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Somewhere Dense Sets and ST_1-Spaces

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Abstract. In this article, the main properties of a somewhere dense set [13] on topological spaces are studied and then it is used to generalize the notions of interior, closure and boundary operators. The class of somewhere dense sets contains all α-open, pre-open, semi-open, β-open and b-open sets except for the empty set. We investigate under what conditions the union of cs-dense sets and the intersection of somewhere dense sets are cs-dense and somewhere dense, respectively. A concept of ST_1-space is defined and its various properties are discussed. Theorem (4.1) and Corollary (4.14) give the answer for why we do not define ST_0-spaces, $ST_{\frac{1}{2}}$-spaces, ST_2-spaces and $ST_{2\frac{1}{2}}$-spaces. Also, we point out that the product of ST_1-spaces is always an ST_1-space and present some examples to illustrate the main results.

AMS (MOS) Subject Classification Codes: 54D05; 54D10
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1. Introduction

The generalizations of open sets play an important role in topology by using them to define and investigate some generalizations of continuous maps, compact spaces, separation axioms, etc. see for example [8].

In this work, we study somewhere dense sets as a new kind of generalized open sets and consider many of its properties. We verify that the union (intersection) of somewhere dense (cs-dense) sets is also somewhere dense (cs-dense) and then we present necessary and sufficient conditions under which the union (intersection) of cs-dense (somewhere dense) sets is also cs-dense (somewhere dense). Some concepts related to somewhere dense sets like S-interior, S-closure and S-boundary operators are investigated in detail. Finally, we introduce a concept of ST_1-space and study sufficient conditions for some maps to preserve this concept. Two of the important results obtained in the last section are Theorem (4.1) and Corollary (4.14) which give reasons of why we do not define ST_0-space, $ST_{\frac{1}{2}}$-space, ST_2-space and $ST_2\frac{1}{2}$-space. Also, we point out that the product of somewhere dense sets (ST_1-spaces) is somewhere dense (ST_1-space) as well.

2. Preliminaries

The following celebrated five notions are defined by interior and closure operators as follows.

Definition 2.1. *A subset E of a topological space (Z,τ) is called:*

(1) *Semi-open* [10] *if $E \subseteq cl(int(E))$.*
(2) *α-open* [13] *if $E \subseteq int(cl(int(E)))$.*
(3) *Pre-open* [11] *if $E \subseteq int(cl(E))$.*
(4) *β-open* [1] *if $E \subseteq cl(int(cl(E)))$.*
(5) *b-open* [4] *if $E \subseteq int(cl(E)) \bigcup cl(int(E))$.*

Remark 2.2. *We note that Corson and Michael* [6] *used the term locally dense for pre-open sets.*

These kinds of generalized open sets similarly are introduced and investigated in soft topological spaces ([2], [3], [5], [7], [9]). Also, these kinds share common properties for example a class which consists of h-open sets in a topological space (Z,τ) forms a supra topology on Z, for each $h \in \{\alpha, \beta, b, pre, semi\}$.

Theorem 2.3. [12] *If M is an open subset of a topological space (Z,τ), then $M \bigcap cl(B) \subseteq cl(M \bigcap B)$, for each $B \subseteq Z$.*

Definition 2.4. *A topological space (Z,τ) with no mutually disjoint non-empty open sets is called hyperconnected.*

Theorem 2.5. [12] *If $\prod_{i\in I} M_i$ is a subset of a product topological space $(\prod_{i\in I} Z_i, T)$, then $cl(\prod_{i\in I} M_i) = \prod_{i\in I} cl(M_i)$.*

Definition 2.6. [14] *A subset E of a topological space (Z,τ) is called somewhere dense if $int(cl(E)) \neq \emptyset$. In other words, A subset E of a topological space (Z,τ) is called somewhere dense if there exists a non-empty open set G such that $G \subseteq cl(E)$.*

Throughout this article, (Z,τ) and (Y,θ) indicate topological spaces, G refers to a non-empty open subset of (Z,τ) and $\mathcal{R}$ stands for the set of real numbers.

