided the concept of fuzzy quasi-coincident and

Q-neighborhood. They also discussed important properties of fuzzy topology by using fuzzy Q-neighborhood. Many authors have been used fuzzy sets in decision analysis problems. In (1999), Molodstov [35] presented the concept of soft sets in order to deal with unpredictability. The theory of soft set is basically generalized form of fuzzy set theory. The theory was presented to handle uncertain data in parametric form. Soft sets are also useful in the area of medical diagnosis system. Maji *et al.* [28, 29, 30] discussed some important operations of soft sets and its implementation in decision information. Ali *et al.* [1] discussed some modified operations of soft sets. Ç$a\breve{g}man$ *et al.* [10], Shabir and Naz [52] independently brought out the concept of soft topology. Kharal and Ahmad [22] presented the idea of mappings of soft classes. Akram *et al.* [4] studied soft intersection Lie algebras. Das [14, 15] presented the abstraction of soft real set and soft metric spaces. Riaz and Naeem [38, 39] established the novel ideas of measurable soft sets and measurable soft mappings. Many authors have been successfully applied soft set theory in various fields (See [4, 5, 6],[7],[10], [19], [22],[29],[32],[52]). Maji *et al.* [31] presented the idea of fuzzy soft set. In (2010), Feng *et al.* [16, 18, 17] solved some problems of decision-making in fuzzy soft sets and provided amplification of soft set with rough set and fuzzy set. Ç$a\breve{g}man$ *et al.* [10, 12] studied fuzzy soft set and its use in decision analysis. Varol and Aygun [55] interpreted fuzzy soft topology. Zorlutuna and Atmaca [60] brought out the abstraction of fuzzy parameterized fuzzy soft(FPFS) topology and FPFS Q-neighborhood. Riaz and Masoomah [41, 42, 43] extended the idea of fuzzy parameterized fuzzy soft topology and proved important results which do hold in classical topology but do not hold in fuzzy parameterized fuzzy soft topology. Kharal and Ahmad [23] defined fuzzy soft mappings. Fuzzy set theory and its applications in decision making have studied by many researchers (see [3, 20, 25, 40, 49, 53, 54, 56]). Malik and Riaz [8, 9] studied modluar group action on real quadratic fields. In (1998), Zhang [59] introduced the extension of fuzzy set with bipolarity, called, bipolar-valued fuzzy sets. In bipolar-valued fuzzy set interval of membership value is $[-1, 1]$. The bipolar fuzzy set involves positive and negative memberships. The positive membership degrees represents the possibilities of something to be happened whereas the negative membership degrees represent the impossibilities. The elements with 0 membership indicate that they are not satisfying the specific property, the interval $(0, 1]$ indicates elements satisfying property with different degrees of membership, whereas $[-1, 0)$ shows that elements satisfying implicit counter property. In (2000), Lee [26] discussed some basic operations of bipolar-valued fuzzy set. Yang [57] introduced the extension of bipolar fuzzy set with soft set called bipolar-value fuzzy soft set. Aslam *et al.* [2] brought out the abstraction of bipolar fuzzy soft set. They discussed the operations of BFS-sets and a problem of decision-making. Zhang [59] introduced lie subalgebra on bipolar-valued fuzzy soft set. In this paper, we study concept of BFS quasi-coincident and BFS Q-neighborhood. We use BFS Q-neighborhood in certain properties of BFS-topology with relevant examples. We also discuss a decision-making problem in project management by using BFS quasi-coincident and BFS-AND, BFS-OR operations.

2. Preliminaries

In present section, we review some basic definitions including BFS-set and BFS-topology. Throughout this work, $V \neq \emptyset$ represents universal set and $\mathcal{D}$ represents relevant set of decision variables or attributes.

Definition 2.1. [58] Consider the universal set V and membership function $f : V \to [0, 1]$. Then V_f is a fuzzy set on V if each element $\xi \in V$ is associated with degree of membership, which is a real number in $[0, 1]$ and it is denoted by f_ξ.

Definition 2.2. [35] Consider the universal set V and set of decision variables $\mathcal{D}$. Let $\mathcal{A}_1 \subseteq \mathcal{D}$ and $\mathcal{K} : \mathcal{A}_1 \to \mathcal{P}(V)$ be the set-valued function, where $\mathcal{P}(V)$ be the power sets of V. Then $\mathcal{K}_{\mathcal{A}_1}$ or $(\mathcal{K}, \mathcal{A}_1)$ denotes a soft set on V.

Definition 2.3. [31] Consider the universal set V and set of decision variables $\mathcal{D}$. Let $\mathcal{A}_1 \subseteq \mathcal{D}$ and $\mathcal{F}(V)$ be the family of all fuzzy subsets of V. If $\mathcal{K} : \mathcal{A}_1 \to \mathcal{F}(V)$ is a set-valued mapping. Then $\mathcal{K}_{\mathcal{A}_1}$ denotes a fuzzy soft set on V.

Definition 2.4. [26] A bipolar fuzzy set on V is of the form

$$\mathcal{K} = \{(\xi_i,\ \delta^+_{\mathcal{K}}(\xi_i),\ \delta^-_{\mathcal{K}}(\xi_i)\ ) \text{: for all } \xi_i \in V\},$$

where $\delta^+_{\mathcal{K}}(\xi_i)$ denotes the positive memberships ranges over $[0, 1]$ and $\delta^-_{\mathcal{K}}(\xi_i)$ denotes the negative memberships ranges over $[-1, 0]$.

Definition 2.5. [2] Let $\mathcal{A}_1 \subset \mathcal{D}$ and define a mapping $\mathcal{K} : \mathcal{A}_1 \to \mathcal{BF}(V)$, where $\mathcal{BF}(V)$ represents the family of all bipolar fuzzy subsets of V. Then $\mathcal{K}_{\mathcal{A}_1}$ or $(\mathcal{K}, \mathcal{A}_1)$ is called a bipolar fuzzy set (BFS-set) on V. A BFS-set can be defined as

$$\mathcal{K}_{\mathcal{A}_1} = \left\{\mathcal{K}(p_j) = (\xi_i,\ \delta^+_{p_j}(\xi_i),\ \delta^-_{p_j}(\xi_i)\ ) \text{: for all } \xi_i{\in}V \text{ and } p_j{\in}\mathcal{A}_1\right\}$$

Definition 2.6. [2] The family of all BFS-sets on V with decision variables from $\mathcal{D}$ is denoted by $\mathcal{BF}(V_{\mathcal{D}})$.

Example 2.7. Consider $V = \{\xi_1, \xi_2, \xi_3\}$ be the set of three companies of home appliances and $\mathcal{D} = \{p_1, p_2, p_3\}$ be the set of decision variables related to their productivity, where
p_1 represents durability.
p_2 represents expensive.
p_3 represents economical.
Suppose that $\mathcal{A}_1 = \{p_1, p_3\}{\subset}\mathcal{D}$, now a BFS-set $\mathcal{K}_{\mathcal{A}_1}$ can be written as follows:

$$\mathcal{K}_{\mathcal{A}_1} = \left\{ \begin{array}{l} \mathcal{K}_{p_1} = \{\ (\xi_1, 0.12, -0.53), (\xi_2, 0.31, -0.62), (\xi_3, 0.41, -0.23)\ \}, \\ \mathcal{K}_{p_3} = \{\ (\xi_1, 0.82, -0.11), (\xi_2, 0.33, -0.61), (\xi_3, 0.42, -0.32)\ \} \end{array} \right\}$$

Definition 2.8. [59] A BFS-set $\mathcal{K}_{\mathcal{D}}$ is called a null BFS-set, if $\delta^+_{p_j}(\xi_i) = 0$ and $\delta^-_{p_j}(\xi_i) = 0$, for each $p_j{\in}\mathcal{D}$ and $\xi_i{\in}V$ and we write it as $\phi_{\mathcal{D}}$.

Definition 2.9. [59] A BFS-set $\mathcal{K}_{\mathcal{D}}$ is called an absolute BFS-set, if $\delta^+_{p_j}(\xi_i) = 1$ and $\delta^-_{p_j}(\xi_i) = -1$, for each $p_j{\in}\mathcal{D}$ and $\xi_i{\in}V$ and we write it as $V_{\mathcal{D}}$

Definition 2.10. [2] The complement of a BFS-set $\mathcal{K}_{\mathcal{A}_1}$ is represented by $(\mathcal{K}_{\mathcal{A}_1})^c$ and is defined by $(\mathcal{K}_{\mathcal{A}_1})^c = \left\{\mathcal{K}(p_j) = (\xi_i, 1 - \delta^+_{\mathcal{A}_1}(\xi_i), -1 - \delta^-_{\mathcal{A}_1}(\xi_i)) : \xi_i \in V, p_j \in \mathcal{A}_1\right\}$.

Definition 2.11. [59] Consider two BFS-sets $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$ on V. Then $\mathcal{K}_{\mathcal{A}_1}$ is a BFS-subset of $\mathcal{K}_{\mathcal{A}_2}$, if
(i) $\mathcal{A}_1 \subseteq \mathcal{A}_2$
(ii) $\delta^+_{\mathcal{A}_1}(\xi_i)\tilde{\leq}\delta^+_{\mathcal{A}_2}(\xi_i)$, $\delta^-_{\mathcal{A}_1}(\xi_i)\tilde{\geq}\delta^-_{\mathcal{A}_2}(\xi_i)$.
Then we can write it as $\mathcal{K}_{\mathcal{A}_1}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_2}$.

Definition 2.12. [2] Let $\mathcal{K}^1_{\mathcal{A}_1}$ and $\mathcal{K}^2_{\mathcal{A}_2}\tilde{\in}\mathcal{BF}(V_{\mathcal{D}})$. Then intersection of $\mathcal{K}^1_{\mathcal{A}_1}$ and $\mathcal{K}^2_{\mathcal{A}_2}$ is a BFS-set $\mathcal{K}_{\mathcal{A}}$, where $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$, $\mathcal{K} : \mathcal{A} \to \mathcal{BF}(V)$ is a mapping defined by $\mathcal{K}_{p_j} = \mathcal{K}^1_{p_j}\tilde{\cap}\mathcal{K}^2_{p_j}$ $\forall\, p_j\in\mathcal{A}$ and it is written as $\mathcal{K}_{\mathcal{A}} = \mathcal{K}^1_{\mathcal{A}_1}\tilde{\cap}\mathcal{K}^2_{\mathcal{A}_2}$.

Definition 2.13. [2] Let $\mathcal{K}^1_{\mathcal{A}_1}$ and $\mathcal{K}^2_{\mathcal{A}_2}\tilde{\in}\mathcal{BF}(V_{\mathcal{D}})$. The union of $\mathcal{K}^1_{\mathcal{A}_1}$ and $\mathcal{K}^2_{\mathcal{A}_2}$ is a BFS-set $\mathcal{K}_{\mathcal{A}}$, where $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{K} : \mathcal{A} \to \mathcal{BF}(V)$ is a mapping defined as

$$\begin{aligned} \mathcal{K}_{p_j} &= \mathcal{K}^1_{p_j} \text{ if } p_j\in\mathcal{A}_1 \setminus \mathcal{A}_2 \\ &= \mathcal{K}^2_{p_j} \text{ if } p_j\in\mathcal{A}_2 \setminus \mathcal{A}_1 \\ &= \mathcal{K}^1_{p_j}\tilde{\cup}\mathcal{K}^2_{p_j} \text{ if } p_j\in\mathcal{A}_1 \cap \mathcal{A}_2 \end{aligned}$$

and it is written as $\mathcal{K}_{\mathcal{A}} = \mathcal{K}^1_{\mathcal{A}_1}\tilde{\cup}\mathcal{K}^2_{\mathcal{A}_2}$.

Definition 2.14. [44] Consider an absolute BFS-set $V_{\mathcal{D}}\tilde{\in}\mathcal{BF}(V_{\mathcal{D}})$. Let $\mathfrak{P}(V_{\mathcal{D}})$ be the class of all BFS-subsets of $V_{\mathcal{D}}$ and $\tilde{\tau}$ be the subclass of $\mathfrak{P}(V_{\mathcal{D}})$. Then $\tilde{\tau}$ is called BFS-topology, if

i. $\phi_{\mathcal{D}}$ and $V_{\mathcal{D}}$ both are in $\tilde{\tau}$.
ii. $\mathcal{K}^1_{\mathcal{D}}$,$\mathcal{K}^2_{\mathcal{D}}\tilde{\in}\tilde{\tau} \Rightarrow \mathcal{K}^1_{\mathcal{D}}\tilde{\cap}\mathcal{K}^2_{\mathcal{D}}\tilde{\in}\tilde{\tau}$.
iii. $\mathcal{K}^{\ell}_{\mathcal{D}}\tilde{\in}\tilde{\tau}$, where $\ell\in\Psi \Rightarrow \tilde{\bigcup}_{\ell\in\Psi}\mathcal{K}^{\ell}_{\mathcal{D}}\tilde{\in}\tilde{\tau}$.

