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On Upper and Lower Contra-Continuous Fuzzy Multifunctions

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Abstract. This paper is devoted to the concepts of fuzzy upper and fuzzy lower contra-continuous, contra-irresolute and contra semi-continuous multifunctions. Several characterizations and properties of these multifunctions along with their mutual relationships are established in L-fuzzy topological spaces. Later, composition and union between these multifunctions have been studied.

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1. Introduction and preliminaries

Kubiak [17] and Sostak [28] introduced the notion of (L-)fuzzy topological space as a generalization of L-topological spaces (originally called (L-)fuzzy topological spaces by Chang [8] and Goguen [10]). It is the grade of openness of an L-fuzzy set. A general approach to the study of topological type structures on fuzzy powersets was developed in [11-13,17,18,28-30].

Berge [7] introduced the concept multimapping $F : X \to Y$ where X and Y are topological spaces and Popa [24,25] introduced the notion of irresolute multimapping. After Chang introduced the concept of fuzzy topology [8], continuity of multifunctions in fuzzy topological spaces have been defined and studied by many authors from different view points (e.g. see [3,4,21-23]). Tsiporkova et. al., [31,32] introduced the Continuity of fuzzy multivalued mappings in the Chang's fuzzy topology [8]. Later, Abbas et al., [1] introduced the concepts of fuzzy upper and fuzzy lower semi-continuous multifunctions in L-fuzzy topological spaces.

Throughout this paper, nonempty sets will be denoted by X, Y etc.. Let a complete lattice L = (L,≤,∨,∧,′) be a complete distributive complete lattice with an order-reversing involution on it, and with a smallest element $\perp$ and largest element $\top$ ($\perp \neq \top$). The family of all L-fuzzy sets in X is denoted by L^X and $L_\circ = L - \{0\}$. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. The complement of an L-fuzzy set λ is denoted by λ^c. This symbol $\multimap$ for a multifunction. All other notations are standard notations of L-fuzzy set theory.

Definition 1. 1. [1] Let $F : X \multimap Y$, then F is called a fuzzy multifunction (FM, for short) iff $F(x) \in L^Y$ for each $x \in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y) = G_F(x, y)$ for any $(x, y) \in X \times Y$.

The domain of F, denoted by $dom(F)$ and the range of F, denoted by $rng(F)$, for any $x \in X$ and $y \in Y$, are defined by:

$$dom(F)(x) = \bigvee_{y \in Y} G_F(x, y) \qquad and \qquad rng(F)(y) = \bigvee_{x \in X} G_F(x, y).$$

Definition 1. 2. [1] Let $F : X \multimap Y$ be a FM. Then F is called:
(1) Normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_F(x, y_0) = \top$.
(2) A crisp iff $G_F(x, y) = \top$ for each $x \in X$ and $y \in Y$.

Definition 1. 3. [1] Let $F : X \multimap Y$ be a FM. Then,
(1) The image of $\lambda \in L^X$ is an L-fuzzy set $F(\lambda) \in L^Y$ defined by:

$$F(\lambda)(y) = \bigvee_{x \in X} [G_F(x, y) \wedge \lambda(x)].$$

(2) The lower inverse of $\mu \in L^Y$ is an L-fuzzy set $F^l(\mu) \in L^X$ defined by:

$$F^l(\mu)(x) = \bigvee_{y \in Y} [G_F(x, y) \wedge \mu(y)].$$

(3) The upper inverse of $\mu \in L^Y$ is an L-fuzzy set $F^u(\mu) \in L^X$ defined by:

$$F^u(\mu)(x) = \bigwedge_{y \in Y} [G^c_F(x, y) \vee \mu(y)].$$

Theorem 1. 4. [1] Let $F : X \multimap Y$ be a FM. Then,
(1) $F(\lambda_1) \leq F(\lambda_2)$ if $\lambda_1 \leq \lambda_2$.
(2) $F^l(\mu_1) \leq F^l(\mu_2)$ and $F^u(\mu_1) \leq F^u(\mu_2)$ if $\mu_1 \leq \mu_2$.
(3) $F^l(\mu^c) = (F^u(\mu))^c$.
(4) $F^u(\mu^c) = (F^l(\mu))^c$.
(5) $F(F^u(\mu)) \leq \mu$ if F is a crisp.
(6) $F^u(F(\lambda)) \geq \lambda$ if F is a crisp.

Definition 1. 5. [1] Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's. Then the composition $H \circ F$ is defined by: $((H \circ F)(x))(z) = \bigvee_{y \in Y} [G_F(x, y) \wedge G_H(y, z)]$.

Theorem 1. 6. [1] Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's. Then we have the following:
(1) $(H \circ F) = F(H)$.
(2) $(H \circ F)^u = F^u(H^u)$.
(3) $(H \circ F)^l = F^l(H^l)$.

Theorem 1. 7. [1] Let $F_i : X \multimap Y$ be a FM. Then,
(1) $(\bigcup_{i\in\Gamma} F_i)(\lambda) = \bigvee_{i\in\Gamma} F_i(\lambda)$.
(2) $(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma} F_i^l(\mu)$.
(3) $(\bigcup_{i\in\Gamma} F_i)^u(\mu) = \bigwedge_{i\in\Gamma} F_i^u(\mu)$.

Definition 1. 8. [13,17,20,28] An L-fuzzy topological space (L-fts, in short) is a pair (X, τ), where X is a nonempty set and $\tau : L^X \to L$ is a mapping satisfying the following properties:
(O1) $\tau(\underline{\top}) = \tau(\underline{\perp}) = \top$,
(O2) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$, for any $\lambda_1, \lambda_2 \in L^X$,
(O3) $\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$, for any $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.
Then τ is called an L-fuzzy topology on X. For every $\lambda \in L^X$, $\tau(\lambda)$ is called the degree of openness of the L-fuzzy set λ.

A mapping $f : (X, \tau) \to (Y, \eta)$ is said to be continuous with respect to L-fuzzy topologies τ and η iff $\tau(f^{-1}(\mu)) \geq \eta(\mu)$ for each $\mu \in L^Y$.