3. Somewhere dense sets

In this section, we investigate the properties of somewhere dense sets and point out its relationships with some famous generalized open sets. Also, we derive various results concerning somewhere dense sets such as that the product of somewhere dense sets is always somewhere dense. Finally, we initiate the concepts of S-interior, S-closure and S-boundary operators and present several of their properties.

Definition 3.1. *The complement of somewhere dense subset B of (Z,τ) is called a cs-dense set.*

Remark 3.2. *Henceforth, $S(\tau)$ is used to denote the collection of all somewhere dense sets in (Z, τ).*

Theorem 3.3. *A subset B of (Z, τ) is cs-dense if and only if there exists a proper closed subset F of Z such that $int(B) \subseteq F$.*

Proof. To prove the "if" part, consider that a set $B \subset Z$ is cs-dense. Then B^c is somewhere dense. Therefore there is a G such that $G \subseteq cl(B^c)$. Thus $int(B) = (cl(B^c))^c \subseteq G^c$ and $G^c \neq Z$. Taking $F = G^c$, hence $int(B) \subseteq F \neq Z$.
To prove the "only if" part, suppose $B \subset Z$ and there is a closed set $F \neq Z$ such that $int(B) \subseteq F$. Then $F^c \subseteq (int(B))^c = cl(B^c)$ and $F^c \neq \emptyset$. Therefore B^c is somewhere dense. This completes the proof. □

Theorem 3.4. *Any non-empty β-open set is somewhere dense.*

Proof. Suppose that E is a non-empty β-open set.
Then $E \subseteq cl(int(cl(E))) \subseteq cl(cl(E)) = cl(E)$. Therefore the set $int(cl(E))$ is non-empty open and $int(cl(E)) \subseteq cl(E)$. Thus E is a somewhere dense set. □

The following example shows that the converse of Theorem (3.4) fails.

Example 3.5. *Let $Z = \{7, 8, 9\}$ and $\tau = \{\emptyset, \{7\}, \{8\}, \{7, 8\}, \{8, 9\}, Z\}$ be a topology on Z. Then $cl(int(cl(\{7, 9\}))) = \{7\}$. Therefore the set $\{7, 9\}$ is not β-open. On the other hand, $\{7, 9\}$ contains a non-empty open set $\{7\}$. This implies that $\{7, 9\}$ is a somewhere dense set.*

Remark 3.6. *Abd El-Monsef et al.*[1] *proved that every open α-open, semi-open, pre-open) set is β-open and Andrijevic* [4] *proved that every b-open set is β-open. Then we can deduce that they are somewhere dense except for the empty set.*

The relationships which were discussed in the previous theorem and remark are presented in the next Figure.

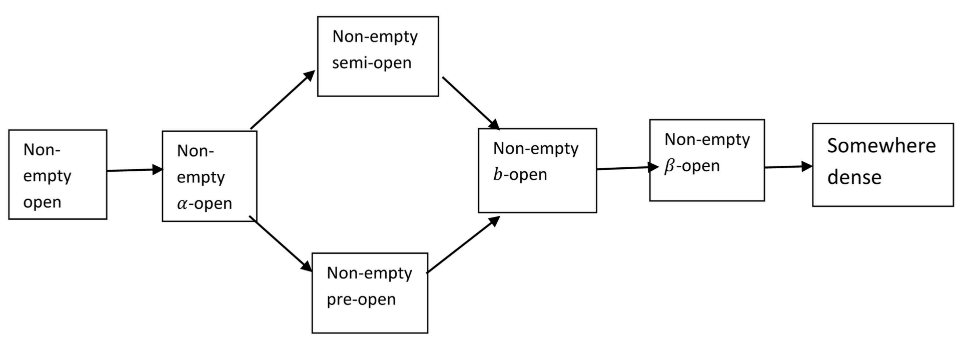

Fig. (1) The relationships between somewhere dense sets and some generalized non-empty open sets.