If $\tilde{\tau}$ be a BFS-topology on $V_{\mathcal{D}}$, then the trinity $(V, \tilde{\tau}, \mathcal{D})$ denotes BFS-topological space over $V_{\mathcal{D}}$. Whereas all the BFS-sets in $\tilde{\tau}$ denote BFS-open sets.

Definition 2.15. [44] Consider a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$ over $V_{\mathcal{D}}$, a BFS-set $\mathcal{K}^1_{\mathcal{D}}\tilde{\in}\mathfrak{P}(V_{\mathcal{D}})$ is called a BFS-closed in $(V, \tilde{\tau}, \mathcal{D})$, if its relative BFS-compliment $(\mathcal{K}^1_{\mathcal{D}})^c$ is BFS-open.

Definition 2.16. [44] Let $\mathcal{Q}$ be a singleton subset of $\mathcal{D}$, $p \in \mathcal{Q}$. Let $\mathcal{K}_{\mathcal{Q}}\tilde{\in}\mathcal{BF}(V_{\mathcal{D}})$ be a BFS-set, where $\delta^{p(+)}_{\mathcal{Q}}(\xi_i)$ and $\delta^{p(-)}_{\mathcal{Q}}(\xi_i)$ are positive membership degrees and negative membership degrees of BFS-set $\mathcal{K}_{\mathcal{Q}}$ and $\delta^{p(+)}_{\mathcal{Q}}(\xi_i), \delta^{p(-)}_{\mathcal{Q}}(\xi_i) \neq \phi$, $\delta^{p(+)}_{\mathcal{Q}}(\xi_i), \delta^{p(-)}_{\mathcal{Q}}(\xi_i) = \phi$ only if $p^c \in \mathcal{D} - \{p\}$. Then $\mathcal{K}_{\mathcal{Q}}$ is called BFS-point and denoted by $\beta(\mathcal{K}_{\mathcal{Q}})$.

Definition 2.17. [44] Consider a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$ and $\mathcal{K}_{\mathcal{D}}\tilde{\in}\mathfrak{P}(V_{\mathcal{D}})$. The BFS-closure of $\mathcal{K}_{\mathcal{D}}$ is expressed as the BFS-intersection of all BFS-closed supersets of $\mathcal{K}_{\mathcal{D}}$. It is denoted by $\widetilde{Cl}(\mathcal{K}_{\mathcal{D}})$. It is to be noted that BFS-closure of a BFS-subset is the smallest BFS-closed superset of that BFS-subset.

Definition 2.18. [44] Consider a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$, $\mathcal{N}_{\mathcal{D}}\tilde{\in}\mathfrak{P}(V_{\mathcal{D}})$ and $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\mathfrak{P}(V_{\mathcal{D}})$. Let $\mathcal{K}_{\mathcal{A}}$ be a BFS-open set. If $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\mathcal{K}_{\mathcal{A}}\tilde{\subseteq}\mathcal{N}_{\mathcal{D}}$ then $\mathcal{N}_{\mathcal{D}}$ is called BFS-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})$. The set of all BFS-neighborhoods of a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ is called BFS-neighborhood system of $\beta(\mathcal{K}_{\mathcal{Q}})$ and is denoted by $\widetilde{\mathcal{N}}(\beta(\mathcal{K}_{\mathcal{Q}}))$.

3. Main Results

In this section we discuss some properties of BFS-topology by using BFS quasi-coincident and BFS Q-neighborhood.

Definition 3.1. Let $\beta(\mathcal{K}_{\mathcal{Q}}), \mathcal{K}_{\mathcal{A}_1}\tilde{\in}\mathcal{BF}(V_{\mathcal{D}})$, then $\beta(\mathcal{K}_{\mathcal{Q}})$ is called BFS quasi-coincident with BFS-set $\mathcal{K}_{\mathcal{A}_1}$ if and only if $\exists\ p \in \mathcal{D}|\ \delta^{p(+)}_{\mathcal{Q}}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_1}(\xi_i) > 1$ and $\delta^{p(-)}_{\mathcal{Q}}(\xi_i) +$

$\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) < -1$ for some $\xi_i \in V$ and it is denoted by $\beta(\mathcal{K}_\mathcal{Q}) q \mathcal{K}_{\mathcal{A}_1}$. If $\beta(\mathcal{K}_\mathcal{Q})$ is not BFS quasi-coincident with $\mathcal{K}_{\mathcal{A}_1}$ then it is denoted by $\beta(\mathcal{K}_\mathcal{Q}) \bar{q} \mathcal{K}_{\mathcal{A}_1}$.

Example 3.2. Consider a universal set $V = \{\xi_1, \xi_2, \xi_3\}$ and $\mathcal{D} = \{p_1, p_2, p_3, p_4\}$ be the set of decision variables. Let $\mathcal{Q} = \{p_1\} \subset \mathcal{D}$ and $\mathcal{A}_1 = \{p_1, p_2\} \subset \mathcal{D}$. We have BFS-point $\beta(\mathcal{K}_\mathcal{Q})$ and BFS-set $\mathcal{K}_{\mathcal{A}_1}$ as follow:

$$\beta(\mathcal{K}_\mathcal{Q}) = \{\ \mathcal{K}(p_1) = \{\ (\xi_1, 0.91, -0.82), (\xi_2, 0.84, -0.75), (\xi_3, 0.63, -0.69)\ \}\ \}$$

and

$$\mathcal{K}_{\mathcal{A}_1} = \left\{ \begin{array}{l} \mathcal{K}(p_1) = \{\ (\xi_1, 0.23, -0.34), (\xi_2, 0.33, -0.43), (\xi_3, 0.53, -0.61)\ \}, \\ \mathcal{K}(p_2) = \{\ (\xi_1, 0.33, -0.44), (\xi_2, 0.47, -0.52), (\xi_3, 0.54, -0.62)\ \} \end{array} \right\}$$

It can be seen that $\beta(\mathcal{K}_\mathcal{Q})$ is BFS quasi-coincident with $\mathcal{K}_{\mathcal{A}_1}$. Because $\delta^{p_1(+)}_{\mathcal{Q}}(\xi_i) + \delta^{p_1(+)}_{\mathcal{A}_1}(\xi_i) > 1$ and $\delta^{p_1(-)}_{\mathcal{Q}}(\xi_i) + \delta^{p_1(-)}_{\mathcal{A}_1}(\xi_i) < -1$, for $p_1 \in \mathcal{D}$ and $\xi_i \in V$.

Definition 3.3. Let $\mathcal{K}_{\mathcal{A}_1}, \mathcal{K}_{\mathcal{A}_2} \tilde{\in} \mathcal{BF}(V_\mathcal{D})$, then $\mathcal{K}_{\mathcal{A}_1}$ is called BFS quasi-coincident with BFS-set $\mathcal{K}_{\mathcal{A}_2}$, if and only if $\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) > 1$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) < -1$ for some $\xi_i \in V$ and $p \in \mathcal{A}_1 \cap \mathcal{A}_2$, and it is denoted by $\mathcal{K}_{\mathcal{A}_1} q \mathcal{K}_{\mathcal{A}_2}$. If $\mathcal{K}_{\mathcal{A}_1}$ is not BFS quasi-coincident with $\mathcal{K}_{\mathcal{A}_2}$ then it is denoted by $\mathcal{K}_{\mathcal{A}_1} \bar{q} \mathcal{K}_{\mathcal{A}_2}$.

Example 3.4. Let $V = \{\xi_1, \xi_2, \xi_3\}$ be the initial universe and $\mathcal{D} = \{p_1, p_2, p_3, p_4\}$ be the set of decision variables. Let $\mathcal{A}_1 = \{p_1, p_2, p_3\} \subset \mathcal{D}$ and $\mathcal{A}_2 = \{p_2, p_3, p_4\} \subset \mathcal{D}$. We have BFS-sets $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$ as follow:

$$\mathcal{K}_{\mathcal{A}_1} = \left\{ \begin{array}{l} \mathcal{K}(p_1) = \{\ (\xi_1, 0.21, -0.33), (\xi_2, 0.36, -0.42), (\xi_3, 0.52, -0.60)\ \}, \\ \mathcal{K}(p_2) = \{\ (\xi_1, 0.32, -0.48), (\xi_2, 0.42, -0.54), (\xi_3, 0.56, -0.64)\ \}, \\ \mathcal{K}(p_3) = \{\ (\xi_1, 0.31, -0.44), (\xi_2, 0.41, -0.57), (\xi_3, 0.52, -0.61)\ \} \end{array} \right\}$$

and

$$\mathcal{K}_{\mathcal{A}_2} = \left\{ \begin{array}{l} \mathcal{K}(p_2) = \{\ (\xi_1, 0.82, -0.71), (\xi_2, 0.73, -0.64), (\xi_3, 0.67, -0.68)\ \}, \\ \mathcal{K}(p_3) = \{\ (\xi_1, 0.81, -0.86), (\xi_2, 0.82, -0.63), (\xi_3, 0.62, -0.71)\ \}, \\ \mathcal{K}(p_4) = \{\ (\xi_1, 0.32, -0.45), (\xi_2, 0.41, -0.52), (\xi_3, 0.55, -0.67)\ \} \end{array} \right\}$$

Now it is ease to see that $\mathcal{K}_{\mathcal{A}_1}$ is BFS quasi-coincident with $\mathcal{K}_{\mathcal{A}_2}$. Because $\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) > 1$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) < -1$, for $p \in \mathcal{A}_1 \cap \mathcal{A}_2$ and $\xi_i \in V$.

Proposition 3.5. *Let $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2} \tilde{\in} \mathcal{BF}(V_\mathcal{D})$, then the following results hold.*

i. $\mathcal{K}_{\mathcal{A}_1} \tilde{\subseteq} \mathcal{K}_{\mathcal{A}_2} \Leftrightarrow \mathcal{K}_{\mathcal{A}_1} \bar{q} \mathcal{K}^c_{\mathcal{A}_2}$.
ii. $\mathcal{K}_{\mathcal{A}_1} q \mathcal{K}_{\mathcal{A}_2} \Rightarrow \mathcal{K}_{\mathcal{A}_1} \tilde{\cap} \mathcal{K}_{\mathcal{A}_2} \neq \phi_\mathcal{D}$.
iii. $\mathcal{K}_{\mathcal{A}_1} \bar{q} \mathcal{K}^c_{\mathcal{A}_1}$.
iv. $\mathcal{K}_{\mathcal{A}_1} q \mathcal{K}_{\mathcal{A}_2} \Leftrightarrow$ *there is* $\beta(\mathcal{K}_\mathcal{Q}) \tilde{\in} \mathcal{K}_{\mathcal{A}_1}$ *such that* $\beta(\mathcal{K}_\mathcal{Q}) q \mathcal{K}_{\mathcal{A}_2}$.
v. *For all* $\beta(\mathcal{K}_\mathcal{Q}) \tilde{\in} \mathcal{BF}(V_\mathcal{D})$, $\beta(\mathcal{K}_\mathcal{Q}) \tilde{\in} \mathcal{K}^c_{\mathcal{A}_1} \Leftrightarrow \beta(\mathcal{K}_\mathcal{Q}) \bar{q} \mathcal{K}_{\mathcal{A}_1}$.
vi. $\mathcal{K}_{\mathcal{A}_1} \tilde{\subseteq} \mathcal{K}_{\mathcal{A}_2} \Rightarrow$ *if* $\beta(\mathcal{K}_\mathcal{Q}) q \mathcal{K}_{\mathcal{A}_1}$, *then* $\beta(\mathcal{K}_\mathcal{Q}) q \mathcal{K}_{\mathcal{A}_2}$ $\forall \beta(\mathcal{K}_\mathcal{Q}) \tilde{\in} \mathcal{BF}(V_\mathcal{D})$.

