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A Note on UNAR LA-Semigroup

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Abstract. The concept of left (resp. right) unar groupoid is extended to introduce left (resp. right) unar LA-semigroup as new subclasses of LA-semigroup. These subclasses are enumerated up to order 6 and some basic relations of these classes with other known subclasses of LA-semigroup and with other relevant algebraic structures are investigated. Furthermore, a variety of examples and counterexamples are provided using the latest computational techniques of GAP, Mace-4 and Prover-9.

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1. Introduction

Right (resp. left) unar groupoids are defined and investigated in [12, 13]. We extend the concept of unar groupoids as new subclasses of LA-semigroups which we call left (resp. right) unar and unar LA-semigroups. A magma S that satisfies the law: $uv \cdot w = wv \cdot u$ is called an LA-semigroup. In addition, if S also satisfies the idenity $xz = yz$, we say S is a right unar LA-semigroup. It is interesting to note that though the identity for right unar is quite simple however, the enumeration table of right unar LA-semigroup shows that non-associative right LA-semigroups of any order do not exist, hence we prove that right LA-semigroup is a commutative semigroup. In Section 2 we define left unar LA-semigroups and enumerate them in Section 4 up to order 6 using GAP. We also find some of the relations of left unar LA-semigroups with the already known subclasses of LA-semigroups and that with semigroups. Section 3 is devoted to unar LA-semigroups, we also prove that non-associative unar LA-semigroups of any order do not exist and prove that every unar LA-semigroup is a commutative semigroup.

Left almost semigroups (LA-semigroups) is a well worked area of non-associative algebra, that was introduced by Kazim and Naseeruddin [11]. The structure is also known as AG-groupoid and modular groupoid and has a variety of applications in topology, matrices, flock theory, finite mathematics and geometry[3, 11, 18, 2, 10]. A considerable work in the area is done for developing its theory of ideals and fuzzification by various researchers in a variety of papers [4, 14, 15, 16, 19]. It is known that an LA-semigroup S always satisfies the medial law: $ab\cdot cd = ac\cdot bd$, and satisfes the paramedial law: $ab\cdot cd = db\cdot ca$ if it contains a left identity element. It is also easy to prove that if S contains a right identity element then it is a commutative semigroup. In the following, we list some fundamental definitions required for the investigation of our new subclasses.

An LA-semigroup S, is called –

(i) transitively commutative if, $ab = ba$ and $bc = cb$ implies $ac = ca, \forall a, b, c \in S$.[18]
(ii) left repeated if, $ab = cd$ implies $aa = cc, \forall a, b, c, d \in S$.[17]
(iii) T^4_f if, $ab = cd$ implies $ad = cb, \forall a, b, c, d \in S$.[1]
(iv) T^4_b if, $ab = cd$ implies $da = bc, \forall a, b, c, d \in S$.[1]
(v) T^3_r if, $ba = ca$ implies $ab = ac, \forall a, b, c \in S$.[18]
(vi) T^3_l if, $ab = ac$ implies $ba = ca, \forall a, b, c \in S$.[18]
(vii) left nuclear square if, $a^2 \cdot bc = a^2 b \cdot c, \forall a, b, c \in S$.[18]
(viii) outer repeated if, $ab \cdot cd = aa \cdot dd, \forall a, b, c, d \in S$.[17]
(ix) inner repeated if, $ab \cdot cd = bb \cdot cc, \forall a, b, c, d \in S$.[17]
(x) repeated if, S is both outer and inner repeated.[17]
(xi) left regular if, $ac = bc$ implies $ad = bd, \forall a, b, c, d \in S$.[17]
(xii) right regular if, $ca = cb$ implies $da = db, \forall a, b, c, d \in S$.[17]
(xiii) regular if, S is both left and right regular.[17]
(xiv) right permutable if, $ab \cdot c = ac \cdot b, \forall a, b, c \in S$.[17]
(xv) paramedial if, $ab \cdot cd = db \cdot ca, \forall a, b, c, d \in S$.[17]
(xvi) weak commutative if, $ab \cdot cd = dc \cdot ba, \forall a, b, c, d \in S$.[17]
(xvii) AG^* groupoid if, $ab \cdot c = b \cdot ac, \forall a, b, c \in S$.[15]
(xviii) flexible if, $ab \cdot a = a \cdot ba, \forall a, b \in S$.[18]
(xix) right alternative if, $ab \cdot b = a \cdot bb, \forall a, b \in S$. [18]
(xx) LA-band if, $a \cdot a = a, \forall a \in S$. [17]
(xxi) LC-LA-semigroup if, $uv \cdot w = vu \cdot w, \forall u, v, w \in S$. [17]
(xxii) T^1-LA-semigroup if $ab = cd \Rightarrow ba = dc, \forall a, b, c, d \in S$. [18]