Proposition 3.7. *If (Z, τ) is an indiscrete topological space, then $S(\tau) \bigcup \{\emptyset\}$ is the discrete topology on Z.*

Proof. For any non-empty subset A of an indiscrete topological space (Z, τ), we have that $cl(A) = Z$. Then all non-empty subsets of Z are somewhere dense. This completes the proof. □

Theorem 3.8. *Every subset of (Z, τ) is somewhere dense or cs-dense.*

Proof. Suppose that A is a subset of Z that is not somewhere dense. Then $cl(A)$ has empty interior, so we cannot have $cl(A) = Z$. Then $(cl(A))^c$ is a non-empty open subset of A^c, hence of $cl(A^c)$, so A^c is somewhere dense and A is cs-dense. □

Proposition 3.9. *let $\{M_k : k \in K\}$ be a class of subsets of (Z, τ). Then $\bigcup_{k\in K} M_k$ is somewhere dense if and only if $\bigcup_{k\in K} \overline{M_k}$ is somewhere dense.*

Proof. Since $\overline{\bigcup_{k\in K} \overline{M_k}} = \overline{\bigcup_{k\in K} M_k}$, then the proposition holds. □

Proposition 3.10. *The union of an arbitrary non-empty family of somewhere dense subsets of (Z, τ) is somewhere dense.*

Proof. Assume that $\{E_k : k \in K \neq \emptyset\}$ is a family of somewhere dense sets. Then there exists a non-empty open set G_{k_0} such that $G_{k_0} \subseteq cl(E_{k_0}) \subseteq cl(\bigcup_{k\in K} E_k)$. Hence $\bigcup_{k\in K} E_k$ is a somewhere dense set. □

Corollary 3.11. *The intersection of an arbitrary non-empty family of cs-dense subsets of (Z, τ) is cs-dense.*

Proof. Let $\{B_k : k \in K \neq \emptyset\}$ be a family of cs-dense sets. Then $\{B_k^c : k \in K \neq \emptyset\}$ is a family of somewhere dense sets. Therefore $\bigcup_{k\in K} B_k^c$ is a somewhere dense set. Hence $\bigcap_{k\in K} B_k$ is cs-dense. □

Corollary 3.12. *The collection $S(\tau) \bigcup \{\emptyset\}$ forms a supra topology on Z.*

Remark 3.13. *The intersection (union) of a finite family of somewhere dense (cs-dense) sets is not somewhere dense (cs-dense) in general as the next example illustrates.*

Example 3.14. *Let τ be the cofinite topology on $\mathcal{R}$. Then $(-\infty, 1)$ and $(1, \infty)$ are cs-dense sets, but the union of them $\mathcal{R} \setminus \{1\}$ is not cs-dense. Also, $(-\infty, 1]$ and $[1, \infty)$ are somewhere dense sets, but the intersection of them $\{1\}$ is not somewhere dense.*

Theorem 3.15. *If M is open and N is somewhere dense in a hyperconnected space (Z, τ), then $M \bigcap N$ is somewhere dense.*

Proof. Consider that N is a somewhere dense subset of (Z, τ). Then there is a G which is contained in $cl(N)$. Therefore $M \bigcap G \subseteq M \bigcap cl(N) \subseteq cl(M \bigcap N)$. Since (Z, τ) is hyperconnected, then $M \bigcap G \neq \emptyset$. Thus $M \bigcap N$ is somewhere dense. □

Corollary 3.16. *If M is closed and N is cs-dense in a hyperconnected space (Z, τ), then $M \bigcup N$ is cs-dense.*

Definition 3.17. *A topological space (Z, τ) is called strongly hyperconnected provided that a subset of Z is dense if and only if it is non-empty and open.*

One can directly notice that every strongly hyperconnected space is hyperconnected, however the converse need not be true as shown by the cofinite topology which defined in Example (3.14).

Theorem 3.18. *Let M and N be subsets of a strongly hyperconnected space (Z, τ). If $int(M) = int(N) = \emptyset$, then $int(M \bigcup N) = \emptyset$.*