Proof. i. Consider $\mathcal{K}_{\mathcal{A}_1} \tilde{\subseteq} \mathcal{K}_{\mathcal{A}_2}$,
$\Leftrightarrow \forall p \in \mathcal{D}$ we have $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) \tilde{\leq} \delta^{p(+)}_{\mathcal{A}_2}(\xi_i)$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) \tilde{\geq} \delta^{p(-)}_{\mathcal{A}_2}(\xi_i)$, where $\xi_i \in V$.
$\Leftrightarrow \forall p \in \mathcal{D}$ and $\xi_i \in V$, we get $\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) - \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \tilde{\leq} 0$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) - \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \tilde{\geq} 0$

$\Leftrightarrow \forall p \in \mathcal{D}$ and $\xi_i \in V$, we have $\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) + 1 - \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \tilde{\leq} 1$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) - 1 - \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \tilde{\geq} -1$
$\Leftrightarrow \mathcal{K}_{\mathcal{A}_1} \bar{q} \mathcal{K}^c_{\mathcal{A}_2}$.
ii. Suppose that $\mathcal{K}_{\mathcal{A}_1} q \mathcal{K}_{\mathcal{A}_2}$, then by definition, for $p \in \mathcal{A}_1 \cap \mathcal{A}_2$ and $\xi_i \in V$,$\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \tilde{>} 1$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \tilde{<} -1$ this shows that $\delta^{p(+)}_{\mathcal{A}_1}(\xi_i)$, $\delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \neq 0$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i)$, $\delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \neq 0$ for $p \in \mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$. So, we get $\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) \tilde{\cap} \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \neq \tilde{\phi}$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) \tilde{\cap} \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \neq \tilde{\phi}$. this shows that $\mathcal{K}_{\mathcal{A}_1} \tilde{\cap} \mathcal{K}_{\mathcal{A}_2} \neq \phi_{\mathcal{D}}$.
iii. Suppose that $\mathcal{K}_{\mathcal{A}_1} q \mathcal{K}^c_{\mathcal{A}_2}$ then we have $p \in \mathcal{A}_1 \cap \mathcal{A}_2{}^c$ and $\xi_i \in V$,$\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) + 1 - \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \tilde{>} 1$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) - 1 - \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \tilde{<} -1$. Which is a contradiction.
iv. Suppose that $\mathcal{K}_{\mathcal{A}_1} q \mathcal{K}_{\mathcal{A}_2}$ then we have for $p \tilde{\in} \mathcal{A}_1 \cap \mathcal{A}_2$ and $\xi_i \in V$,$\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \tilde{>} 1$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \tilde{<} -1$. Now put $\delta^{p(+)}_{\mathcal{Q}}(\xi_i) = \delta^{p(+)}_{\mathcal{A}_1}(\xi_i)$ and $\delta^{p(-)}_{\mathcal{Q}}(\xi_i) = \delta^{p(-)}_{\mathcal{A}_1}(\xi_i)$. This shows that $\beta(\mathcal{K}_{\mathcal{Q}}) \tilde{\in} \mathcal{K}_{\mathcal{A}_1}$ and $\beta(\mathcal{K}_{\mathcal{Q}}) q \mathcal{K}_{\mathcal{A}_2}$. Conversely, suppose that $\beta(\mathcal{K}_{\mathcal{Q}}) \tilde{\in} \mathcal{K}_{\mathcal{A}_1}$ such that $\beta(\mathcal{K}_{\mathcal{Q}}) q \mathcal{K}_{\mathcal{A}_2}$, then $\delta^{p(+)}_{\mathcal{Q}}(\xi_i) \tilde{\leq} \delta^{p(+)}_{\mathcal{A}_1}(\xi_i)$ and $\delta^{p(-)}_{\mathcal{Q}}(\xi_i) \tilde{\geq} \delta^{p(-)}_{\mathcal{A}_1}(\xi_i)$,
since $\beta(\mathcal{K}_{\mathcal{Q}}) q \mathcal{K}_{\mathcal{A}_2}$, $\delta^{p(+)}_{\mathcal{Q}}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \tilde{>} 1$ and $\delta^{p(-)}_{\mathcal{Q}}(\xi_i) + \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \tilde{<} -1$. We get, $\delta^{p(+)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \tilde{>} 1$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi_i) + \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \tilde{<} -1$. So, $\mathcal{K}_{\mathcal{A}_1} q \mathcal{K}_{\mathcal{A}_2}$.
v. Let $\beta(\mathcal{K}_{\mathcal{Q}}) \tilde{\in} \mathcal{K}^c_{\mathcal{A}_1}$,
$\Leftrightarrow \delta^{p(+)}_{\mathcal{Q}}(\xi_i) \tilde{\leq} 1 - \delta^{p(+)}_{\mathcal{A}_1}(\xi_i)$ and $\delta^{p(-)}_{\mathcal{Q}}(\xi_i) \tilde{\geq} -1 - \delta^{p(-)}_{\mathcal{A}_1}(\xi_i)$.
$\Leftrightarrow \delta^{p(+)}_{\mathcal{Q}}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_1}(\xi_i) \tilde{\leq} 1$ and $\delta^{p(+)}_{\mathcal{Q}}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_1}(\xi_i) \tilde{\geq} -1$.
$\Leftrightarrow \beta(\mathcal{K}_{\mathcal{Q}}) \bar{q} \mathcal{K}_{\mathcal{A}_1}$.
vi. Let $\beta(\mathcal{K}_{\mathcal{Q}}), \mathcal{K}_{\mathcal{A}_1} \tilde{\in} \mathcal{BF}(V_{\mathcal{D}})$ and $\beta(\mathcal{K}_{\mathcal{Q}}) q \mathcal{K}_{\mathcal{A}_1} \Rightarrow \delta^{p(+)}_{\mathcal{Q}}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_1}(\xi_i) \tilde{>} 1$ and $\delta^{p(-)}_{\mathcal{Q}}(\xi_i) + \delta^{p(-)}_{\mathcal{A}_1}(\xi_i) \tilde{<} -1$. Since $\mathcal{K}_{\mathcal{A}_1} \tilde{\subseteq} \mathcal{K}_{\mathcal{A}_2}$ therefore, $\delta^{p(+)}_{\mathcal{Q}}(\xi_i) + \delta^{p(+)}_{\mathcal{A}_2}(\xi_i) \tilde{>} 1$ and $\delta^{p(-)}_{\mathcal{Q}}(\xi_i) + \delta^{p(-)}_{\mathcal{A}_2}(\xi_i) \tilde{<} -1$. Hence $\beta(\mathcal{K}_{\mathcal{Q}}) q \mathcal{K}_{\mathcal{A}_2}$. □

Proposition 3.6. *Consider a collection of BFS-sets $\{\mathcal{K}_{\mathcal{A}_i}\}_{i \in I}$ in a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$. Then a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}}) q \tilde{\bigcup\limits_{i \in I}} \mathcal{K}_{\mathcal{A}_i} \Leftrightarrow \beta(\mathcal{K}_{\mathcal{Q}}) q \mathcal{K}_{\mathcal{A}_i}$.*

Proof. Straightforward. □

Definition 3.7. Consider a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$. A BFS-set $\mathcal{K}_{\mathcal{A}}$ is called BFS Q-neighborhood of a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$, if $\exists\, \mathcal{K}_{\mathcal{A}_1} \tilde{\in} \tilde{\tau}$ such that $\beta(\mathcal{K}_{\mathcal{Q}}) q \mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_1} \tilde{\subseteq} \mathcal{K}_{\mathcal{A}}$. The collection of all BFS Q-neighborhoods of a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ is called system of BFS Q-neighborhoods of $\beta(\mathcal{K}_{\mathcal{Q}})$. We write it as BFS $Q\mathcal{N}(\beta(\mathcal{K}_{\mathcal{Q}}))$.

Example 3.8. Let us consider the initial universal set $V = \{\xi_1, \xi_2, \xi_3, \xi_4\}$,
$\mathcal{D} = \{p_1, p_2, p_3, p_4\}$ be the set of decision variables. Now choose subsets $\mathcal{A}_1 = \{p_1, p_2\}$ and $\mathcal{A}_2 = \{p_1, p_2, p_4\}$ of $\mathcal{D}$, construct BFS-sets $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$

$$\mathcal{K}_{\mathcal{A}_1} = \left\{ \begin{array}{l} \mathcal{K}_{p_1} = \{ \ (\xi_1, 0.21, -0.42), (\xi_2, 0.41, -0.52), (\xi_3, 0.51, -0.32), (\xi_4, 0.61, -0.32) \ \}, \\ \mathcal{K}_{p_2} = \{ \ (\xi_1, 0.41, -0.32), (\xi_2, 0.51, -0.42), (\xi_3, 0.61, -0.42), (\xi_4, 0.41, -0.32) \ \} \end{array} \right\}$$

$$\mathcal{K}_{\mathcal{A}_2} = \left\{ \begin{array}{l} \mathcal{K}_{p_1} = \{ \ (\xi_1, 0.33, -0.54), (\xi_2, 0.53, -0.64), (\xi_3, 0.63, -0.44), (\xi_4, 0.73, -0.44) \ \}, \\ \mathcal{K}_{p_2} = \{ \ (\xi_1, 0.53, -0.44), (\xi_2, 0.63, -0.54), (\xi_3, 0.73, -0.54), (\xi_4, 0.53, -0.44) \ \}, \\ \mathcal{K}_{p_3} = \{ \ (\xi_1, 0.63, -0.24), (\xi_2, 0.23, -0.14), (\xi_3, 0.43, -0.24), (\xi_4, 0.43, -0.14) \ \} \end{array} \right\}$$

Now with these BFS-sets, we define a BFS-topology $\tilde{\tau} = \{\phi_{\mathcal{D}}, V_{\mathcal{D}}, \mathcal{K}_{\mathcal{A}_1}, \mathcal{K}_{\mathcal{A}_2}\}$, where $\phi_{\mathcal{D}}$ and $V_{\mathcal{D}}$ are null BFS-set and absolute BFS-set respectively. Let us consider a BFS-point,

$$\beta(\mathcal{K}_{\mathcal{Q}}) = \quad \mathcal{K}_{p_1} = \{ \ (\xi_1, 0.80, -0.60), (\xi_2, 0.62, -0.50), (\xi_3, 0.53, -0.73), (\xi_4, 0.43, -0.70) \ \}$$

and a BFS-set

$$\mathcal{K}_{\mathcal{A}} = \left\{ \begin{array}{l} \mathcal{K}_{p_1} = \{ \ (\xi_1, 0.51, -0.62), (\xi_2, 0.63, -0.74), (\xi_3, 0.71, -0.62), (\xi_4, 0.81, -0.62) \ \}, \\ \mathcal{K}_{p_2} = \{ \ (\xi_1, 0.61, -0.52), (\xi_2, 0.73, -0.64), (\xi_3, 0.93, -0.74), (\xi_4, 0.61, -0.62) \ \}, \\ \mathcal{K}_{p_3} = \{ \ (\xi_1, 0.71, -0.51), (\xi_2, 0.42, -0.33), (\xi_3, 0.65, -0.66), (\xi_4, 0.52, -0.35) \ \} \end{array} \right\}$$

It can be seen that $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_2}$, for $p\in\mathcal{Q}\cap\mathcal{A}_2$ and $\xi_i\in V$, we have $\delta_{\mathcal{Q}}^{p_1(+)}(\xi_i)+\delta_{\mathcal{A}_1}^{p_1(+)}(\xi_i)\tilde{>}1$ and $\delta_{\mathcal{Q}}^{p_1(-)}(\xi_i)+\delta_{\mathcal{A}_1}^{p_1(-)}(\xi_i)\tilde{<}-1$ also $\mathcal{K}_{\mathcal{A}_2}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$. So, $\mathcal{K}_{\mathcal{A}}$ is BFS Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})$.