Theorem 1. 9. [9,14,16,20] Let (X, τ) be an L-fts. Then for each $\lambda \in L^X$, $r \in L_\circ$ we define L-fuzzy operators C_τ and $I_\tau : L^X \times L_\circ \to L^X$ as follows:

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in L^X \ : \lambda \leq \mu, \tau(\mu^c) \geq r\}.$$

$$I_\tau(\lambda, r) = \bigvee\{\mu \in L^X \ : \mu \leq \lambda, \tau(\mu) \geq r\}.$$

For λ, $\mu \in L^X$ and $r, s \in L_\circ$ the operator C_τ satisfies the following statements:
(C1) $C_\tau(\underline{\perp}, r) = \underline{\perp}$.
(C2) $\lambda \leq C_\tau(\lambda, r)$.
(C3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.
(C4) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.
(C5) $C_\tau(\lambda, r) = \lambda$ iff $\tau(\lambda^c) \geq r$.
(C6) $C_\tau(\lambda^c, r) = (I_\tau(\lambda, r))^c$ and $I_\tau(\lambda^c, r) = (C_\tau(\lambda, r))^c$.

Definition 1. 10. [6,14,27] Let (X, τ) be an L-fts. Then for each $\lambda, \mu \in L^X$ and $r \in L_\circ$. Then λ is called:
(1) r-fuzzy semi-open (r-fso, in short) iff $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$.
(2) r-fuzzy semi-closed (r-fsc, in short) iff $I_\tau(C_\tau(\lambda, r), r) \leq \lambda$.

Theorem 1. 11. [14] Let (X, τ) be an L-fts. Then for each $\lambda \in L^X$, $r \in L_\circ$ we define L-fuzzy operators SC_τ and $SI_\tau : L^X \times L_\circ \to L^X$ as follows:

$$SC_\tau(\lambda, r) = \bigwedge\{\mu \in L^X \ : \lambda \leq \mu, \ \mu \ is \ r - fsc\}.$$

$$SI_\tau(\lambda, r) = \bigvee\{\mu \in L^X \ : \mu \leq \lambda, \ \mu \ is \ r - fso\}.$$

Theorem 1. 12. [1] Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η) and $\mu \in L^Y$. Then we have the following:
(1) F is FLS-continuous iff $\tau(F^l(\mu)) \geq \eta(\mu)$.
(2) If F is normalized, then F is FUS-continuous iff $\tau(F^u(\mu)) \geq \eta(\mu)$.
(3) F is FLS-continuous iff $\tau((F^u(\mu))^c) \geq \eta(\mu^c)$.

(4) If F is normalized, then F is FUS-continuous iff $\tau((F^l(\mu))^c) \geq \eta(\mu^c)$.

Definition 1. 13. [2] Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η) and $r \in L_\circ$. Then F is called:

(1) FUW-continuous (resp. FLW-continuous) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ (resp. $x_t \in F^l(\mu)$) for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(C_\eta(\mu, r))$ (resp. $\lambda \leq F^l(C_\eta(\mu, r))$).

(2) FUW-continuous (resp. FLW-continuous) iff it is FUW-continuous (resp. FLW-continuous) at every $x_t \in dom(F)$.

Proposition 1. 14. [2] F is normalized implies F is FUW-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_\eta(\mu, r))$.

2. Fuzzy upper and lower contra-continuous multifunctions

Definition 2. 1. Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η) and $r \in L_\circ$. Then F is called:

(1) Fuzzy upper contra-continuous (FUC-continuous, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu^c) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(\mu)$.

(2) Fuzzy lower contra-continuous (FLC-continuous, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu^c) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.

(3) FUC-continuous (resp. FLC-continuous) iff it is FUC-continuous (resp. FLC-continuous) at every $x_t \in dom(F)$.

Proposition 2. 2. F is normalized implies F is FUC-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu^c) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$.

Remark 2. 3. The notions of FUC-continuous multifunctions and FUS-continu ous multifunctions are independent as shown in the following Examples 2.6 and 2.7.

Theorem 2. 4. Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η) and $\mu \in L^Y$, then the following are equivalent:

(1) F is FLC-continuous.

(2) $\tau(F^l(\mu)) \geq r$, if $\eta(\mu^c) \geq r$.

(3) $\tau((F^u(\mu))^c) \geq r$, if $\eta(\mu) \geq r$.

Proof. (1) $\Rightarrow$ (2) Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu^c) \geq r$ and $x_t \in F^l(\mu)$ then, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$ and hence $x_t \in I_\tau(F^l(\mu), r)$. Therefore, we obtain $F^l(\mu) \leq I_\tau(F^l(\mu), r)$. Thus $\tau(F^l(\mu)) \geq r$.

(2) $\Rightarrow$ (3) Let $\mu \in L^Y$ and $\eta(\mu) \geq r$ hence by (2),

$$\tau(F^l(\mu^c)) = \tau((F^u(\mu))^c) \geq r.$$

(3) $\Rightarrow$ (2) It is similar to that of (2) $\Rightarrow$ (3).

(2) $\Rightarrow$ (1) Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu^c) \geq r$ with $x_t \in F^l(\mu)$ we have by (2), $\tau(F^l(\mu)) \geq r$. Let $F^l(\mu) = \lambda$ (say) then, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$. Thus F is FLC-continuous.

We state the following result without proof in view of above theorem.

Theorem 2. 5. Let $F : X \multimap Y$ be a FM and normalized between two L-fts's (X, τ), (Y, η) and $\mu \in L^Y$, then the following are equivalent:
(1) F is FUC-continuous.
(2) $\tau(F^u(\mu)) \geq r$, if $\eta(\mu^c) \geq r$.
(3) $\tau((F^l(\mu))^c) \geq r$, if $\eta(\mu) \geq r$.