Proof. If M or N are empty, then the proof is trivial.
Let M and N be non-empty sets. Suppose, to the contrary, that $int(M \bigcup N) \neq \emptyset$. Then there is $x \in int(M \bigcup N)$ and this implies there is a G containing x such that G is contained in $M \bigcup N$. Since (Z, τ) is strongly hyperconnected, then we get

$$cl(M) \bigcup cl(N) = Z \tag{3. 1}$$

If $cl(M)$ is dense, then M is non-empty open. But this contradicts that $int(M) = \emptyset$. Therefore $\emptyset \subset cl(M) \subset Z$. Similarly, $\emptyset \subset cl(N) \subset Z$. Thus $\emptyset \subset (cl(M))^c \subset Z$ and $\emptyset \subset (cl(N))^c \subset Z$. From (3. 1), we obtain that $(cl(M))^c \bigcap (cl(N))^c = \emptyset$. But $(cl(M))^c$ and $(cl(N))^c$ are disjoint non-empty open sets and this contradicts that (Z, τ) is strongly hyperconnected. As a contradiction arises by assuming that $int(M \bigcup N) \neq \emptyset$, then the theorem holds. □

We now investigate under what conditions the union of cs-dense sets is cs-dense.

Lemma 3.19. *If M is a cs-dense subset of a strongly hyperconnected space (Z, τ), then $int(M) = \emptyset$.*

Proof. Let M be a cs-dense subset of (Z, τ). Then there is a closed set $F \neq Z$ containing $int(M)$. Suppose that $int(M) \neq \emptyset$. Then $cl(F) = Z$ and this implies that the set F is open. But this contradicts that (Z, τ) is strongly hyperconnected. Therefore $int(M) = \emptyset$. □

Theorem 3.20. *If M and N are cs-dense subsets of a strongly hyperconnected space (Z, τ), then $M \bigcup N$ is cs-dense.*

Proof. Consider that M and N are cs-dense subsets of (Z, τ). Then there are two closed sets $F \neq Z$ and $H \neq Z$ such that $int(M) \subseteq F$ and $int(N) \subseteq H$. Since (Z, τ) is strongly hyperconnected, then $int(M) = \emptyset$, $int(N) = \emptyset$ and $F \bigcup H \neq Z$. From Theorem (3.18), we obtain that $int(M) \bigcup int(N) = int(M \bigcup N) = \emptyset$. Consequently, $int(M \bigcup N) \subseteq F \bigcup H$. Hence the proof is completed. □

Corollary 3.21. *If M and N are somewhere dense subsets of a strongly hyperconnected space (Z, τ), then $M \bigcap N$ is somewhere dense.*

Corollary 3.22. *If (Z, τ) is strongly hyperconnected, then $S(\tau) \bigcup \{\emptyset\}$ forms a topology on Z.*

Theorem 3.23. *Let* $(\prod_{i=1}^{i=s} Z_i, T)$ *be a finite product topological space. Then* M_i *is a somewhere dense subset of* (Z_i, τ_i)*, for each* $i = 1, 2, ..., s$*, if and only if* $\prod_{i=1}^{i=s} M_i$ *is a somewhere dense subset of* $(\prod_{i=1}^{i=s} Z_i, T)$.

Proof. Necessity: Let M_i be a somewhere dense subset of (Z_i, τ_i). Then there is an open set $G_i \neq \emptyset$ such that $G_i \subseteq cl(M_i)$. Therefore $G_1 \times G_2 \times ... \times G_s \subseteq cl(M_1) \times cl(M_2) \times ... \times cl(M_s) = \prod_{i=1}^{i=s} cl(M_i) = cl(\prod_{i=1}^{i=s} M_i)$. Thus $\prod_{i=1}^{i=s} M_i$ is a somewhere dense subset of $(\prod_{i=1}^{i=s} Z_i, T)$.

Sufficiency: Let $\prod_{i=1}^{i=s} M_i$ be a somewhere dense subset of $(\prod_{i=1}^{i=s} Z_i, T)$. Then there is a non-empty open set $G_1 \times G_2 \times ... \times G_s$ of $(\prod_{i=1}^{i=s} Z_i, T)$ such that $G_1 \times G_2 \times ... \times G_s \subseteq cl(\prod_{i=1}^{i=s} M_i)$. Therefore $G_i \subseteq cl(M_i)$, for each $i = 1, 2, ..., s$. Thus M_i is a somewhere dense subset of (Z_i, τ_i). □

Corollary 3.24. *Let* $(\prod_{i=1}^{i=s} Z_i, T)$ *be a finite product topological space. Then* B_i *is a cs-dense subset of* (Z_i, τ_i)*, for each* $i = 1, 2, ..., s$ *if and only if* $\bigcup_{i=1}^{i=s} (B_i \times \prod_{j=1, j\neq i}^{j=s} Z_j)$ *is a cs-dense subset of* $(\prod_{i=1}^{i=s} Z_i, T)$.