Theorem 3.9. *Suppose that $\mathcal{U}_\beta$ be the family of BFS Q-neighborhoods of a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ in a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$. Then the following results are valid*

i. *If $\mathcal{K}_{\mathcal{A}}\tilde{\in}\mathcal{U}_\beta$, then $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}}$.*
ii. *If $\mathcal{K}_{\mathcal{A}}\tilde{\in}\mathcal{U}_\beta$ and $\mathcal{K}_{\mathcal{A}}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_1}$, $\mathcal{K}_{\mathcal{A}_1}\tilde{\in}\mathcal{U}_\beta$.*
iii. *If $\mathcal{K}_{\mathcal{A}}\tilde{\in}\mathcal{U}_\beta$, then $\exists\ \mathcal{K}_{\mathcal{A}_1}\tilde{\in}\mathcal{U}_\beta$ such that $\mathcal{K}_{\mathcal{A}_1}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$ and $\mathcal{K}_{\mathcal{A}_1}\tilde{\in}\mathcal{U}_\alpha$ for each BFS-point $\alpha(\mathcal{K}_{\mathcal{Q}})$, which is quasi-coincident with $\mathcal{K}_{\mathcal{A}_1}$.*

Proof. i. Let $\mathcal{K}_{\mathcal{A}}\tilde{\in}\mathcal{U}_\beta$. Then by definition of BFS Q-neighborhood $\exists\ \mathcal{K}_{\mathcal{A}_3}\tilde{\in}\tilde{\tau}$ such that $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_3}$ and $\mathcal{K}_{\mathcal{A}_3}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$. From these conditions we get, $\delta_{\mathcal{Q}}^{p(+)}(\xi_i) + \delta_{\mathcal{A}_3}^{p(+)}(\xi_i)\tilde{>}1$ and $\delta_{\mathcal{Q}}^{p(-)}(\xi_i)+\delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{<}-1$ for $p\in\mathcal{D}$, $\xi_i\in V$ also $\delta_{\mathcal{A}_3}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}}^{p(+)}(\xi_i)$ and $\delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}}^{p(-)}(\xi_i)$ for $p\in\mathcal{D}$, $\xi_i\in V$. Therefore, $\delta_{\mathcal{Q}}^{p(+)}(\xi_i)+\delta_{\mathcal{A}}^{p(+)}(\xi_i)\tilde{\geq}\delta_{\mathcal{Q}}^{p(+)}(\xi_i)+\delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{>}1$ and $\delta_{\mathcal{Q}}^{p(-)}(\xi_i)+\delta_{\mathcal{A}}^{p(-)}(\xi_i)\tilde{\leq}\delta_{\mathcal{Q}}^{p(-)}(\xi_i) + \delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{<} - 1$, for $p\in\mathcal{D}$, $\xi_i\in V$. Hence $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}}$.

ii. Let $\mathcal{K}_{\mathcal{A}}\tilde{\in}\mathcal{U}_\beta$. Then by definition of BFS Q-neighborhood $\exists\ \mathcal{K}_{\mathcal{A}_3}\tilde{\in}\tilde{\tau}$ such that $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_3}$ and $\mathcal{K}_{\mathcal{A}_3}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$. From above conditions we get, $\delta_{\mathcal{Q}}^{p(+)}(\xi_i) + \delta_{\mathcal{A}_3}^{p(+)}(\xi_i)\tilde{>}1$ and $\delta_{\mathcal{Q}}^{p(-)}(\xi_i) + \delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{<} - 1$ for $p\in\mathcal{D}$, $\xi_i\in V$ also

$$\delta_{\mathcal{A}_3}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}}^{p(+)}(\xi_i), \delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}}^{p(-)}(\xi_i) \tag{3. 1}$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Since it is given that $\mathcal{K}_{\mathcal{A}}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_1}$, then we have

$$\delta_{\mathcal{A}}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}_1}^{p(+)}(\xi_i), \delta_{\mathcal{A}}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}_1}^{p(-)}(\xi_i) \tag{3. 2}$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Since $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_3}$, so we only need to prove that $\mathcal{K}_{\mathcal{A}_3}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_1}$.
Since $\delta_{\mathcal{A}_3}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}}^{p(+)}(\xi_i)$ and $\delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}}^{p(-)}(\xi_i)$ for $p\in\mathcal{D}$, $\xi_i\in V$. By comparing the equation (1) and (2), we get $\delta_{\mathcal{A}_3}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}_1}^{p(+)}(\xi_i)$ and
$\delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}}^{p(-)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}_1}^{p(-)}(\xi_i)$ for $p\in\mathcal{D}$, $\xi_i\in V$. Hence $\mathcal{K}_{\mathcal{A}_1}\tilde{\in}\mathcal{U}_\beta$.

iii. Let $\mathcal{K}_{\mathcal{A}}\tilde{\in}\mathcal{U}_\beta$, then $\exists\ \mathcal{K}_{\mathcal{A}_1}\tilde{\in}\tilde{\tau}$ such that $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_1}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$. So, $\exists\ \mathcal{K}_{\mathcal{A}_1}\tilde{\in}\mathcal{U}_\beta$ such that $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_1}$, $\mathcal{K}_{\mathcal{A}_1}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$. Now let $\alpha(\mathcal{K}_{\mathcal{Q}})$ be any BFS-point and $\alpha(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_1}$. Hence $\mathcal{K}_{\mathcal{A}_1}\tilde{\in}\mathcal{U}_\alpha$. □

Theorem 3.10. *If $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$ be two BFS Q-neighborhoods of a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ in a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$, then $\mathcal{K}_{\mathcal{A}_1}\tilde{\cap}\mathcal{K}_{\mathcal{A}_2}$ is also a BFS Q-neighborhood.*

Proof. Let $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$ be two BFS Q-neighborhoods of a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$, then by definition of BFS Q-neighborhood $\mathcal{K}_{\mathcal{A}_1}$, $\exists\ \mathcal{K}_{\mathcal{A}_3}\tilde{\in}\tilde{\tau}$ such that $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_3}$ and $\mathcal{K}_{\mathcal{A}_3}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_1}$. From these conditions we get,

$$\delta_{\mathcal{Q}}^{p(+)}(\xi_i) + \delta_{\mathcal{A}_3}^{p(+)}(\xi_i)\tilde{>}1,\ \delta_{\mathcal{Q}}^{p(-)}(\xi_i) + \delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{<} - 1$$

for $p\in\mathcal{D}$, $\xi_i\in V$ also

$$\delta_{\mathcal{A}_3}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}_1}^{p(+)}(\xi_i),\ \delta_{\mathcal{A}_3}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}_1}^{p(-)}(\xi_i)$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Similarly, for BFS Q-neighborhood $\mathcal{K}_{\mathcal{A}_2}$, $\exists\, \mathcal{K}_{\mathcal{A}_4}\tilde{\in}\tilde{\tau}$ such that $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_4}$ and $\mathcal{K}_{\mathcal{A}_4}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_2}$. From above conditions we get,

$$\delta_{\mathcal{Q}}^{p(+)}(\xi_i)+\delta_{\mathcal{A}_4}^{p(+)}(\xi_i)\tilde{>}1,\ \delta_{\mathcal{Q}}^{p(-)}(\xi_i)+\delta_{\mathcal{A}_4}^{p(-)}(\xi_i)\tilde{<}-1$$

for $p\in\mathcal{D}$, $\xi_i\in V$ also

$$\delta_{\mathcal{A}_4}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}_2}^{p(+)}(\xi_i),\ \delta_{\mathcal{A}_4}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}_2}^{p(-)}(\xi_i)$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Now $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$ are BFS-sets and their BFS-intersection is again a BFS-set. Let $\mathcal{K}_{\mathcal{A}_1}\tilde{\cap}\mathcal{K}_{\mathcal{A}_2}=\mathcal{K}_{\mathcal{A}}$. This shows that

$$\delta_{\mathcal{A}}^{p(+)}(\xi_i)=min\{\delta_{\mathcal{A}_1}^{p(+)}(\xi_i),\delta_{\mathcal{A}_2}^{p(+)}(\xi_i)\},\ \delta_{\mathcal{A}}^{p(-)}(\xi_i)=max\{\delta_{\mathcal{A}_1}^{p(-)}(\xi_i),\delta_{\mathcal{A}_2}^{p(-)}(\xi_i)\}$$

Since $\mathcal{K}_{\mathcal{A}_3}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_4}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_2}\Rightarrow\mathcal{K}_{\mathcal{A}_3}\tilde{\cap}\mathcal{K}_{\mathcal{A}_4}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_1}\tilde{\cap}\mathcal{K}_{\mathcal{A}_2}$. If $\mathcal{K}_{\mathcal{A}_3}\tilde{\cap}\mathcal{K}_{\mathcal{A}_4}=\mathcal{K}_{\mathcal{A}_5}$, then $\mathcal{K}_{\mathcal{A}_5}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$. So,

$$\delta_{\mathcal{A}_5}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}}^{p(+)}(\xi_i),\ \delta_{\mathcal{A}_5}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}}^{p(-)}(\xi_i)$$

for $p\in\mathcal{D}$, $\xi_i\in V$. As we have, $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_3}$ and $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_4}$, this shows that $\beta(\mathcal{K}_{\mathcal{Q}})q[\mathcal{K}_{\mathcal{A}_3}\tilde{\cap}\mathcal{K}_{\mathcal{A}_4}]=\mathcal{K}_{\mathcal{A}_5}$, where

$$\delta_{\mathcal{A}_5}^{p(+)}(\xi_i)=min\{\delta_{\mathcal{A}_3}^{p(+)}(\xi_i),\delta_{\mathcal{A}_4}^{p(+)}(\xi_i)\},\ \delta_{\mathcal{A}_5}^{p(+)}(\xi_i)=max\{\delta_{\mathcal{A}_3}^{p(-)}(\xi_i),\delta_{\mathcal{A}_4}^{p(-)}(\xi_i)\}$$

and

$$\delta_{\mathcal{Q}}^{p(+)}(\xi_i)+\delta_{\mathcal{A}_5}^{p(+)}(\xi_i)\tilde{>}1,\ \delta_{\mathcal{Q}}^{p(-)}(\xi_i)+\delta_{\mathcal{A}_5}^{p(-)}(\xi_i)\tilde{<}-1$$

for $p\in\mathcal{D}$, $\xi_i\in V$, with the condition $\mathcal{K}_{\mathcal{A}_5}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$. Hence, $\mathcal{K}_{\mathcal{A}}$ is BFS Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})$. □

Theorem 3.11. *A BFS-point* $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})\Leftrightarrow$ *BFS Q-neighborhood of* $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}}$.

Proof. $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})\Leftrightarrow$ every BFS-closed set $\mathcal{K}_{\mathcal{A}_1}$ which containing $\mathcal{K}_{\mathcal{A}}$ also fulfil the condition $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\mathcal{K}_{\mathcal{A}_1}$. By this condition we have

$$\delta_{\mathcal{Q}}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}_1}^{p(+)}(\xi_i),\ \delta_{\mathcal{Q}}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}_1}^{p(-)}(\xi_i)$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Now $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})\Leftrightarrow\forall$ BFS-closed sets $\mathcal{K}_{\mathcal{A}}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_1}$

$$1-\delta_{\mathcal{Q}}^{p(+)}(\xi_i)\tilde{\geq}1-\delta_{\mathcal{A}_1}^{p(+)}(\xi_i),\ -1-\delta_{\mathcal{Q}}^{p(-)}(\xi_i)\tilde{\leq}-1-\delta_{\mathcal{A}_1}^{p(-)}(\xi_i)$$

Now, $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})\Leftrightarrow$ for any BFS-open set $\mathcal{K}_{\mathcal{A}_2}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}^c$

$$\delta_{\mathcal{A}_2}^{p(+)}(\xi_i)\tilde{\leq}1-\delta_{\mathcal{Q}}^{p(+)}(\xi_i),\ \delta_{\mathcal{A}_2}^{p(-)}(\xi_i)\tilde{\geq}-1-\delta_{\mathcal{Q}}^{p(-)}(\xi_i)$$

for $p\in\mathcal{D}$, $\xi_i\in V$. More precisely it can be stated as, $\forall$ BFS-open set $\mathcal{K}_{\mathcal{A}_2}$ satisfying

$$\delta_{\mathcal{A}_2}^{p(+)}(\xi_i)\tilde{>}1-\delta_{\mathcal{Q}}^{p(+)}(\xi_i),\ \delta_{\mathcal{A}_2}^{p(-)}(\xi_i)\tilde{<}-1-\delta_{\mathcal{Q}}^{p(-)}(\xi_i)$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Which shows that $\mathcal{K}_{\mathcal{A}_2}\tilde{\nsubseteq}\mathcal{K}_{\mathcal{A}}^c$ again

$$\mathcal{K}_{\mathcal{A}_2}\tilde{\nsubseteq}\mathcal{K}_{\mathcal{A}}^c\Leftrightarrow\mathcal{K}_{\mathcal{A}_2}q\mathcal{K}_{\mathcal{A}}$$

Hence it is shown that $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})\ \Leftrightarrow$ every BFS-open Q-neighborhood $\mathcal{K}_{\mathcal{A}_2}$ of $\beta(\mathcal{K}_{\mathcal{Q}})$ is BFS quasi-coincident with $\mathcal{K}_{\mathcal{A}}$. Which is perfectly equivalent what we want to show. □

Definition 3.12. A BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ is said to be a BFS-adherence point of a BFS-set $\mathcal{K}_{\mathcal{A}}$, if each BFS Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})$ is quasi-coincident with $\mathcal{K}_{\mathcal{A}}$.