Example 2. 6. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = \top$, $G_F(x_1, y_3) = \bot$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = \bot$ and $G_F(x_2, y_3) = \top$. We assume that $\top = 1$ and $\bot = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} \top, & \text{if} \quad \lambda \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{\underline{0.5}, \underline{0.6}\}, \\ \bot, & \text{otherwise}, \end{cases}$$

$$\eta(\mu) = \begin{cases} \top, & \text{if} \quad \mu \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if} \quad \mu = \underline{0.5}, \\ \frac{1}{3}, & \text{if} \quad \mu = \underline{0.4}, \\ \bot, & \text{otherwise}. \end{cases}$$

(1) F is FUC-continuous but not FUS-continuous because $\eta(\underline{0.4}) = \frac{1}{3}$ in (Y, η), $F^u(\underline{0.4}) = \underline{0.4}$ and $\tau(F^u(\underline{0.4})) = \bot$. Hence, $\tau(F^u(\underline{0.4})) \ngeq \eta(\underline{0.4})$.
(2) F is FLC-continuous but not FLS-continuous because $\eta(\underline{0.4}) = \frac{1}{3}$ in (Y, η), $F^l(\underline{0.4}) = \underline{0.4}$ and $\tau(F^l(\underline{0.4})) = \bot$. Hence, $\tau(F^l(\underline{0.4})) \ngeq \eta(\underline{0.4})$.

Example 2. 7. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1, y_1) = 0.1$, $G_F(x_1, y_2) = \top$, $G_F(x_1, y_3) = \bot$, $G_F(x_2, y_1) = 0.5$, $G_F(x_2, y_2) = \bot$ and $G_F(x_2, y_3) = \top$. We assume that $\top = 1$ and $\bot = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} \top, & \text{if} \quad \lambda \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{\underline{0.4}, \underline{0.5}\}, \\ \bot, & \text{otherwise}, \end{cases}$$

$$\eta(\mu) = \begin{cases} \top, & \text{if} \quad \mu \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if} \quad \mu = \underline{0.5}, \\ \frac{1}{3}, & \text{if} \quad \mu = \underline{0.4}, \\ \bot, & \text{otherwise}. \end{cases}$$

(1) F is FUS-continuous but not FUC-continuous because $\eta(\underline{0.4}) = \frac{1}{3}$ in (Y, η), $F^l(\underline{0.4}) = \underline{0.4}$ and $\tau((F^l(\underline{0.4}))^c) = \bot$. Thus, $\tau((F^l(\underline{0.4}))^c) \ngeq \frac{1}{3}$.
(2) F is FLS-continuous but not FLC-continuous because $\eta(\underline{0.4}) = \frac{1}{3}$ in (Y, η), $F^u(\underline{0.4}) = \underline{0.4}$ and $\tau((F^u(\underline{0.4}))^c) = \bot$. Thus, $\tau((F^u(\underline{0.4}))^c) \ngeq \frac{1}{3}$.

Definition 2. 8. Let (X,τ) be an L-fts. Then for each $\lambda\in L^X$ and $r\in L_\circ$ we define L-fuzzy operator $Ker_\tau : L^X\times L_\circ\to L^X$ as follows:

$$Ker_\tau(\lambda,r)=\bigwedge\{\mu\in L^X\ :\lambda\le\mu,\ \tau(\mu)\ge r\}.$$

Lemma 2. 9. For λ in an L-fts (X,τ), if $\tau(\lambda)\ge r$ then $\lambda=Ker_\tau(\lambda,r)$.

Theorem 2. 10. Let $F:X\multimap Y$ be a FM between two L-fts's (X,τ) and (Y,η). If $C_\tau(F^u(\mu),r)\le F^u(Ker_\eta(\mu,r))$ for any $\mu\in L^Y$, then F is FLC-continuous.

Proof. Suppose that $C_\tau(F^u(\mu),r)\le F^u(Ker_\eta(\mu,r))$ for any $\mu\in L^Y$. Let $\nu\in L^Y$ and $\eta(\nu)\ge r$ by Lemma 2.9, we have $C_\tau(F^u(\nu),r)\le F^u(Ker_\eta(\nu,r))=F^u(\nu)$. This implies that $C_\tau(F^u(\nu),r)=F^u(\nu)$ and hence $\tau((F^u(\nu))^c)\ge r$. Thus, by Theorem 2.4(3), F is FLC-continuous.

Theorem 2. 11. Let $F:X\multimap Y$ be a FM and normalized between two L-fts's (X,τ) and (Y,η). If $C_\tau(F^l(\mu),r)\le F^l(Ker_\eta(\mu,r))$ for any $\mu\in L^Y$, then F is FUC-continuous.

Proof. Suppose that $C_\tau(F^l(\mu),r)\le F^l(Ker_\eta(\mu,r))$ for any $\mu\in L^Y$. Let $\nu\in L^Y$ and $\eta(\nu)\ge r$ by Lemma 2.9, we have $C_\tau(F^l(\nu),r)\le F^l(Ker_\eta(\nu,r))=F^l(\nu)$. This implies that $C_\tau(F^l(\nu),r)=F^l(\nu)$ and hence $\tau((F^l(\nu))^c)\ge r$. Thus, by Theorem 2.5(3), F is FUC-continuous.

Theorem 2. 12. Let $\{F_i\}_{i\in\Gamma}$ be a family of FLC-continuous between two L-fts's (X,τ) and (Y,η). Then $\bigcup_{i\in\Gamma}F_i$ is FLC-continuous.

Proof. Let $\mu\in L^Y$ and $\eta(\mu^c)\ge r$ then $(\bigcup_{i\in\Gamma}F_i)^l(\mu)=\bigvee_{i\in\Gamma}(F_i^l(\mu))$ by Theorem 1.7(2). Since $\{F_i\}_{i\in\Gamma}$ is a family of FLC-continuous between two L-fts's (X,τ) and (Y,η), then $\tau(F_i^l(\mu))\ge r$ for each $i\in\Gamma$. Then for each $\mu\in L^Y$ and $\eta(\mu^c)\ge r$, we have $\tau((\bigcup_{i\in\Gamma}F_i)^l(\mu))=\tau(\bigvee_{i\in\Gamma}(F_i^l(\mu))\ge\bigwedge_{i\in\Gamma}\tau(F_i^l(\mu))\ge r$. Hence $\bigcup_{i\in\Gamma}F_i$ is FLC-continuous.