Theorem 3.25. *If a map* $q : (Z, \tau) \to (Y, \theta)$ *is open and continuous, then the image of each somewhere dense set is somewhere dense.*

Proof. Let E be a somewhere dense subset of (Z, τ). Then there is a G such that $G \subseteq cl(E)$. Now, $q(G) \subseteq q(cl(E))$. Because q is open and continuous, then $q(G)$ is open and $q(cl(E)) \subseteq cl(q(E))$. Therefore $q(E)$ is somewhere dense. □

Corollary 3.26. *If* $\prod_{i\in I} M_i$ *is a somewhere dense subset of a product topological space* $(\prod_{i\in I} Z_i, T)$*, then* M_i *is a somewhere dense subset of* (Z_i, τ_i)*, for each* $i \in I$.

Definition 3.27. *Let* M *be a subset of* (Z, τ)*. Then:*

(1) *The* S*-interior of* M *(for short,* $Sint(M)$*) is the union of all somewhere dense sets contained in* M.
(2) *The* S*-closure of* M *(for short,* $Scl(M)$*) is the intersection of all cs-dense sets containing* M.
(3) *The* S*-boundary of* M *(for short,* $Sb(M)$*) is the set of elements which belong to* $(Sint(M) \bigcup Sint(M^c))^c$.

Proposition 3.28. *Consider a subset* M *of* (Z, τ)*. Then:*

(1) $M \subseteq Scl(M)$*; and a set* $M \neq Z$ *is* cs*-dense if and only if* $M = Scl(M)$.
(2) $Sint(M) \subseteq M$*; and a non-empty set* M *is somewhere dense if and only if* $M = Sint(M)$.
(3) $(Sint(M))^c = Scl(M^c)$.
(4) $(Scl(M))^c = Sint(M^c)$.

Proof. (1) and (2): The proofs of (1) and (2) come immediately from Definition (3.27) and Definition (2.6).
(3) $X - Sint(M) = (Sint(M))^c = \{\cup E : E$ is a somewhere dense set included in $M\}^c = \cap\{E^c : E^c$ is a cs-dense set including $M^c\} = Scl(M^c)$.
By analogy with (3), one can prove (4). □

For the sake of economy, the proof of the next proposition will be omitted.

Proposition 3.29. *Suppose* M *and* N *are subsets of* (Z, τ)*. Then:*
(1) $Sint(M) \bigcup Sint(N) \subseteq Sint(M \bigcup N)$ *and*
$Sint(M \bigcap N) \subseteq Sint(M) \bigcap Sint(N)$.
(2) $Scl(M \bigcap N) \subseteq Scl(M) \bigcap Scl(N)$ *and*
$Scl(M) \bigcup Scl(N) \subseteq Scl(M \bigcup N)$.
(3) $Sb(Sint(N)) \subseteq Sb(N)$.

In the following, we point out that the inclusion relation in the above proposition can be proper.

Example 3.30. *Assume that* (Z, τ) *is the same as in Example* (3.14)*. Let* $M = \mathcal{R} \setminus \{13, 14\}$*,* $N = \{13, 14, 19, 20\}$*,* $O = (-\infty, 1]$ *and* $P = [1, \infty)$*. Then*
(1) $Sint(M) = M$*,* $Sint(N) = \emptyset$ *and* $Sint(M \bigcup N) = \mathcal{R}$*. Also,* $Scl(O) = (-\infty, 1]$*,* $Scl(P) = [1, \infty)$ *and* $Sint(O \bigcap P) = \emptyset$*.*
(2) $Scl(M) = \mathcal{R}$*,* $Scl(N) = N$ *and* $Scl(M \bigcap N) = \{19, 20\}$*. Also,* $Scl(O \setminus \{1\}) = O \setminus \{1\}$*,* $Scl(P \setminus \{1\}) = P \setminus \{1\}$ *and* $Scl((O \setminus \{1\}) \bigcup (P \setminus \{1\})) = \mathcal{R}$*.*
(3) $Sb(Sint(N)) = \emptyset$ *and* $Sb(N) = N$*.*

Proposition 3.31. *Assume that* M *is a subset of* (Z, τ)*. Then:*
(1) $Sb(M) = Scl(M) \bigcap Scl(M^c)$.
(2) $Sb(M) = Scl(M) \setminus Sint(M)$.