Theorem 3.13. *Every BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ of a BFS-set $\mathcal{K}_{\mathcal{A}}$ is a BFS-adherence point of $\mathcal{K}_{\mathcal{A}}$.*

Proof. Consider an arbitrary BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\mathcal{K}_{\mathcal{A}}$, by this condition we have

$$\delta_{\mathcal{Q}}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}}^{p(+)}(\xi_i), \delta_{\mathcal{Q}}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}}^{p(-)}(\xi_i) \tag{3. 3}$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Now consider a BFS Q-neighborhood $\mathcal{K}_{\mathcal{A}_1}$ of $\beta(\mathcal{K}_{\mathcal{Q}})$, then by definition of BFS Q-neighborhood, $\exists\ \mathcal{K}_{\mathcal{A}_2}\tilde{\in}\tilde{\tau}\ |\ \beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_2}$ and $\mathcal{K}_{\mathcal{A}_2}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}_1}$, then by these conditions we get

$$\delta_{\mathcal{Q}}^{p(+)}(\xi_i)+\delta_{\mathcal{A}_2}^{p(+)}(\xi_i)\tilde{>}1, \delta_{\mathcal{Q}}^{p(-)}(\xi_i)+\delta_{\mathcal{A}_2}^{p(-)}(\xi_i)\tilde{<}-1 \tag{3. 4}$$

and

$$\delta_{\mathcal{A}_2}^{p(+)}(\xi_i)\tilde{\leq}\delta_{\mathcal{A}_1}^{p(+)}(\xi_i), \delta_{\mathcal{A}_2}^{p(-)}(\xi_i)\tilde{\geq}\delta_{\mathcal{A}_1}^{p(-)}(\xi_i) \tag{3. 5}$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Now by adding the equations (3), (5) and then comparing the resultant equations with (4) we get,

$$\delta_{\mathcal{A}_1}^{p(+)}(\xi_i)+\delta_{\mathcal{A}}^{p(+)}(\xi_i)\tilde{\geq}\delta_{\mathcal{Q}}^{p(+)}(\xi_i)+\delta_{\mathcal{A}_2}^{p(+)}(\xi_i)\tilde{>}1$$
$$\delta_{\mathcal{A}_1}^{p(-)}(\xi_i)+\delta_{\mathcal{A}}^{p(-)}(\xi_i)\tilde{\leq}\delta_{\mathcal{Q}}^{p(-)}(\xi_i)+\delta_{\mathcal{A}_2}^{p(-)}(\xi_i)\tilde{<}-1$$

for $p\in\mathcal{D}$, $\xi_i\in V$. Hence $\mathcal{K}_{\mathcal{A}_1}q\mathcal{K}_{\mathcal{A}}$. Since $\mathcal{K}_{\mathcal{A}_1}$ is a BFS Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})$ and $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}}$. So $\beta(\mathcal{K}_{\mathcal{Q}})$ is a BFS-adherence point of $\mathcal{K}_{\mathcal{A}}$. □

Definition 3.14. A BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ is said to be a BFS-accumulation point of a BFS-set $\mathcal{K}_{\mathcal{A}}$ if $\beta(\mathcal{K}_{\mathcal{Q}})$ is a BFS-adherence point of $\mathcal{K}_{\mathcal{A}}$ and every BFS Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})$ and $\mathcal{K}_{\mathcal{A}}$ are BFS quasi-coincident at some BFS-point different from β, $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\mathcal{K}_{\mathcal{A}}$.

Note that the BFS-union of BFS-accumulation points of $\mathcal{K}_{\mathcal{A}}$ is called BFS-derived set of $\mathcal{K}_{\mathcal{A}}$ and we write it as $\mathcal{K}_{\mathcal{A}}^{\mathrm{d}}$.

Theorem 3.15. $\widetilde{Cl}\mathcal{K}_{\mathcal{A}}=\mathcal{K}_{\mathcal{A}}\tilde{\cup}\mathcal{K}_{\mathcal{A}}^{\mathrm{d}}$

Proof. Consider a collection $\sum=\{\beta(\mathcal{K}_{\mathcal{Q}})\}$ of BFS-adherence point $\beta(\mathcal{K}_{\mathcal{Q}})$ of $\mathcal{K}_{\mathcal{A}}$, then by previous theorem, " A BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})$ $\Leftrightarrow$ every BFS Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}}$ ". We have $\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})=\tilde{\cup}\sum$. So $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\sum$ $\Leftrightarrow$ either $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\mathcal{K}_{\mathcal{A}}$ or $\beta(\mathcal{K}_{\mathcal{Q}})\tilde{\in}\mathcal{K}_{\mathcal{A}}^{\mathrm{d}}$. Hence $\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})=\tilde{\cup}\sum=\mathcal{K}_{\mathcal{A}}\tilde{\cup}\mathcal{K}_{\mathcal{A}}^{\mathrm{d}}$. □

Theorem 3.16. *A BFS-set $\mathcal{K}_{\mathcal{A}}$ is closed $\Leftrightarrow$ $\mathcal{K}_{\mathcal{A}}$ contains all of its BFS-accumulation points.*

Proof. Suppose that $\mathcal{K}_{\mathcal{A}}$ be a BFS-set then by the result " $\widetilde{Cl}(\mathcal{K}_{\mathcal{A}})=\mathcal{K}_{\mathcal{A}}\tilde{\cup}\mathcal{K}_{\mathcal{A}}^{\mathrm{d}}$ ".
$\mathcal{K}_{\mathcal{A}}$ is BFS-closed
$\Leftrightarrow \mathcal{K}_{\mathcal{A}}=Cl(\mathcal{K}_{\mathcal{A}})$
$\Leftrightarrow \widetilde{Cl}(\mathcal{K}_{\mathcal{A}})=\mathcal{K}_{\mathcal{A}}\tilde{\cup}\mathcal{K}_{\mathcal{A}}^{\mathrm{d}}$
$\Leftrightarrow \mathcal{K}_{\mathcal{A}}=\mathcal{K}_{\mathcal{A}}\tilde{\cup}\mathcal{K}_{\mathcal{A}}^{\mathrm{d}}$
$\Leftrightarrow \mathcal{K}_{\mathcal{A}}^{\mathrm{d}}\tilde{\subseteq}\mathcal{K}_{\mathcal{A}}$
$\Leftrightarrow \mathcal{K}_{\mathcal{A}}$ contains all of its BFS-accumulation points. □

Definition 3.17. A collection $\sum = \{\beta(\mathcal{K_Q})\}$ of BFS-points is called BFS-dense in a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$ if each nonempty BFS-open set contains some members of $\sum$. $\sum$ is called BFS Q-dense $\Leftrightarrow$ each nonempty BFS-open set is BFS quasi-coincident with some members of $\sum$.

Note that both the concepts (being BFS-dense and BFS Q-dense) do not imply each other. It can be seen by the counter examples given below.

Example 3.18. Let $\sum = \{\alpha(\mathcal{K}_{\mathcal{Q}_1}), \beta(\mathcal{K}_{\mathcal{Q}_2}), \gamma(\mathcal{K}_{\mathcal{Q}_3})\}$ be the collection of BFS-points. Where

$$\alpha(\mathcal{K}_{\mathcal{Q}_1}) = \{\ \mathcal{K}_{p_1} = \{\ (\xi_1, 0.23, -0.32), (\xi_2, 0.21, -0.34), (\xi_3, 0.21, -0.23), (\xi_4, 0.31, -0.26)\ \}\ \}$$
$$\beta(\mathcal{K}_{\mathcal{Q}_2}) = \{\ \mathcal{K}_{p_2} = \{\ (\xi_1, 0.31, -0.42), (\xi_2, 0.32, -0.24), (\xi_3, 0.32, -0.41), (\xi_4, 0.31, -0.37)\ \}\ \}$$
$$\gamma(\mathcal{K}_{\mathcal{Q}_3}) = \{\ \mathcal{K}_{p_3} = \{\ (\xi_1, 0.11, -0.22), (\xi_2, 0.21, -0.42), (\xi_3, 0.17, -0.28), (\xi_4, 0.11, -0.22)\ \}\ \}$$

Let us consider BFS-topology $\tilde{\tau} = \{\phi_{\mathcal{D}}, V_{\mathcal{D}}, \mathcal{K}_{\mathcal{A}_1}, \mathcal{K}_{\mathcal{A}_2}\}$, where $\phi_{\mathcal{D}}$ and $V_{\mathcal{D}}$ are null BFS-set and absolute BFS-set respectively and

$$\mathcal{K}_{\mathcal{A}_1} = \left\{ \begin{array}{l} \mathcal{K}_{p_1} = \{\ (\xi_1, 0.51, -0.42), (\xi_2, 0.56, -0.52), (\xi_3, 0.51, -0.42), (\xi_4, 0.51, -0.33)\ \}, \\ \mathcal{K}_{p_2} = \{\ (\xi_1, 0.61, -0.65), (\xi_2, 0.71, -0.84), (\xi_3, 0.72, -0.62), (\xi_4, 0.69, -0.63)\ \}, \\ \mathcal{K}_{p_3} = \{\ (\xi_1, 0.81, -0.72), (\xi_2, 0.84, -0.64), (\xi_3, 0.86, -0.51), (\xi_4, 0.73, -0.70)\ \} \end{array} \right\}$$
$$\mathcal{K}_{\mathcal{A}_2} = \begin{array}{l} \mathcal{K}_{p_2} = \{\ (\xi_1, 0.51, -0.53), (\xi_2, 0.70, -0.74), (\xi_3, 0.62, -0.53), (\xi_4, 0.53, -0.56)\ \}, \\ \mathcal{K}_{p_3} = \{\ (\xi_1, 0.71, -0.65), (\xi_2, 0.83, -0.54), (\xi_3, 0.84, -0.41), (\xi_4, 0.62, -0.65)\ \} \end{array}$$

It is ease to see that every nonempty BFS-open set contains some members of $\sum$. So $\sum$ is BFS-dense in $(V, \tilde{\tau}, \mathcal{D})$ but it is not BFS Q-dense because not every BFS-open set is quasi-coincident with some members of $\sum$.

Example 3.19. Let us consider now $\sum = \{\alpha(\mathcal{K}_{\mathcal{Q}_1}), \beta(\mathcal{K}_{\mathcal{Q}_2}), \gamma(\mathcal{K}_{\mathcal{Q}_3})\}$ be the collection of BFS-points. Where

$$\alpha(\mathcal{K}_{\mathcal{Q}_1}) = \{\ \mathcal{K}_{p_1} = \{\ (\xi_1, 0.81, -0.74), (\xi_2, 0.92, -0.85), (\xi_3, 0.72, -0.87), (\xi_4, 0.82, -0.74)\ \}\ \}$$
$$\beta(\mathcal{K}_{\mathcal{Q}_2}) = \{\ \mathcal{K}_{p_2} = \{\ (\xi_1, 0.82, -0.62), (\xi_2, 0.71, -0.53), (\xi_3, 0.84, -0.91), (\xi_4, 0.79, -0.82)\ \}\ \}$$
$$\gamma(\mathcal{K}_{\mathcal{Q}_3}) = \{\ \mathcal{K}_{p_3} = \{\ (\xi_1, 0.91, -0.83), (\xi_2, 0.87, -0.73), (\xi_3, 0.85, -0.91), (\xi_4, 0.81, -0.84)\ \}\ \}$$

Suppose that $\tilde{\tau} = \{\phi_{\mathcal{D}}, V_{\mathcal{D}}, \mathcal{K}_{\mathcal{A}_1}, \mathcal{K}_{\mathcal{A}_2}\}$ be a BFS-topology, where $\phi_{\mathcal{D}}$ and $V_{\mathcal{D}}$ are null BFS-set and absolute BFS-set respectively and

$$\mathcal{K}_{\mathcal{A}_1} = \left\{ \begin{array}{l} \mathcal{K}_{p_1} = \{\ (\xi_1, 0.51, -0.42), (\xi_2, 0.56, -0.52), (\xi_3, 0.51, -0.42), (\xi_4, 0.51, -0.33)\ \}, \\ \mathcal{K}_{p_2} = \{\ (\xi_1, 0.61, -0.65), (\xi_2, 0.71, -0.84), (\xi_3, 0.72, -0.62), (\xi_4, 0.69, -0.63)\ \}, \\ \mathcal{K}_{p_3} = \{\ (\xi_1, 0.81, -0.72), (\xi_2, 0.84, -0.64), (\xi_3, 0.86, -0.51), (\xi_4, 0.73, -0.70)\ \} \end{array} \right\}$$
$$\mathcal{K}_{\mathcal{A}_2} = \left\{ \begin{array}{l} \mathcal{K}_{p_2} = \{\ (\xi_1, 0.51, -0.53), (\xi_2, 0.70, -0.74), (\xi_3, 0.62, -0.53), (\xi_4, 0.53, -0.56)\ \}, \\ \mathcal{K}_{p_3} = \{\ (\xi_1, 0.71, -0.65), (\xi_2, 0.83, -0.54), (\xi_3, 0.84, -0.41), (\xi_4, 0.62, -0.65)\ \} \end{array} \right\}$$

It is ease to see that every nonempty BFS-open set is quasi-coincident with some members of $\sum$. So $\sum$ is BFS Q-dense in $(V, \tilde{\tau}, \mathcal{D})$ but it is not BFS-dense because not every BFS-open set contains some members of $\sum$.