Theorem 2. 13. Let F_1 and F_2 be two normalized FUC-continuous between two L-fts's (X,τ) and (Y,η). Then $F_1\bigcup F_2$ is FUC-continuous.

Proof. Let $\mu\in L^Y$ and $\eta(\mu^c)\ge r$ then $(F_1\bigcup F_2)^u(\mu)=F_1^u(\mu)\wedge F_2^u(\mu)$ by Theorem 1.7(3). Since F_1 and F_2 be two normalized FUC-continuous between two L-fts's (X,τ) and (Y,η), then $\tau(F_i^u(\mu))\ge r$ for each $i\in\{1,2\}$. Then for each $\mu\in L^Y$ and $\eta(\mu^c)\ge r$, we have $\tau((F_1\bigcup F_2)^u(\mu))=\tau(F_1^u(\mu)\wedge F_2^u(\mu))\ge\tau(F_1^u(\mu))\wedge\tau(F_2^u(\mu))\ge r$. Hence $F_1\bigcup F_2$ is FUC-continuous.

Theorem 2. 14. Let $F:X\multimap Y$ and $H:Y\multimap Z$ be two $FM's$ and let (X,τ), (Y,η) and (Z,δ) be three L-fts's. If F is FLS-continuous and H is FLC-continuous, then $H\circ F$ is FLC-continuous.

Proof. Let F be FLS-continuous, H be FLC-continuous and $\gamma \in L^Z$, $\delta(\gamma^c) \geq r$. Then from Theorem 1.12(1) and Theorem 2.4(2), we have $(H \circ F)^l(\gamma) = F^l(H^l(\gamma))$ and $\tau(F^l(H^l(\gamma))) \geq \eta(H^l(\gamma)) \geq r$. Thus $H \circ F$ is FLC-continuous.

We state the following result without proof in view of above theorem.

Theorem 2. 15. Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let (X, τ), (Y, η) and (Z, δ) be three L-fts's. If F and H are normalized, F is FUS-continuous and H is FUC-continuous, then $H \circ F$ is FUC-continuous.

Theorem 2. 16. Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let (X, τ), (Y, η) and (Z, δ) be three L-fts's. If H is normalized, H is FUS-continuous and F is FLC-continuous, then $H \circ F$ is FLC-continuous.

Proof. Let F be FLC-continuous, H be FUS-continuous and $\gamma \in L^Z$, $\delta(\gamma) \geq r$. Then from Theorem 1.12(2) and Theorem 2.4(3), we have $(H \circ F)^u(\gamma) = F^u(H^u(\gamma))$ and $\tau([F^u(H^u(\gamma))]^c) \geq r$ with $\eta(H^u(\gamma)) \geq r$. Thus $H \circ F$ is FLC-continuous.

We state the following result without proof in view of above theorem.

Theorem 2. 17. Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let (X, τ), (Y, η) and (Z, δ) be three L-fts's. If F is normalized, F is FUC-continuous and H is FLS-continuous, then $H \circ F$ is FUC-continuous.

Definition 2. 18. [5,15,19,26] An L-fuzzy set λ in an L-fts (X, τ) is called r-fuzzy compact iff every family in $\{\mu : \ \tau(\mu) > r, \ \mu \in L^X\}$, where $r \in L_\circ$ covering λ has a finite subcover.

Definition 2. 19. An L-fuzzy set λ in an L-fts (X, τ) is called r-fuzzy strongly S-closed iff every family in $\{\mu : \ \tau(\mu^c) > r, \ \mu \in L^X\}$, where $r \in L_\circ$ covering λ has a finite subcover.

Theorem 2. 20. Let $F : X \multimap Y$ be a crisp FUC-continuous between two L-fts's (X, τ) and (Y, η). Suppose that $F(x_t)$ is r-fuzzy strongly S-closed for each $x_t \in dom(F)$. If an L-fuzzy set λ in an L-fts (X, τ) is r-fuzzy compact, then $F(\lambda)$ is r-fuzzy strongly S-closed.

Proof. Let λ be r-fuzzy compact set in X and $\{\gamma_i : \ \eta(\gamma_i^c) \geq r, \ i \in \Gamma\}$ be a family covering of $F(\lambda)$ i.e., $F(\lambda) \leq \bigvee_{i \in \Gamma} \gamma_i$. Since $\lambda = \bigvee_{x_t \in \lambda} x_t$, we have

$$F(\lambda) = F(\bigvee_{x_t \in \lambda} x_t) = \bigvee_{x_t \in \lambda} F(x_t) \leq \bigvee_{i \in \Gamma} \gamma_i.$$

It follows that for each $x_t \in \lambda$, $F(x_t) \leq \bigvee_{i \in \Gamma} \gamma_i$. Since $F(x_t)$ is r-fuzzy strongly S-closed for each $x_t \in dom(F)$, then there exists finite subset Γ_{x_t} of Γ such that $F(x_t) \leq \bigvee_{n \in \Gamma_{x_t}} \gamma_n = \gamma_{x_t}$. By Theorem 1.4(6), we have $x_t \leq F^u(F(x_t)) \leq F^u(\gamma_{x_t})$ and

$$\lambda = \bigvee_{x_t \in \lambda} x_t \leq \bigvee_{x_t \in \lambda} F^u(\gamma_{x_t}).$$

From Theorem 2.5(2), we have $\tau(F^u(\gamma_{x_t})) \geq r$. Hence $\{F^u(\gamma_{x_t}) : \tau(F^u(\gamma_{x_t})) \geq r,\ x_t \in \lambda\}$ is a family covering the set λ. Since λ is compact, then there exists finite index set N such that $\lambda \leq \bigvee_{n\in N} F^u(\gamma_{x_{t_n}})$. From Theorem 1.4(5), we have

$$F(\lambda) \leq F(\bigvee_{n\in N} F^u(\gamma_{x_{t_n}})) = \bigvee_{n\in N} F(F^u(\gamma_{x_{t_n}})) \leq \bigvee_{n\in N} \gamma_{x_{t_n}}.$$

Then, $F(\lambda)$ is r-fuzzy strongly S-closed.