Proof. (1) $Sb(M) = (Sint(M) \bigcup Sint(M^c))^c$
$= (Sint(M))^c \bigcup (Sint(M^c))^c$ (De Morgan's law)
$= Scl(M) \bigcap Scl(M^c)$ (Proposition (3.29)(iii))
(2) $Sb(M) = Scl(M) \bigcap Scl(M^c) = Scl(M) \bigcap (Sint(M))^c = Scl(M) \setminus Sint(M)$.
□

Corollary 3.32. $Sb(M) = Sb(M^c)$*, for every subset* M *of* (Z, τ)*.*

Lemma 3.33. *Let* M *be a subset of* (Z, τ)*. If* $Scl(M) = Z$*, then* $Scl(M^c) \neq Z$*.*

Proof. $Scl(M) = Z \Rightarrow cl(M) = Z \Rightarrow int(M^c) = \emptyset \Rightarrow M^c \neq Z$ and M^c is cs-dense $\Rightarrow Scl(M^c) = M^c \neq Z$. □

Proposition 3.34. $Sb(M)$ *is* cs*-dense, for every subset* M *of* (Z, τ)*.*

Proof. Let M be a subset of (Z,τ). Then we have the following two cases:

(1) $Scl(M) \neq Z$ and $Scl(M^c) \neq Z$. Then $Scl(M) \bigcap Scl(M^c)$ is cs-dense.
(2) $Scl(M) \neq Z$ and $Scl(M^c) = Z$ or $Scl(M) = Z$ and $Scl(M^c) \neq Z$. Say, $Scl(M) \neq Z$ and $Scl(M^c) = Z$. Then $Scl(M) \bigcap Scl(M^c) = Scl(M)$ is cs-dense.

Thus $Sb(M)$ is cs-dense. □

Proposition 3.35. *The following two statements hold.*

(1) *A non-empty subset M of (Z,τ) is somewhere dense if and only if $Sb(M) \bigcap M = \emptyset$.*
(2) *A proper subset M of (Z,τ) is cs-dense if and only if $Sb(M) \subseteq M$.*

Proof. (1) Necessity: $Sb(M) \bigcap M = Sb(M) \bigcap Sint(M) = \emptyset$.
Sufficiency: Let $x \in M$. Then $x \in Sint(M)$ or $x \in Sb(M)$. As $Sb(M) \bigcap M = \emptyset$, then $x \in Sint(M)$. Therefore $M \subseteq Sint(M)$. Thus M is somewhere dense.
(2) $M \neq \emptyset$ is cs-dense $\Leftrightarrow$ M^c is somewhere dense $\Leftrightarrow Sb(M^c) \bigcap M^c = \emptyset \Leftrightarrow Sb(M) \bigcap M^c = \emptyset \Leftrightarrow Sb(M) \subseteq M$.

□

Corollary 3.36. *Let M be a non-empty proper subset of (Z,τ). Then M is both somewhere dense and cs-dense if and only if $Sb(M) = \emptyset$.*

Proposition 3.37. *If M is a subset of (Z,τ), then $Sb(M) \subseteq M$ or $Sb(M) \subseteq M^c$.*

Proof. Let M be a subset of (Z,τ). Then from Theorem (3.8), one of M and M^c is cs-dense. By Proposition (3.35)(ii), either $Sb(M^c) = Sb(M) \subseteq M$ or $Sb(M) = Sb(M^c) \subseteq M^c$. □

4. ST_1-SPACES

We devote this section to defining a new separation axiom in topological spaces namely, RT_1-space and to studying its fundamental properties. Also, we derive some important results such as that the product of ST_1-spaces is also an ST_1-space.