Theorem 3.20. *Let $(V, \tilde{\tau}, \mathcal{D})$ be a BFS-topological space. A collection $\sum$ of BFS-points $\{\beta(\mathcal{K}_{\mathcal{Q}_2})\} \tilde{\in} V_{\mathcal{D}}$ is BFS Q-dense $\Leftrightarrow \tilde{\bigcup} \sum = V_{\mathcal{D}}$.*

Proof. It is obvious that $\tilde{\bigcup} \sum \tilde{\subseteq} V_{\mathcal{D}}$. Let $\beta(\mathcal{K}_{\mathcal{Q}_2}) \tilde{\in} V_{\mathcal{D}}$ be a BFS-point, then by the result, " Every BFS-point $\beta(\mathcal{K_Q})$ of a BFS-set $\mathcal{K_A}$ is a BFS-adherence point of $\mathcal{K_A}$ ". Therefore each BFS-point in $\tilde{\bigcup} \sum \tilde{\subseteq} V_{\mathcal{D}}$ is a BFS-adherence point of $\tilde{\bigcup} \sum$ and therefore belongs to $\widetilde{Cl}(\tilde{\bigcup} \sum)$. Hence the condition is necessary. Now for sufficient condition let $\mathcal{K}_{\mathcal{A}_1}$ be a nonempty BFS-open set and since $\beta(\mathcal{K_Q}) \tilde{\in} \widetilde{Cl}(\tilde{\bigcup} \sum) = V_{\mathcal{D}}$. Then by the theorem, " A BFS-point $\beta(\mathcal{K_Q}) \tilde{\in} \widetilde{Cl}(\mathcal{K_A}) \Leftrightarrow$ BFS Q-neighborhood of $\beta(\mathcal{K_Q}) q \mathcal{K_A}$ ". It is ease to show which we want to prove. $\square$

Definition 3.21. Consider a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$. A sub-collection $\mathfrak{B}$ of $\tilde{\tau}$ is said to be a BFS-base for $\tilde{\tau} \Leftrightarrow \exists\ (\mathcal{H}_i)_{i\in I} \tilde{\subset} \mathfrak{B}$, for each $\mathcal{K}_{\mathcal{A}} \tilde{\in} \tilde{\tau}$ such that $\mathcal{K}_{\mathcal{A}} = \tilde{\bigcup\limits_{i\in I}} \mathcal{H}_i$.

Definition 3.22. A sub-collection $\mathbb{B} \tilde{\in} \mathcal{N}(\beta(\mathcal{K}_{\mathcal{Q}}))$ is said to be a BFS-neighborhood base of $\mathcal{N}(\beta(\mathcal{K}_{\mathcal{Q}}))$ if $\exists\ \mathbf{B} \tilde{\in} \mathbb{B}$ for each $\mathbb{A} \tilde{\in} \mathcal{N}(\beta(\mathcal{K}_{\mathcal{Q}}))|\mathbf{B} \tilde{\subset} \mathbb{A}$.

Definition 3.23. A sub-collection $\mathbb{B} \tilde{\in} Q\mathcal{N}(\beta(\mathcal{K}_{\mathcal{Q}}))$ is said to be a BFS Q-neighborhood base of $Q\mathcal{N}(\beta(\mathcal{K}_{\mathcal{Q}}))$ if $\exists\ \mathbf{B} \tilde{\in} \mathbb{B}$ for each $\mathbb{A} \tilde{\in} Q\mathcal{N}(\beta(\mathcal{K}_{\mathcal{Q}}))|\mathbf{B} \tilde{\subset} \mathbb{A}$.

Definition 3.24. A BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$ satisfies BFS first axiom of countability if there is a countable BFS-neighborhood base for each BFS-point $\beta(\mathcal{K}_{\mathcal{Q}}) \tilde{\in} (V, \tilde{\tau}, \mathcal{D})$. We denote it by BFS-$\mathcal{C}_I$.

Definition 3.25. A BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$ satisfies BFS second axiom of countability if there is a countable BFS-base for each BFS-point $\beta(\mathcal{K}_{\mathcal{Q}}) \tilde{\in} (V, \tilde{\tau}, \mathcal{D})$. We denote it by BFS-$\mathcal{C}_{II}$.

Definition 3.26. A BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$ satisfies BFS Q-first axiom of countability if there is a countable BFS Q-neighborhood base for each BFS-point $\beta(\mathcal{K}_{\mathcal{Q}}) \tilde{\in} (V, \tilde{\tau}, \mathcal{D})$. We denote it by BFS Q-$\mathcal{C}_I$.

Proposition 3.27. *If a BFS-topological space is BFS-$\mathcal{C}_I$ then it is also BFS Q-$\mathcal{C}_I$.*

Proof. Consider a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})_{(\alpha,\gamma)} \tilde{\in} (V, \tilde{\tau}, \mathcal{D})$, where $\alpha = \delta_Q^{p(+)}(\xi)$ and $\gamma = \delta_Q^{p(-)}(\xi)$. Let $\{\gamma_n\} = \delta_{Q_n}^{p(-)}(\xi)$ and $\{\alpha_n\} = \delta_{Q_n}^{p(+)}(\xi)$ be two sequences in [-1, $-1-\gamma$) and $(1-\alpha$, 1] respectively, for each $n \in \mathbf{N}$. Assume that $\{\alpha_n\}$ converges to $1-\alpha$ and $\{\gamma_n\}$ converges to $-1-\gamma$, then $\beta(\mathcal{K}_{\mathcal{Q}})_{(\alpha_n,\gamma_n)}$ be a BFS-point in $(V, \tilde{\tau}, \mathcal{D})$, for each $n \in \mathbf{N}$. As $(V, \tilde{\tau}, \mathcal{D})$ is BFS-$\mathcal{C}_I$, then $\exists$ a countable BFS-neighborhood base $\mathbb{B}_n$ of the BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})_{(\alpha_n,\gamma_n)}$ (obviously we assume here, that each member of $\mathbb{B}_n$ is open) , for each $n \in \mathbf{N}$. Now suppose that there is $\mathbf{B}_n \tilde{\in} \mathbb{B}_n$ such that $\beta(\mathcal{K}_{\mathcal{Q}})_{(\alpha_n,\gamma_n)} \tilde{\in} \mathbf{B}_n$, for each $n \in \mathbf{N}$. Therefore, $\alpha_n \tilde{\leq} \delta_{\mathbf{B}_n}^{p(+)}(\xi)$ and $\gamma_n \tilde{\geq} \delta_{\mathbf{B}_n}^{p(-)}(\xi)$, for each $n \in \mathbf{N}$. We get, $\delta_{\mathbf{B}_n}^{p(+)}(\xi) \tilde{\geq} \alpha \tilde{>} 1-\alpha$ and $\delta_{\mathbf{B}_n}^{p(-)}(\xi) \tilde{\leq} \gamma \tilde{<} -1-\gamma$. This shows that $\beta(\mathcal{K}_{\mathcal{Q}})_{(\alpha,\gamma)} q \mathbf{B}_n$, for each $n \in \mathbf{N}$. Consider the family $\mathbb{B}$ of all members of all $\mathbf{B}_n$, then obviously $\mathbb{B}$ is BFS-open Q-neighborhood base of $\beta(\mathcal{K}_{\mathcal{Q}})_{(\alpha,\gamma)}$. Suppose that $\mathbb{A}$ is an arbitrary BFS-open Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})_{\alpha,\gamma}$, then we have $\delta_{\mathbb{A}}^{p(+)}(\xi) \tilde{>} \delta_Q^{p(+)}(\xi) \tilde{>} 1-\alpha$ and $\delta_{\mathbb{A}}^{p(-)}(\xi) \tilde{<} \delta_Q^{p(-)}(\xi) \tilde{<} -1-\gamma$. As we have $\alpha_n \in (1-\alpha, 1]$ and $\gamma_n \in [-1, -1-\gamma)$, for each $n \in \mathbf{N}$, $\exists\ m \in \mathbf{N} | \delta_{Q_m}^{p(+)}(\xi) \tilde{\leq} \delta_{\mathbb{A}}^{p(+)}(\xi)$ and $\delta_{Q_m}^{p(-)}(\xi) \tilde{\geq} \delta_{\mathbb{A}}^{p(-)}(\xi)$ and $\delta_{\mathbb{A}}^{p(+)}(\xi) \tilde{\geq} \alpha_m \tilde{>} 1-\alpha$ and $\delta_{\mathbb{A}}^{p(-)}(\xi) \tilde{\leq} \gamma_m \tilde{<} -1-\gamma$. This shows that $\beta(\mathcal{K}_{\mathcal{Q}})_{(\alpha_m,\gamma_m)} q \mathbb{A}$ this implies that $\mathbb{A}$ is a BFS-open neighborhood base of (α_m, γ_m). Therefore, $\exists\ \mathbf{B} \tilde{\in} \mathbb{B}_n \tilde{\subset} \mathbb{B} | \mathbf{B} \tilde{\subset} \mathbb{A}$. Further, $\beta(\mathcal{K}_{\mathcal{Q}}) q \mathbf{B}$. This implies that $\mathbb{B}$ is countable BFS Q-neighborhood base of $\beta(\mathcal{K}_{\mathcal{Q}})$. So $(V, \tilde{\tau}, \mathcal{D})$ is BFS Q-$\mathcal{C}_I$.

□

Theorem 3.28. *Let $(V, \tilde{\tau}, \mathcal{D})$ be a BFS-topological space. A sub-collection $\mathfrak{B}$ in $\tilde{\tau}$ is said to be A BFS-base for $\tilde{\tau} \Leftrightarrow$ for each BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ and BFS-open Q-neighborhood $\mathcal{K}_{\mathcal{A}_1}$ of $\beta(\mathcal{K}_{\mathcal{Q}})$ $\exists\ \mathcal{H}_{\mathfrak{B}} \tilde{\in} \mathfrak{B}\ |\beta(\mathcal{K}_{\mathcal{Q}}) q \mathcal{H}_{\mathfrak{B}}$, $\mathcal{H}_{\mathfrak{B}} \tilde{\subset} \mathcal{K}_{\mathcal{A}}$.*

Proof. The necessity of the theorem can be directly follow from the the definition of BFS-base and by the proposition " Consider a collection of BFS-sets $\{\mathcal{K}_{\mathcal{A}_i}\}_{i\in I}$ in a BFS-topological space $(V,\tilde{\tau},\mathcal{D})$. Then a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})q\tilde{\bigcup\limits_{i\in I}}\mathcal{K}_{\mathcal{A}_i} \Leftrightarrow \beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}_i}$". We have to prove for sufficiency. Suppose on contrary that $\mathfrak{B}$ is not BFS-base for $\tilde{\tau}$, then $\exists\ \mathcal{K}_{\mathcal{A}}\tilde{\in}\tilde{\tau}$ such that $\mathcal{A}_1 = \tilde{\bigcup}\{\mathcal{H}_{\mathfrak{B}}\tilde{\in}\mathfrak{B} : \mathcal{H}_{\mathfrak{B}}\tilde{\subset}\mathcal{K}_{\mathcal{A}}\} \neq \mathcal{K}_{\mathcal{A}}$. Thus $\exists\ p \in \mathcal{D}$ such that $\delta^{p(+)}_{\mathcal{A}_1}(\xi)\tilde{<}\delta^{p(+)}_{\mathcal{A}}(\xi)$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi)\tilde{>}\delta^{p(-)}_{\mathcal{A}}(\xi)$, for $\xi \in V$. Let $\delta^{p(+)}_{\mathcal{A}_1}(\xi) = 1-\alpha$ and $\delta^{p(-)}_{\mathcal{A}_1}(\xi) = -1-\gamma$, then we obtain a fuzzy point $\beta(\mathcal{K}_{\mathcal{Q}})$. Since $\delta^{p(+)}_{\mathcal{A}}(\xi)+\alpha\tilde{>}\delta^{p(+)}_{\mathcal{A}_1}(\xi)+\alpha = 1$ and $\delta^{p(-)}_{\mathcal{A}}(\xi)+\gamma\tilde{>}\delta^{p(-)}_{\mathcal{A}_1}(\xi)+\gamma = -1$ i.e. $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}}$. This implies that $\exists$ $\mathcal{H}_{\mathfrak{B}}\tilde{\in}\mathfrak{B}|\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{H}_{\mathfrak{B}}$, $\mathcal{H}_{\mathfrak{B}}\tilde{\subset}\mathcal{A}$. Since $\mathcal{H}_{\mathfrak{B}}\tilde{\in}\mathcal{A}_1$. This shows that $\delta^{p(+)}_{\mathcal{H}_{\mathfrak{B}}}(\xi)\tilde{\leq}\delta^{p(+)}_{\mathcal{A}_1}(\xi)$ and $\delta^{p(-)}_{\mathcal{H}_{\mathfrak{B}}}(\xi)\tilde{\geq}\delta^{p(-)}_{\mathcal{A}_1}(\xi)$ i.e. $\delta^{p(+)}_{\mathcal{H}_{\mathfrak{B}}}(\xi)+\alpha\tilde{\leq}1$ and $\delta^{p(-)}_{\mathcal{H}_{\mathfrak{B}}}(\xi)+\gamma\tilde{\geq}-1$, a contradiction to the fact that $\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{H}_{\mathfrak{B}}$. Hence the theorem. □