Theorem 2. 21. Let $F : X \multimap Y$ be a FM between two L-fts's (X,τ), (Y,η). If F is FLC-continuous then, F is FLW-continuous.

Proof. Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Since F is FLC-continuous, $\eta([C_\eta(\mu,r)]^c) \geq r$ and $x_t \in F^l(C_\eta(\mu,r)$ then, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_\eta(\mu,r))$. Hence FLW-continuous.

We state the following result without proof in view of above theorem.

Theorem 2. 22. Let $F : X \multimap Y$ be a FM and normalized between two L-fts's (X,τ), (Y,η). If F is FUC-continuous then, F is FUW-continuous.

Remark 2. 23. [4,33] Let (X,τ) and (Y,η) be an L-fts's . An L-fuzzy sets of the form $\lambda \times \mu$ with $\tau(\lambda) \geq r$ and $\eta(\mu) \geq r$ form a basis for the product L-fuzzy topology $\tau \times \eta$ on $X \times Y$, where for any $(x,y) \in X \times Y$, $(\lambda \times \mu)(x,y) = min\{\lambda(x), \mu(y)\}$.

Theorem 2. 24. Let (X,τ) and (X_i,τ_i) be L-fts's $(i \in I)$. If a FM $F : X \multimap \prod_{i\in I} X_i$ is FLC-continuous (where $\prod_{i\in I} X_i$ is the product space), then $P_i \circ F$ is FLC-continuous for each $i \in I$, where $P_i : \prod_{i\in I} X_i \multimap X_i$ is the projection multifunction which is defined by $P_k((x_i)) = \{x_i\}$ for each $k \in I$.

Proof. Let $\mu_{i_0} \in L^{X_{i_0}}$ and $\tau_{i_0}(\mu^c_{i_0}) \geq r$. Then $(P_{i_0} \circ F)^l(\mu_{i_0}) = F^l(P^l_{i_0}(\mu_{i_0})) = F^l(\mu_{i_0} \times \prod_{i\neq i_0} X_i)$. Since F is FLC-continuous and $\tau_i((\mu_{i_0} \times \prod_{i\neq i_0} X_i)^c) \geq r$, it follows that $\tau(F^l(\mu_{i_0} \times \prod_{i\neq i_0} X_i)) \geq r$. Then $P_i \circ F$ is an FLC-continuous.

We state the following result without proof in view of above theorem.

Theorem 2. 25. Let (X,τ) and (X_i,τ_i) be L-fts's $(i \in I)$. If a FM $F : X \multimap \prod_{i\in I} X_i$ is FUC-continuous (where $\prod_{i\in I} X_i$ is the product space), then $P_i \circ F$ is FUC-continuous for each $i \in I$, where $P_i : \prod_{i\in I} X_i \multimap X_i$ is the projection multifunction which is defined by $P_k((x_i)) = \{x_i\}$ for each $k \in I$.

Theorem 2. 26. Let (X_i,τ_i) and (Y_i,η_i) be L-fts's and $F_i : X_i \multimap Y_i$ be a FM for each $i \in I$. Suppose that $F : \prod_{i\in I} X_i \multimap \prod_{i\in I} Y_i$ is defined by $F((x_i)) = \prod_{i\in I} F_i(x_i)$. If F is FLC-continuous, then F_i is FLC-continuous for each $i \in I$.

Proof. Let $\mu_i \in L^{Y_i}$ and $\eta_i(\mu^c_i) \geq r$. Then $\eta_i((\mu_i \times \prod_{i\neq j} Y_j)^c) \geq r$. Since F is FLC-continuous, it follows that $\tau_i(F^l(\mu_i \times \prod_{i\neq j} Y_j)) \geq r$ and $F^l(\mu_i \times \prod_{i\neq j} Y_j) = F^l(\mu_i) \times \prod_{i\neq j} X_j$. Consequently, we obtain that $\tau_i(F^l(\mu_i)) \geq r$ for each $i \in I$.
Thus, F_i is FLC-continuous.

We state the following result without proof in view of above theorem.

Theorem 2. 27. Let (X_i, τ_i) and (Y_i, η_i) be L-fts's and $F_i : X_i \multimap Y_i$ be a FM for each $i \in I$. Suppose that $F : \prod_{i \in I} X_i \multimap \prod_{i \in I} Y_i$ is defined by $F((x_i)) = \prod_{i \in I} F_i(x_i)$. If F is FUC-continuous, then F_i is FUC-continuous for each $i \in I$.

3. Fuzzy upper and lower contra semi-continuous multifunctions

Definition 3. 1. Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η) and $r \in L_\circ$. Then F is called:

(1) Fuzzy upper contra semi-continuous ($FUCS$-continuous, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu^c) \geq r$ there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(\mu)$.

(2) Fuzzy lower contra semi-continuous ($FLCS$-continuous, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu^c) \geq r$ there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.

(3) $FUCS$-continuous (resp. $FLCS$-continuous) iff it is $FUCS$-continuous (resp. $FLCS$-continuous) at every $x_t \in dom(F)$.

Definition 3. 2. Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η) and $r \in L_\circ$. Then F is called:

(1) Fuzzy upper contra-irresolute (FUC-irresolute, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ is r-fsc there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(\mu)$.

(2) Fuzzy lower contra-irresolute (FLC-irresolute, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ is r-fsc there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.

(3) FUC-irresolute (resp. FLC-irresolute) iff it is FUC-irresolute (resp. FLC-irresolute) at every $x_t \in dom(F)$.

Proposition 3. 3. F is normalized implies F is $FUCS$-continuous (resp. FUC-irresolute) at $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu^c) \geq r$ (resp. μ is r-fsc) there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$.

Remark 3. 4. The notions of FUC-continuous multifunctions and FUC-irresolute multifunctions are independent as shown in the following Examples 3.9 and 3.10.