Definition 1. *A magma S is called an LA-semigroup, if*

$$xy \cdot z = zy \cdot x, \forall x, y, z \in S. \tag{1.1}$$

2. Unar LA-semigroups

We define our new subclasses of LA-semigroup as under:

2.1. Right unar LA-semigroup.

Definition 2. *A Right unar LA-semigroup is an LA-semigroup S such that $\forall x, y, z \in S$,*

$$xz = yz. \tag{2.1}$$

Example 1. *Associative right unar LA-semigroup.*

$*$	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

Example 1 depicts that right unar LA-semigroups are single valued, and hence are commutative semigroups. This is verified in the following theorem.

Theorem 1. *Every right unar LA-semigroup is a commutative semigroup.*

Proof. Let S be a right unar LA-semigroup, and x, y, z be any elements of S. Then by (2.1)

$$xy = zy = u(\text{say for instance}, u \in S) \tag{2.2}$$

Then by (1.1) and (2.2),

$$\begin{aligned} xy \cdot z &= zy \cdot x \\ \Rightarrow uz &= ux \end{aligned} \tag{2.3}$$

Thus

$$\begin{aligned} xy &= zy = zx, \qquad \text{by (2.3)} \\ \Rightarrow xy = zx &= yx, \qquad \text{by (2.2,2.3)} \\ \Rightarrow xy &= yx. \end{aligned}$$

Hence S is commutative. Since commutativity implies associativity in LA-semigroups, thus S is a commutative semigroup. □

Since we are interested only in non-associative structure, thus we avoid this class of LA-semigroups and consider the next candidate, left unar LA-semigroups for further exploration.

2.2. **Left unar LA-semigroups.**

Definition 3. *A left unar LA-semigroup is an LA-semigroup S that satisfies the identity*

$$xy = xz \quad \forall x, y, z \in S. \tag{2.4}$$

Example 2. *A left unar LA-semigroups of order* 4.

$*$	u	v	w	x
u	u	u	u	u
v	u	u	u	u
w	v	v	v	v
x	v	v	v	v

Remark. It is worth mentioning that finite left unar LA-semigroups are very rare, as can be seen in Section 4. It is investigated in Section 4 that this type of LA-semigroups are only $1, 2, 4,$ and 6 out of $8, 269, 31467,$ and 40097003 respectively of order 3 to 6. Next, we discuss the relation of left unar LA-semigroup with some known subclasses of LA-semigroups. Throughout this article left unar LA-semigroup will be abbreviated by LU-LA-semigroup. Example 1 can be generalized to arbitrary sets S (of finite or infinite order). Choose an element $s_0 \in S$ and denote

$x \cdot y = s_0$ for all $x, y \in S$. Then these groupoids are single valued or 'trivial'. In Theorem 1, one can actually show that a right unar LA- semigroup is trivial (from which it follows that it is a commutative semigroup). This is another aspect of the present Theorem 1. It further also shows that right unar LA-semigroup is not an interesting notion because there is only one such LA-semigroup, up to isomorphism, in each order. In a similar way Example 2 should be generalized to arbitrary sets S (of finite or infinite order) involving a function $f : S \to S$ with the property that f^2 is constant. The LU-LA-semigroups are then precisely those groupoids that are represented by such a function f. With this notion many (but not all) of the proofs of Theorems 2, 3 and 4 become easy. However, we present our proof in the usual way as presented, the alternative way of proof, declare the status of the results more authentic. Furthermore, Theorems 3 and Theorem 4 shall alternatively be more strengthened to the following.
An LU-LA-semigroup is trivial if it is either $T_b^3, or T_l^3$, or flexible or a right alternative LA-semigroup. Furthermore, an LU-LA-semigroup that has a left identity element or is an LA-band, has order 1 or it is trivial. We start with the following theorem.