Proposition 3.29. *If a BFS-topological space is BFS-$\mathcal{C}_{II}$ then it is also BFS Q-$\mathcal{C}_I$.*

Proof. Since $(V,\tilde{\tau},\mathcal{D})$ is BFS-$\mathcal{C}_{II}$ space, so $\mathfrak{B}$ be a countable BFS-base for $\tilde{\tau}$. Consider a BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$, then for BFS-open set $\mathcal{K}_{\mathcal{A}}|\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{K}_{\mathcal{A}}$ and by the result " Let $(V,\tilde{\tau},\mathcal{D})$ be a BFS-topological space. A sub-collection $\mathfrak{B}$ in $\tilde{\tau}$ is said to be A BFS-base for $\tilde{\tau}$ $\Leftrightarrow$ for each BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ and BFS-open Q-neighborhood $\mathcal{K}_{\mathcal{A}_1}$ of $\beta(\mathcal{K}_{\mathcal{Q}})$ $\exists$ $\mathcal{H}_{\mathfrak{B}}\tilde{\in}\mathfrak{B}$ $|\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{H}\mathfrak{B}$, $\mathcal{H}_{\mathfrak{B}}\tilde{\subset}\mathcal{K}_{\mathcal{A}}$". We get $\mathcal{H}_{\mathfrak{B}}\tilde{\in}\mathfrak{B}|\beta(\mathcal{K}_{\mathcal{Q}})q\mathcal{H}_{\mathfrak{B}}\tilde{\subset}\mathcal{K}_{\mathcal{A}}$. So it is evident that $\mathcal{H}_{\mathfrak{B}}$ is BFS Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})$. Now suppose that $\mathcal{U}$ be the family of all such members $\mathcal{H}_{\mathfrak{B}}\tilde{\in}\mathfrak{B}$. Then obviously this family is countable family of BFS Q-neighborhood of $\beta(\mathcal{K}_{\mathcal{Q}})$ i.e. the BFS-point $\beta(\mathcal{K}_{\mathcal{Q}})$ has countable BFS Q-neighborhood base. Hence the theorem. □

Definition 3.30. Consider a BFS-topological space $(V,\tilde{\tau},\mathcal{D})$. Suppose that $\mathcal{K}_{\mathcal{A}}$ and $\mathcal{K}_{\mathcal{A}_i}$ where $i \in I$ be BFS-sets in $(V,\tilde{\tau},\mathcal{D})$. Then $\{\mathcal{K}_{\mathcal{A}_i}|i\in I\}$ is said to be a BFS-cover of $\mathcal{K}_{\mathcal{A}}$ $\Leftrightarrow \mathcal{K}_{\mathcal{A}}\tilde{\subset}\tilde{\bigvee\limits_{i\in I}}\mathcal{K}_{\mathcal{A}_i}$. if $\exists$ $\{I_1 \subset I|\mathcal{K}_{\mathcal{A}}\tilde{\subset}\tilde{\bigvee\limits_{i\in I}}\mathcal{K}_{\mathcal{A}_i}$ where $i \in I_1\}$, then $\mathcal{K}_{\mathcal{A}_i}$ for $i \in I_1$ is called BFS-subcover(which is a BFS-open cover itself) of $\mathcal{K}_{\mathcal{A}}$.

Definition 3.31. A BFS-set $\mathcal{K}_{\mathcal{A}}$ in a BFS-topological space $(V,\tilde{\tau},\mathcal{D})$ is said to be satisfied BFS-Lindel$\ddot{o}$f property if each BFS-open cover of $\mathcal{K}_{\mathcal{A}}$ has a countable BFS-subcover.

Example 3.32. Let $V = \{\xi_1,\xi_2,\xi_3\}$ be a universal set and $\mathcal{D} = \{p_1,p_2\}$ be the set of decision variables. Let $\mathcal{K}^1_{\mathcal{D}},\mathcal{K}^2_{\mathcal{D}}and\mathcal{K}^3_{\mathcal{D}}\ \tilde{\in}\mathfrak{P}(V_{\mathcal{D}})$. Where

$$\mathcal{K}^1_{\mathcal{D}} = \left\{\begin{array}{l} \mathcal{K}^1_{p_1} = \{\ (\xi_1,0.31,-0.52),(\xi_2,0.21,-0.32),(\xi_3,0.33,-0.34)\ \}, \\ \mathcal{K}^1_{p_2} = \{\ (\xi_1,0.52,-0.32),(\xi_2,0.51,-0.53),(\xi_3,0.55,-0.36)\ \} \end{array}\right\}$$

$$\mathcal{K}^2_{\mathcal{D}} = \left\{\begin{array}{l} \mathcal{K}^2_{p_1} = \{\ (\xi_1,0.51,-0.52),(\xi_2,0.32,-0.32),(\xi_3,0.33,-0.34)\ \}, \\ \mathcal{K}^2_{p_2} = \{\ (\xi_1,0.52,-0.43),(\xi_2,0.51,-0.53),(\xi_3,0.71,-0.51)\ \} \end{array}\right\}$$

$$\mathcal{K}^3_{\mathcal{D}} = \left\{\begin{array}{l} \mathcal{K}^3_{p_1} = \{\ (\xi_1,0.31,-0.33),(\xi_2,0.21,-0.21),(\xi_3,0.22,-0.31)\ \}, \\ \mathcal{K}^3_{p_2} = \{\ (\xi_1,0.32,-0.32),(\xi_2,0.51,-0.33),(\xi_3,0.55,-0.36)\ \} \end{array}\right\}$$

Then $\tilde{\tau} = \{\phi_{\mathcal{D}},V_{\mathcal{D}},\mathcal{K}^1_{\mathcal{D}},\mathcal{K}^2_{\mathcal{D}},\mathcal{K}^3_{\mathcal{D}}\}$. Now suppose that

$$\mathcal{K}_{\mathcal{D}} = \left\{\begin{array}{l} \mathcal{K}_{p_1} = \{\ (\xi_1,0.41,-0.43),(\xi_2,0.22,-0.25),(\xi_3,0.11,-0.23)\ \}, \\ \mathcal{K}_{p_2} = \{\ (\xi_1,0.42,-0.33),(\xi_2,0.44,-0.42),(\xi_3,0.62,-0.15)\ \} \end{array}\right\}$$

Then it can be seen that each BFS-open cover of $\mathcal{K}_{\mathcal{D}}$ has countable BFS-subcover.

Definition 3.33. The support of a BFS-set $\mathcal{K}_{\mathcal{A}}$ is defined as $\{\xi|\delta_p^+(\xi) \neq 0, \delta_p^-(\xi) \neq 0\}$. We denote it by supp$(\mathcal{K}_{\mathcal{A}})$.

Definition 3.34. A BFS-set $\mathcal{K}_{\mathcal{A}}$ in a BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$ is uncountable $\Leftrightarrow$ its support is an uncountable set.

Definition 3.35. A BFS-topological space $(V, \tilde{\tau}, \mathcal{D})$ is called BFS-separable or (BFS Q-separable) $\Leftrightarrow$ $\exists$ a countable collection of BFS-points in $\mathcal{BF}(V_{\mathcal{D}})$ which is BFS-dense or (BFS Q-dense).

Theorem 3.36. *A BFS-topological space is BFS-separable $\Leftrightarrow$ it is BFS Q-separable.*

Proof. Let us suppose that $\sum = \{\beta(\mathcal{K}_{\mathcal{Q}_n})\}$ be a countable collection of BFS-points which is BFS-dense. We assume that Supp$(\beta(\mathcal{K}_{\mathcal{Q}_n})) = \{\xi_n\}$ and let $\beta_1(\mathcal{K}_{\mathcal{Q}_n}) = \{\xi_n\}_{(1,-1)}$, where 1 And -1 are positive and negative membership degrees of the BFS-point $\beta_1(\mathcal{K}_{\mathcal{Q}_n})$ at Support ξ_n. It is obvious that the collection $\widetilde{\sum} = \{\beta_1(\mathcal{K}_{\mathcal{Q}_n})\}$ is countable BFS Q-dense. Now for the converse part let us assume that $\sum = \{\beta(\mathcal{K}_{\mathcal{Q}_n})\}$ be collection of BFS Q-dense with Support ξ_n and let $\beta_{\ell,m}(\mathcal{K}_{\mathcal{Q}_n}) = \{\xi_n\}_{(1/\ell,\text{-1/m})}$, where $1/\ell$ and $-1/m$ are positive and negative membership degrees of the BFS-point $\beta_{\ell,m}(\mathcal{K}_{\mathcal{Q}_n})$ at support ξ_n. Then evidently $\widetilde{\sum} = \{\beta_{\ell,m}(\mathcal{K}_{\mathcal{Q}_n})|\ell, m = 1, 2, 3, ...\}$ is countable BFS-dense. □

4. Application of BFS Quasi-Coincident in Selection of Project Manager

Decision making techniques have become a popular way to solve many real life problems, which involve uncertainties. Many researchers have been developed a number of algorithms to deal with unpredictable data. In the present section, we solve a problem by applying a modified form of the algorithm given in [46]. We use concept of BFS quasi-coincident because it will ensure that each object has nonzero membership degree with some similar properties, i.e. $\delta_p^+(\xi_i) \neq 0$ and $\delta_p^-(\xi_i) \neq 0$ for $p \in \mathcal{A}_i \cap \mathcal{A}_j \neq \emptyset$. We calculate BFS-AND, BFS-OR operations to consider all suitable possibilities from our bipolar information.

Definition 4.1. In a comparison table rows and columns are equal in numbers and are labeled by members of universal set. The entries are denoted by m_{ij}, where m_{ij} represents the number of decision variables for which membership degree of $m_i \geq m_j$.

Algorithm for selection of project manager
Step 1: Select suitable sets of decision variables.
Step 2: Write BFS-sets for suitable subsets of decision variables.
Step 3: Apply BFS-AND operation on the sets, which are BFS quasi-coincident with each other.
Step 4: Calculate BFS-OR operation for suitable set of choice variables.
Step 5: Evaluate comparison tables of positive and negative information of resultant BFS-set.
Step 6: Calculate positive and negative membership scores by subtracting row sum from column sum of comparison tables.
Step 7: Calculate final scores by subtracting positive scores $\mathbf{P}$ from negative scores $\mathbf{N}$.