The following implications hold:
1. FUC-continuous $\Rightarrow$ $FUCS$-continuous $\Leftarrow$ FUC-irresolute.
2. FLC-continuous $\Rightarrow$ $FLCS$-continuous $\Leftarrow$ FLC-irresolute.

In general the converses are not true.

Theorem 3. 5. Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η) and $\mu \in L^Y$, then the following are equivalent:

(1) F is $FLCS$-continuous.

(2) $F^l(\mu)$ is r-fso, if $\eta(\mu^c) \geq r$.

(3) $F^u(\mu)$ is r-fsc, if $\eta(\mu) \geq r$.

Proof. (1) ⇒ (2) Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu^c) \geq r$ and $x_t \in F^l(\mu)$ then, there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$ and hence $x_t \in SI_\tau(F^l(\mu), r)$. Therefore, we obtain $F^l(\mu) \leq SI_\tau(F^l(\mu), r)$. Thus, $F^l(\mu)$ is r-fso.

(2) ⇒ (3) Let $\mu \in L^Y$ and $\eta(\mu) \geq r$ hence by (1), $F^l(\mu^c) = (F^u(\mu))^c$ is r-fso. Then, $F^u(\mu)$ is r-fsc.

(3) ⇒ (2) It is similar to that of (2) ⇒ (3).

(2) ⇒ (1) Let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu^c) \geq r$ with $x_t \in F^l(\mu)$ we have by (2), $F^l(\mu) = \lambda$ (say) is r-fso then, there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$. Thus, F is $FLCS$-continuous.

Theorem 3. 6. Let $F : X \multimap Y$ be a FM between two L-fts's (X, τ), (Y, η) and $\mu \in L^Y$, then the following are equivalent:

(1) F is FLC-irresolute.

(2) $F^l(\mu)$ is r-fso, for any μ is r-fsc.

(3) $F^u(\mu)$ is r-fsc, for any μ is r-fso.

Proof. (1) ⇒ (2) Let $x_t \in dom(F)$, $\mu \in L^Y$, μ be r-fsc and $x_t \in F^l(\mu)$ then, there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$ and hence $x_t \in SI_\tau(F^l(\mu), r)$. Therefore, we obtain $F^l(\mu) \leq SI_\tau(F^l(\mu), r)$. Thus, $F^l(\mu)$ is r-fso.

(2) ⇒ (3) Let $\mu \in L^Y$ and μ be r-fso hence by (1), $F^l(\mu^c) = (F^u(\mu))^c$ is r-fso. Then, $F^u(\mu)$ is r-fsc.

(3) ⇒ (2) It is similar to that of (2) ⇒ (3).

(2) ⇒ (1) Let $x_t \in dom(F)$, $\mu \in L^Y$, μ be r-fsc with $x_t \in F^l(\mu)$ we have by (2), $F^l(\mu) = \lambda$ (say) is r-fso then, there exists r-fso set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$. Thus, F is FLC-irresolute.

We state the following results without proof in view of above theorems.

Theorem 3. 7. Let $F : X \multimap Y$ be a FM and normalized between two L-fts's (X, τ), (Y, η) and $\mu \in L^Y$, then the following are equivalent:

(1) F is $FUCS$-continuous.

(2) $F^u(\mu)$ is r-fso, if $\eta(\mu^c) \geq r$.

(3) $F^l(\mu)$ is r-fsc, if $\eta(\mu) \geq r$.

Theorem 3. 8. Let $F : X \multimap Y$ be a FM and normalized between two L-fts's (X, τ), (Y, η) and $\mu \in L^Y$, then the following are equivalent:

(1) F is FUC-irresolute.

(2) $F^u(\mu)$ is r-fso, for any μ is r-fsc.

(3) $F^l(\mu)$ is r-fsc, for any μ is r-fso.

Example 3. 9. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1, y_1) = 0.1,\ G_F(x_1, y_2) = \top,\ G_F(x_1, y_3) = \bot,\ G_F(x_2, y_1) = 0.5,\ G_F(x_2, y_2) = \bot$ and $G_F(x_2, y_3) = \top$. We assume that $\top = 1$ and $\bot = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} \top, & \text{if} \quad \lambda \in \{\underline{\bot}, \underline{\top}\}, \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{\underline{0.5}, \underline{0.6}\}, \\ \bot, & \text{otherwise}, \end{cases}$$

$$\eta(\mu)=\begin{cases}\top, & \text{if} \quad \mu\in\{\underline{\perp},\underline{\top}\},\\ \frac{1}{2}, & \text{if} \quad \mu=\underline{0.5},\\ \frac{1}{3}, & \text{if} \quad \mu=\underline{0.4},\\ \perp, & \text{otherwise.}\end{cases}$$

(1) F is $FUCS$-continuous (resp. FUC-continuous) but not FUC-irresolute because $\underline{0.45}$ is $\frac{1}{2}$-fso in (Y,η) and $F^l(\underline{0.45})=\underline{0.45}$ is not $\frac{1}{2}$-fsc.

(2) F is $FLCS$-continuous (resp. FLC-continuous) but not FLC-irresolute because $\underline{0.45}$ is $\frac{1}{2}$-fso in (Y,η) and $F^u(\underline{0.45})=\underline{0.45}$ is not $\frac{1}{2}$-fsc.