In the next theorem, we point out why we did not define ST_0-space and $RT_{\frac{1}{2}}$-space.

Theorem 4.1. *Let (Z,τ) be a topological space. Then $Scl(\{s\}) \neq Scl(\{t\})$ for each pair of distinct points $s,t \in Z$.*

Proof. Let s,t be two distinct points in Z. From Theorem (3.8), we have the next two cases:

(1) A set $\{s\}$ is somewhere dense, then $\{s\} \bigcap \{t\} = \emptyset$. Therefore $s \notin Scl(\{t\})$. Thus $Scl(\{s\}) \neq Scl(\{t\})$.
(2) A set $\{s\}$ is not somewhere dense, then $\{s\}$ is cs-dense. So $Scl(\{s\}) = \{s\}$ and hence $Scl(\{s\}) \neq Scl(\{t\})$.

□

Definition 4.2. *A topological space (Z, τ) is said to be ST_1-space if for any pair of distinct points $a, b \in Z$, there exist two somewhere dense sets one containing a but not b and the other containing b but not a.*

Proposition 4.3. *Every T_1-space is an ST_1-space.*

Proof. Straightforward. □

In the next example, we illustrate that an ST_1-space is not always a T_1-space.

Example 4.4. *The class $\tau = \{\emptyset, \{r, s\}, Z\}$ defines a topology on $Z = \{r, s, t\}$. Observe that $S(\tau) = \{\{r\}, \{s\}, \{r, s\}, \{r, t\}, \{s, t\}, Z\}$. Then (Z, τ) is an ST_1-space, however it is not a T_1-space.*

For the sake of economy, the proof of the next theorem will be omitted.

Theorem 4.5. *The next three conditions are equivalent:*

(1) *(Z, τ) is an ST_1-space;*
(2) *All singleton subsets of (Z, τ) are cs-dense;*
(3) *For each subset U of Z, the intersection of all somewhere dense sets containing U is exactly U.*

Definition 4.6. *A subset W of (Z, τ) is said to be S-neighborhood of $a \in Z$ provided that there is a somewhere dense set E such that $a \in E \subseteq W$.*

Now, it is straightforward to verify the next two propositions.

Proposition 4.7. *Every neighbourhood of any point in (Z, τ) is a somewhere dense set.*

Proposition 4.8. *If E is a somewhere dense set in (Z, τ), then every proper superset of E is somewhere dense.*

Corollary 4.9. *A subset E of (Z, τ) is somewhere dense if and only if it is an S-neighbourhood of at least one point of Z.*

Corollary 4.10. *If the boundary of a closed set F is somewhere dense, then F is somewhere dense.*

Proof. Assume that $b(F)$ is somewhere dense and F is closed. Then $b(F) \neq \emptyset$ and $b(F) \subseteq F$. Hence F is somewhere dense. □

The converses of Propositions(4.7) and Propositions(4.8) do not hold as shown in the next example.

Example 4.11. *Consider the topology $\tau = \{\emptyset, \{r\}, \{t, u\}, \{r, s\}, \{r, t, u\}, Z\}$ on $Z = \{r, s, t, u\}$. Then we have the following:*

(1) *$\{s, u\}$ is somewhere dense, but is not a neighbourhood of any point.*
(2) *For any proper superset M of $\{s\}$, we get that M is somewhere dense. But $\{s\}$ is not somewhere dense.*

Lemma 4.12. *A subset E of (Z, τ) is somewhere dense if and only if $Scl(E)$ is somewhere dense.*

Proof. " $\Rightarrow$ ": Obvious.
" $\Leftarrow$ ": Let $Scl(E)$ be somewhere dense. Since $Scl(E) \subseteq cl(E)$, then $cl(E)$ is also somewhere dense. Therefore E is somewhere dense. □

Theorem 4.13. *A topological space (Z, τ) is an ST_1-space if and only if $\{x\} = \bigcap\{F_i : F_i$ is a cs-dense neighborhood of $x\}$, for each $x \in Z$*