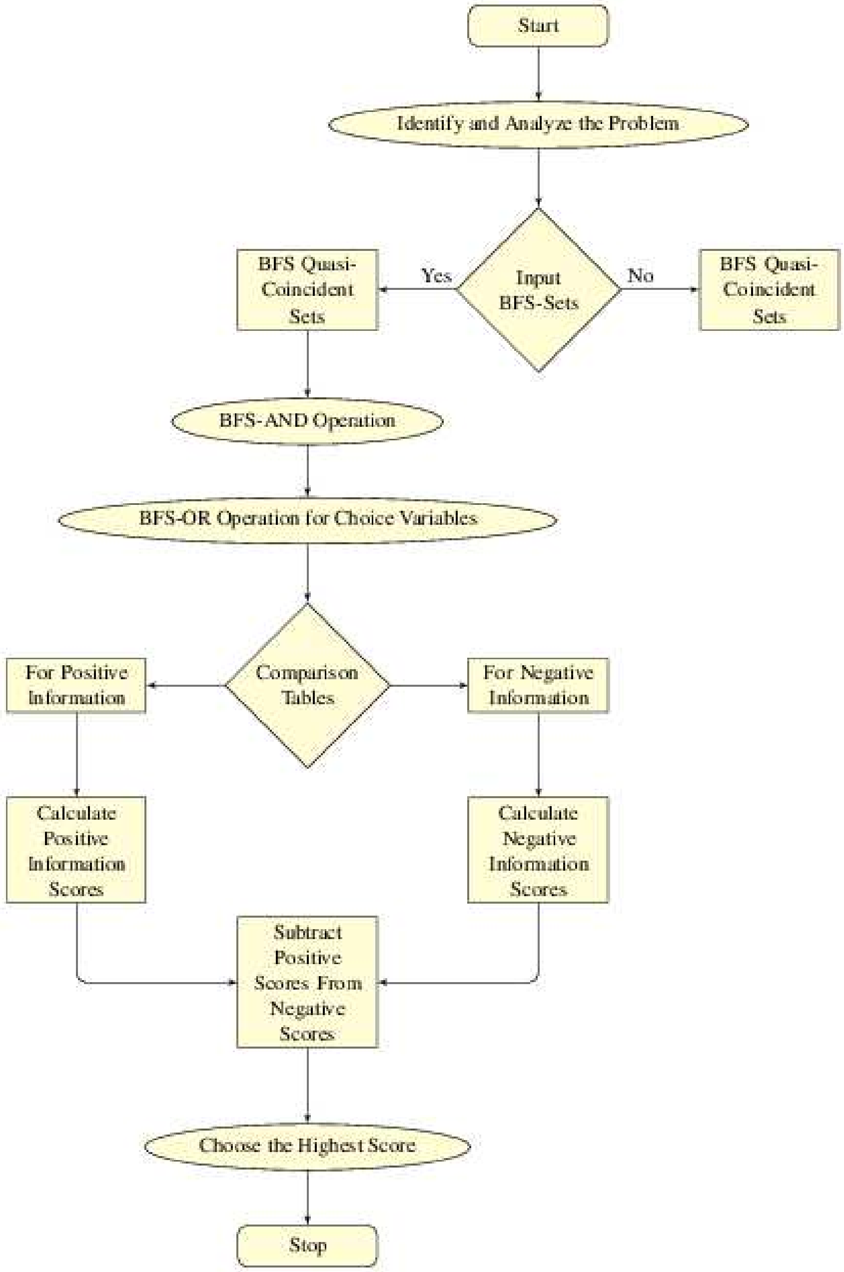

FIGURE 1. Flow Chart of the Algorithm

Step 8: Optimal choice is the maximum score in the column $\mathbf{P}-\mathbf{N}$.
It has become a difficult task to find a good leader but it is more then difficult to find a great project manager. It is not ease to find a reliable, efficient and successful project manager. These kinds of problems involve many ambiguities. The managing of a project is that particular type of leadership position, which requires specific properties and character. Can we find a best project manager who can deliver project within time limit and specific budget set by the company? Here we set an algorithm to deal with such kind of uncertain information.

Example 4.2. Suppose that a company want to choose right candidate for project managing from a list of his employees to assign an important project. A panel of observers short list some most competent candidates on the basis of candidate's previous experience, best performance and good reputation. Let $V=\{\mathfrak{z}_1,\mathfrak{z}_2,\mathfrak{z}_3,\mathfrak{z}_4\}$ be the initial universe, which is the set of short listed candidates. The panel assumes a set of decision variables $D=\{p_1,p_2,p_3,p_4\}$, where
p_1: represents hard working as well as good decision maker.
p_2: represents intelligent and expert in task delegation.
p_3: represents confident and well organized.
p_4: represents passionate and great problem solver.
Let $\mathcal{A}_1=\{p_1,p_2\},\mathcal{A}_2=\{p_1,p_3\}$ and $\mathcal{A}_3=\{p_2,p_4\}\tilde{\subset}\mathcal{D}$. According to these subsets of decision variables the panel constructs some BFS-sets by keeping in view the requirements of the company. The panel assign membership degrees to each set after a careful analysis of each candidate on the basis of screening test and interview. The panel construct each set with respect to chosen subset of parameters.

$$\mathcal{K}_{\mathcal{A}_1}=\left\{\begin{array}{l}\mathcal{K}_{p_1}=\{\ (\xi_1,0.50,-0.40),(\xi_2,0.60,-0.30),(\xi_3,0.70,-0.40),(\xi_4,0.60,-0.20)\ \},\\ \mathcal{K}_{p_2}=\{\ (\xi_1,0.80,-0.20),(\xi_2,0.90,-0.40),(\xi_3,0.70,-0.30),(\xi_4,0.60,-0.40)\ \}\end{array}\right\}$$

$$\mathcal{K}_{\mathcal{A}_2}=\left\{\begin{array}{l}\mathcal{K}_{p_1}=\{\ (\xi_1,0.60,-0.70),(\xi_2,0.60,-0.80),(\xi_3,0.70,-0.70),(\xi_4,0.50,-0.90)\ \},\\ \mathcal{K}_{p_3}=\{\ (\xi_1,0.80,-0.20),(\xi_2,0.90,-0.50),(\xi_3,0.70,-0.30),(\xi_4,0.60,-0.30)\ \}\end{array}\right\}$$

and

$$\mathcal{K}_{\mathcal{A}_3}=\left\{\begin{array}{l}\mathcal{K}_{p_2}=\{\ (\xi_1,0.50,-0.90),(\xi_2,0.20,-0.80),(\xi_3,0.40,-0.90),(\xi_4,0.50,-0.80)\ \},\\ \mathcal{K}_{p_4}=\{\ (\xi_1,0.80,-0.20),(\xi_2,0.70,-0.30),(\xi_3,0.60,-0.30),(\xi_4,0.50,-0.40)\ \}\end{array}\right\}$$

Now we perform BFS-AND operation on the BFS-sets which are BFS quasi-coincident with each other. Since it can be seen that $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$, $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_3}$ are BFS quasi-coincident but $\mathcal{K}_{\mathcal{A}_2}$ and $\mathcal{K}_{\mathcal{A}_3}$ are not BFS quasi-coincident. So, we only need to perform BFS-AND operation between $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_2}$, $\mathcal{K}_{\mathcal{A}_1}$ and $\mathcal{K}_{\mathcal{A}_3}$. The resultant BFS-sets are

$$\mathcal{K}_{\mathcal{A}_1}\widetilde{\wedge}\mathcal{K}_{\mathcal{A}_2}=\left\{\begin{array}{l}\mathcal{K}_{p_{11}}=\{\ (\xi_1,0.50,-0.40),(\xi_2,0.60,-0.30),(\xi_3,0.70,-0.40),(\xi_4,0.50,-0.20)\ \},\\ \mathcal{K}_{p_{13}}=\{\ (\xi_1,0.50,-0.20),(\xi_2,0.60,-0.30),(\xi_3,0.70,-0.30),(\xi_4,0.60,-0.20)\ \},\\ \mathcal{K}_{p_{21}}=\{\ (\xi_1,0.60,-0.20),(\xi_2,0.60,-0.40),(\xi_3,0.70,-0.30),(\xi_4,0.50,-0.40)\ \},\\ \mathcal{K}_{p_{23}}=\{\ (\xi_1,0.80,-0.20),(\xi_2,0.90,-0.40),(\xi_3,0.70,-0.30),(\xi_4,0.60,-0.30)\ \}\end{array}\right\}$$

and

$$\mathcal{K}_{\mathcal{A}_1}\widetilde{\wedge}\mathcal{K}_{\mathcal{A}_3}=\left\{\begin{array}{l}\mathcal{K}_{p_{12}}=\{\ (\xi_1,0.50,-0.40),(\xi_2,0.20,-0.30),(\xi_3,0.40,-0.40),(\xi_4,0.50,-0.20)\ \},\\ \mathcal{K}_{p_{14}}=\{\ (\xi_1,0.50,-0.20),(\xi_2,0.60,-0.30),(\xi_3,0.60,-0.30),(\xi_4,0.50,-0.20)\ \},\\ \mathcal{K}_{p_{22}}=\{\ (\xi_1,0.50,-0.20),(\xi_2,0.20,-0.40),(\xi_3,0.40,-0.30),(\xi_4,0.50,-0.40)\ \},\\ \mathcal{K}_{p_{24}}=\{\ (\xi_1,0.80,-0.20),(\xi_2,0.70,-0.30),(\xi_3,0.60,-0.30),(\xi_4,0.50,-0.40)\ \}\end{array}\right\}$$

Assume that the panel choose the following set of choice variables (CV).
$CV = \{p_{11}\widetilde{\triangledown}p_{12}, p_{13}\widetilde{\triangledown}p_{14}, p_{21}\widetilde{\triangledown}p_{22}, p_{23}\widetilde{\triangledown}p_{22}, p_{23}\widetilde{\triangledown}p_{24}, p_{11}\widetilde{\triangledown}p_{24}\}$. The observers construct a BFS-set $\mathcal{K}_{\mathcal{A}}$ for these choice variables.

$$\mathcal{K}_{\mathcal{A}} = \left\{ \begin{array}{l} \mathcal{K}_{p_{11}\widetilde{\triangledown}p_{12}} = \{ \ (\xi_1, 0.50, -0.40), (\xi_2, 0.60, -0.30), (\xi_3, 0.70, -0.40), (\xi_4, 0.50, -0.20) \ \}, \\ \mathcal{K}_{p_{13}\widetilde{\triangledown}p_{14}} = \{ \ (\xi_1, 0.50, -0.20), (\xi_2, 0.60, -0.30), (\xi_3, 0.70, -0.30), (\xi_4, 0.60, -0.20) \ \}, \\ \mathcal{K}_{p_{21}\widetilde{\triangledown}p_{22}} = \{ \ (\xi_1, 0.60, -0.20), (\xi_2, 0.60, -0.40), (\xi_3, 0.70, -0.30), (\xi_4, 0.50, -0.40) \ \}, \\ \mathcal{K}_{p_{23}\widetilde{\triangledown}p_{22}} = \{ \ (\xi_1, 0.80, -0.20), (\xi_2, 0.90, -0.40), (\xi_3, 0.70, -0.30), (\xi_4, 0.60, -0.40) \ \}, \\ \mathcal{K}_{p_{23}\widetilde{\triangledown}p_{24}} = \{ \ (\xi_1, 0.80, -0.20), (\xi_2, 0.90, -0.40), (\xi_3, 0.70, -0.30), (\xi_4, 0.60, -0.40) \ \}, \\ \mathcal{K}_{p_{11}\widetilde{\triangledown}p_{24}} = \{ \ (\xi_1, 0.80, -0.40), (\xi_2, 0.70, -0.30), (\xi_3, 0.70, -0.40), (\xi_4, 0.50, -0.40) \ \} \end{array} \right\}$$

Comparison table of positive membership degrees of $\mathcal{K}_{\mathcal{A}}$

$m_i \geq m_j$	ξ_1	ξ_2	ξ_3	ξ_4
ξ_1	6	2	3	5
ξ_2	6	6	3	6
ξ_3	3	4	6	6
ξ_4	2	1	0	6

Positive membership degree scores of $\mathcal{K}_{\mathcal{A}}$

	$Sum\ of\ rows(\alpha)$	$Sum\ of\ columns(\beta)$	$\alpha - \beta$
ξ_1	16	17	-1
ξ_2	21	13	8
ξ_3	19	12	7
ξ_4	9	23	-14

Comparison table of negative membership degrees of $\mathcal{K}_{\mathcal{A}}$

$m_i \geq m_j$	ξ_1	ξ_2	ξ_3	ξ_4
ξ_1	6	2	2	3
ξ_2	4	6	4	5
ξ_3	6	3	6	3
ξ_4	5	4	4	6

Negative membership degrees scores of $\mathcal{K}_{\mathcal{A}}$

	$Sum\ of\ rows(\alpha_1)$	$Sum\ of\ columns(\beta_1)$	$\alpha_1 - \beta_1$
ξ_1	13	21	-8
ξ_2	19	15	4
ξ_3	18	16	2
ξ_4	19	17	2

Now we obtain final score by subtracting positive score from negative score

	$Positive(\mathbf{P})$	$Negative(\mathbf{N})$	$(\mathbf{P} - \mathbf{N})$
ξ_1	-1	-8	7
ξ_2	8	4	4
ξ_3	7	2	5
ξ_4	-14	2	-16

Since we choose that candidate which attain maximum value. So, ξ_1 is the best choice because it achieved maximum score.

5. CONCLUSION

In this paper, we discussed properties of BFS-topology by using BFS quasi-coincident and BFS Q-neighborhood. We defined BFS quasi-coincident and BFS Q-neighborhood for BFS-set and proved some certain properties of BFS quasi-coincident and BFS Q-neighborhood. These properties are very useful throughout this work. We studied important notions of BFS-topology including, BFS-adherence point, BFS-accumulation point, BFS Q-countability axioms and BFS Q-separable space by using BFS Q-neighborhood. We distinguished between BFS-denseness and BFS Q-denseness of a subset. Both these terms do not imply each other, the counter examples are given for both terms. We also discussed the concept of BFS-Lindel$\ddot{o}$fspace. Finally, we present an application of BFS-quasi-coincident in decision-making of project management by using BFS-AND, BFS-OR operations. The proposed algorithm is perfect, when we have more then one BFS-sets. The flow chart of the algorithm has been also presented.

6. ACKNOWLEDGMENTS

The authors are highly thankful to the Editor-in-chief and the referees for their valuable comments and suggestions for improving the quality of our paper.

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