Example 3. 10. Let $X=\{x_1,x_2\}$, $Y=\{y_1,y_2,y_3\}$ and $F:X\multimap Y$ be a FM defined by $G_F(x_1,y_1)=0.2,\ G_F(x_1,y_2)=\top,\ G_F(x_1,y_3)=0.3,\ G_F(x_2,y_1)=0.5,\ G_F(x_2,y_2)=0.3$ and $G_F(x_2,y_3)=\top$. We assume that $\top=1$ and $\perp=0$. Define L-fuzzy topologies $\tau:L^X\to L$ and $\eta:L^Y\to L$ as follows:

$$\tau(\lambda)=\begin{cases}\top, & \text{if} \quad \lambda\in\{\underline{\perp},\underline{\top}\},\\ \frac{1}{2}, & \text{if} \quad \lambda=\underline{0.3},\\ \perp, & \text{otherwise,}\end{cases}$$

$$\eta(\mu)=\begin{cases}\top, & \text{if} \quad \mu\in\{\underline{\perp},\underline{\top}\},\\ \frac{1}{2}, & \text{if} \quad \mu=\underline{0.4},\\ \perp, & \text{otherwise.}\end{cases}$$

We can obtain the followings:

$$SC_\tau(\lambda,r)=\begin{cases}\underline{\perp}, & \text{if} \quad \lambda=\underline{\perp}, \quad r\in L_\circ,\\ \lambda, & \text{if} \quad \underline{0.3}\le\lambda\le\underline{0.7}, \quad \perp<r\le\frac{1}{2},\\ \underline{\top}, & \text{otherwise,}\end{cases}$$

$$SC_\eta(\lambda,r)=\begin{cases}\underline{\perp}, & \text{if} \quad \lambda=\underline{\perp}, \quad r\in L_\circ,\\ \lambda, & \text{if} \quad \underline{0.4}\le\lambda\le\underline{0.6}, \quad \perp<r\le\frac{1}{2},\\ \underline{\top}, & \text{otherwise.}\end{cases}$$

(1) F is $FUCS$-continuous (resp. FUC-irresolute) but not FUC-continuous because $\eta(\underline{0.4})=\frac{1}{2}$ in (Y,η), $F^l(\underline{0.4})=\underline{0.4}$ and $\tau([F^l(\underline{0.4})]^c)\not\geq\frac{1}{2}$.

(3) F is $FLCS$-continuous (resp. FLC-irresolute) but not FLC-continuous because $\eta(\underline{0.4})=\frac{1}{2}$ in (Y,η), $F^u(\underline{0.4})=\underline{0.4}$ and $\tau([F^u(\underline{0.4})]^c)\not\geq\frac{1}{2}$.

Theorem 3. 11. Let $F:X\multimap Y$ be a FM between two L-fts's (X,τ), (Y,η) and $\mu\in L^Y$. Suppose that one of the following properties hold:
(1) $SC_\tau(F^u(\mu),r)\le F^u(I_\eta(\mu,r))$.
(2) $F^l(C_\eta(\mu,r))\le SI_\tau(F^l(\mu),r)$.
Then F is $FLCS$-continuous.

Proof. (1) $\Rightarrow$ (2) Let $\mu\in L^Y$ hence by (1), we obtain $[SI_\tau(F^l(\mu),r)]^c=SC_\tau([F^l(\mu)]^c,r)$ $=SC_\tau(F^u(\mu^c),r)\le F^u(I_\eta(\mu^c,r))=[F^l(C_\eta(\mu,r))]^c$. Then, we obtain

$$F^l(C_\eta(\mu,r))\le SI_\tau(F^l(\mu),r).$$

Suppose that (2) holds. Let $\mu\in L^Y$ and $\eta(\mu^c)\geq r$ then by (2), we have $F^l(\mu)\le SI_\tau(F^l(\mu),r)$. Thus $F^l(\mu)$ is r-fso. Then from Theorem 3.5(2), F is $FLCS$-continuous.

Theorem 3. 12. Let $F : X \multimap Y$ be a FM between two L-fts's (X,τ), (Y,η) and $\mu \in L^Y$. Suppose that one of the following properties hold:
(1) $SC_\tau(F^u(\mu),r) \leq F^u(SI_\eta(\mu,r))$.
(2) $F^l(SC_\eta(\mu,r)) \leq SI_\tau(F^l(\mu),r)$.
Then F is FLC-irresolute.

Proof. (1) $\Rightarrow$ (2) Let $\mu \in L^Y$ hence by (1), we obtain $[SI_\tau(F^l(\mu),r)]^c = SC_\tau([F^l(\mu)]^c,r)$ $= SC_\tau(F^u(\mu^c),r) \leq F^u(SI_\eta(\mu^c,r)) = [F^l(SC_\eta(\mu,r))]^c$. Then, we obtain

$$F^l(SC_\eta(\mu,r)) \leq SI_\tau(F^l(\mu),r).$$

Suppose that (2) holds. Let $\mu \in L^Y$ and μ be r-fsc then by (2), we have $F^l(\mu) \leq SI_\tau(F^l(\mu),r)$. Thus $F^l(\mu)$ is r-fso. Then from Theorem 3.6(2), F is FLC-irresolute.

We state the following results without proof in view of above theorems.

Theorem 3. 13. Let $F : X \multimap Y$ be a FM and normalized between two L-fts's (X,τ), (Y,η) and $\mu \in L^Y$. Suppose that one of the following properties hold:
(1) $SC_\tau(F^l(\mu),r) \leq F^l(I_\eta(\mu,r))$.
(2) $F^u(C_\eta(\mu,r)) \leq SI_\tau(F^u(\mu),r)$.
Then F is $FUCS$-continuous.

Theorem 3. 14. Let $F : X \multimap Y$ be a FM and normalized between two L-fts's (X,τ), (Y,η) and $\mu \in L^Y$. Suppose that one of the following properties hold:
(1) $SC_\tau(F^l(\mu),r) \leq F^l(SI_\eta(\mu,r))$.
(2) $F^u(SC_\eta(\mu,r)) \leq SI_\tau(F^u(\mu),r)$.
Then F is FUC-irresolute.

Theorem 3. 15. Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two $FM's$ and let (X,τ), (Y,η) and (Z,δ) be three L-fts's. If H is normalized, H is FUS-continuous and F is $FLCS$-continuous, then $H \circ F$ is $FLCS$-continuous.

Proof. Let F be $FLCS$-continuous, H be FUS-continuous and $\gamma \in L^Z$, $\delta(\gamma^c) \geq r$. Then from Theorem 1.12(4) and Theorem 3.5(2), we have $(H \circ F)^l(\gamma) = F^l(H^l(\gamma))$ and $F^l(H^l(\gamma))$ is r-fso with $\eta((H^l(\gamma))^c) \geq r$. Thus, $H \circ F$ is $FLCS$-continuous.