Proof. " $\Rightarrow$ ": The collection $\{F_i : i \in I\}$ of all cs-dense neighborhoods of x is also the collection of all somewhere dense sets containing x. From Theorem (4.5), we get that $\{x\} = \bigcap\{F_i : i \in I\}$
" $\Leftarrow$ ": Let $x \neq y$. Since $\{x\} = \bigcap\{F_i : F_i$ is a cs-dense neighborhood of $x\}$ and $\{y\} = \bigcap\{H_j : H_j$ is a cs-dense neighborhood of $y\}$, then there exist somewhere dense sets F_{i_0} and H_{j_0} including x and y, respectively, such that $y \notin F_{i_0}$ and $x \notin H_{j_0}$. Therefore (Z, τ) is an ST_1-space. □

Corollary 4.14. *The next five properties are equivalent:*

1. *(Z, τ) is an ST_1-space;*
2. *For each pair of distinct points $a, b \in Z$, there are two disjoint somewhere dense sets one containing a and the other containing b;*
3. *For each pair of distinct points $a, b \in Z$, there are two disjoint sets E and S containing a and b, respectively, such that E and S are both somewhere dense and cs-dense;*
4. *For all $a \in Z$, we have $\{a\} = \bigcap\{cl(E_i) : E_i$ is a somewhere dense set containing $a\}$;*
5. *The subset $\{(z, z) : z \in Z\}$ of $Z \times Z$ is cs-dense.*

Remark 4.15. *The property of being ST_1-space is not a hereditary property as the next example illuminates.*

Example 4.16. *Consider the topology $\tau = \{\emptyset, \{r, s\}, Z\}$ on $Z = \{r, s, t\}$ and let $M = \{s, t\}$. Then (Z, τ) is an ST_1-space, but the subspace (M, τ_M) is not an ST_1-space.*

Theorem 4.17. *A product of ST_1-spaces is always an ST_1-space.*

Proof. Let $\{(Z_j, \tau_j) : j \in J\}$ be a collection of ST_1-spaces and $(\prod\limits_{j\in J} Z_j, T)$ be their product space. Let a, b be two distinct points in $\prod\limits_{j\in J} Z_j$. Without loss of generality, there exists $j_0 \in J$ such that $a_{j_0} \neq b_{j_0}$. Since (Z_{j_0}, τ_{j_0}) is an ST_1-space, then there are disjoint somewhere dense sets V_{j_0} and W_{j_0} including a_{j_0} and b_{j_0}, respectively. Therefore there is an open set G in (Z_{j_0}, τ_{j_0}) such that $G \subseteq cl(V_{j_0})$ and this implies that $\pi^{-1}(G) \subseteq \pi^{-1}(cl(V_{j_0})) = \{\prod\limits_{i\in J} Z_i : i \neq j\} \times cl(V_{j_0}) = cl(\{\prod\limits_{i\in J} Z_i : i \neq j\} \times V_{j_0})$. As π is surjective and continuous, then $\pi^{-1}(G)$ is a non-empty open subset of the product space $(\prod\limits_{j\in J} Z_j, T)$. Therefore $\{\prod\limits_{i\in J} Z_i : i \neq j\} \times V_{j_0}$ is a somewhere dense set and $b \notin \{\prod\limits_{i\in J} X_i : i \neq j\} \times V_{j_0}$. Similarly, $\{\prod\limits_{i\in J} Z_i : i \neq j\} \times W_{j_0}$ is a somewhere dense set containing b and $a \notin \{\prod\limits_{i\in J} Z_i : i \neq j\} \times V_{j_0}$. Hence $(\prod\limits_{j\in J} Z_j, T)$ is an ST_1-space. □

5. Conclusion

In the present paper, we give a concept of somewhere dense sets in topological spaces and derive interesting results such as any subset of (Z, τ) is somewhere dense or cs-dense. We present a notion of strongly hyperconnected space and then this is used to verify that $(Z, S(\tau) \bigcup \{\emptyset\})$ is a topological space if (Z, τ) is strongly hyperconnected. In the end, we define a notion of ST_1-space and derive several properties related to this notion as that the product of ST_1-spaces is an ST_1-space as well. In an upcoming paper, we plan to use an idea of somewhere dense sets to study new types of compactness and connectedness in topological spaces.

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