We state the following result without proof in view of above theorem.

Theorem 3. 16. Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two $FM's$ and let (X,τ), (Y,η) and (Z,δ) be three L-fts's. If F is normalized, F is $FUCS$-continuous and H is FLS-continuous, then $H \circ F$ is $FUCS$-continuous.

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REFERENCES

[1] S. E. Abbas, M. A. Hebeshi and I. M. Taha , *On fuzzy upper and lower semi-continuous multifunctions*, Journal of Fuzzy Mathematics **22**, No. 4 (2014) 951-962.

[2] S. E. Abbas, M. A. Hebeshi and I. M. Taha, *On upper and lower almost weakly continuous fuzzy multifunctions*, Iranian Journal of Fuzzy Systems **12**, No. 1 (2015) 101-114.

[3] K. M. A. Al-hamadi and S. B. Nimse, *On fuzzy α-continuous multifunctions*, Miskolc Mathematical Notes **11**, No. 2 (2010) 105-112.

[4] M. Alimohammady, E. Ekici, S. Jafari and M. Roohi, *On fuzzy upper and lower contra-continuous multifunctions*, Iranian Journal of Fuzzy Systems **8**, No. 3 (2011) 149-158.

[5] H. Aygün, M. W. Warner and S. R. T. Kudri, *On smooth L-fuzzy topological spaces*, J. Fuzzy Math. **5**, No. 2 (1997) 321-338.

[6] H. Aygün and S. E. Abbas, *Some good extensions of compactness in Sostak's L-fuzzy topology*, Hacettepe Journal of Mathematics and Statistics **36**, No. 2 (2007) 115-125.

[7] C. Berge, *Topological Spaces, Including a Treatment of Multi-Valued Functions, Vector Spaces and Convexity*, Oliver, Boyd London, 1963.

[8] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24**, (1968) 182-190.

[9] K. C. Chattopadhyay and S. K. Samanta, *Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness*, Fuzzy Sets and Systems **54**, No. 2 (1993) 207-212.

[10] J. A. Goguen, *The Fuzzy Tychonoff theorem*, J. Math. Anal. Appl. **43**, (1993) 734-742.

[11] U. Höhle, *Upper semicontinuous fuzzy sets and applications*, J. Math. Anall. Appl. **78**, (1980) 659-673.

[12] U. Höhle and A. P. Sostak, *A general theory of fuzzy topological spaces*, Fuzzy Sets and Systems **73**, (1995) 131-149.

[13] U. Höhle and A. P. Sostak, *Axiomatic Foundations of Fixed-Basis fuzzy topology*, The Handbooks of Fuzzy sets series Volume 3, Kluwer Academic Publishers, Dordrecht (Chapter 3), 1999.

[14] Y. C. Kim, A. A. Ramadan and S. E. Abbas, *Weaker forms of continuity in Sostak?s fuzzy topology*, Indian J. Pure Appl. Math. **34**, No. 2 (2003) 311-333.

[15] Y. C. Kim and S. E. Abbas, *On several types of R-fuzzy compactness*, J. Fuzzy Math. **12**, No. 4 (2004) 827-844.

[16] Y. C. Kim, *Initial L-fuzzy closure spaces*, Fuzzy Sets and Systems **133**, (2003) 277-297.

[17] T. Kubiak, *On fuzzy topologies*, Ph. D. Thesis, A. Mickiewicz, Poznan, 1985.

[18] T. Kubiak and A. P. Sostak, *Lower set-valued fuzzy topologies*, Quaestions Math. **20**, No. 3 (1997) 423-429.

[19] S. R. T. Kudri, *Compactness in L-fuzzy topological spaces*, Fuzzy Sets Systems **67**, (1994) 229-236.

[20] Y. Liu and M. Luo, *Fuzzy Topology*, World Scientific Publishing, Singapore, 1997.

[21] R. A. Mahmoud, *An application of continuous fuzzy multifunctions*, Chaos, Solitons and Fractals **17**, (2003) 833-841.

[22] M. N. Mukherjee and S. Malakar, *On almost continuous and weakly continuous fuzzy multifunctions*, Fuzzy Sets and Systems **41**, (1991) 113-125.

[23] N. S. Papageorgiou, *Fuzzy topology and fuzzy multifunctions*, J. Math. Anal. Appl. **109**, (1985) 397-425.

[24] V. Popa, *On characterizations of irresolute multimapping*, J. Univ. Kuwait (Sci). **15**, (1988) 21-25.

[25] V. Popa, *Irresolute multifunctions?*, Internat. J. Math. and Math. Sci. **13**, No. 2 (1990) 275-280.

[26] A. A. Ramadan and S. E. Abbas, *On L-smooth compactness*, J. Fuzzy Math. **9**, No. 1 (2001) 59-73.

[27] A. A. Ramadan, S. E. Abbas and Y. C. Kim, *Fuzzy irresolute mappings in smooth fuzzy topological spaces*, J. Fuzzy Math. **9,** No. 4 (2001) 865-877.

[28] A. P. Sostak, *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Palerms ser II, **11**, (1985) 89-103.

[29] A. P. Sostak, *Two decades of fuzzy topology: basic ideas, notion and results*, Russian Math. Surveys **44**, No. 6 (1989) 125-186.

[30] A. P. Sostak, *Basic structures of fuzzy topology*, J. Math. Sci. **78**, No. 6 (1996) 662-701.

[31] E. Tsiporkova, B. De Baets and E. Kerre, *A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mappings*, Fuzzy Sets and Systems **85**, (1997) 93-108.

[32] E. Tsiporkova, B. De Baets and E. Kerre, *Continuity of fuzzy multivalued mappings*, Fuzzy Sets and Systems **94**, (1998) 335-348.

[33] C. K. Wong, *Fuzzy topology: product and quotient theorems*, J. Math. Anal. Appl. **45**, (1974) 512-521.