Theorem 2. *For an LU-LA-semigroup S, any of the following is true.*
(a) S is transitively commutative LA-semigroup.
(b) S is left repeated LA-semigroup.
(c) S is T_f^4-LA-semigroup.
(d) S is T_r^3-LA-semigroup.
(e) S is left nuclear square LA-semigroup.
(f) S is repeated LA-semigroup.
(g) S is regular LA-semigroup.
(h) S is right permutable LA-semigroup.
(i) S is paramedial LA-semigroup.
(j) S is weak-commutative LA-semigroup.
(k) S is LC-LA-semigroup.

Proof. Since S is LU-LA-semigroup, and $u, v, w, x \in S$, then by Equations (1.1) and (2.4), and the respective assumptions we have:
(a) Assume that for all u, v, w in S such that $uv = vu, vw = wv$, then we have,

$$uw = uv = vu = vw = wv = wu.$$

Thus $uw = wu$. Hence S is transitively commutative.
(b) Assume that $uv = wx$. We have,

$$uu = uv = wx = ww.$$

Thus $uu = ww$. Hence S is left repeated.
(c) Assume that $uv = wx$. Then,

$$ux = uv = wx = wv.$$

Thus $ux = wv$. Hence S is T_f^4.
(d) Assume $vu = wu$. Then,

$$uv = uw.$$

Hence S is T_r^3.

(e) Now, for left nuclear square, we have

$$u^2(vw) = u^2(vu) = (vu \cdot u)u = (uu \cdot v)u = (u^2v)w.$$

Thus $u^2(vw) = (u^2v)w$. So, S is left nuclear square.

(f) For repeated LA-semigroup, let $u, v, w, x \in S$. Then via in turn, we show that S is outer and inner repeated,

$$uv \cdot wx = (wx \cdot v)u = (wx.x)u = ux \cdot wx = uw \cdot xx = uu \cdot xx.$$

Thus $uv \cdot wx = uu \cdot xx$. Hence S is outer repeated. Now, for inner repeated, we have

$$uv \cdot wx = uw \cdot vx = (vx \cdot w)u \quad = \quad (vx \cdot w)v = (wx \cdot v)v = vv \cdot wx = vv \cdot ww.$$

Thus, $uv \cdot wx = vv \cdot ww$. So S is inner repeated. Hence, S is repeated.

(g) To show that S is regular, we show that S is left and right regular via in turn. Now, for left regular let $u, v, w, x \in S$. Let $uw = vw$, so we have,

$$ux = uw = vw = vx.$$

Thus $ux = vx$. Hence S is left regular. For right regular, assume $wu = wv$, then as S is left unar so,

$$xu = xv.$$

Thus, S is right regular. Hence S is regular.

(h) For right permutable,

$$uv \cdot w = wv \cdot u = ww \cdot u = uw \cdot w = uw \cdot v.$$

Thus $uv \cdot w = uw \cdot v$. Hence S is right permutable.

(i) For paramedial LA-semigroup, we have

$$uv \cdot wx = (wx \cdot v)u = (wx \cdot v)x = xv \cdot wx = xv \cdot wu.$$

Thus $uv \cdot wx = xv \cdot wu$. Hence, S is paramedial.

(j) For weak-commutative, we have

$$uv \cdot wx = (wx \cdot v)u = (wx \cdot v)x = (wu \cdot v)x = (vu \cdot w)x = xw \cdot vu.$$

Thus $uv \cdot wx = xw \cdot vu$. Hence, S is weak-commutative.

(k) To Show that S is LC-LA, we have

$$uv \cdot w = uu \cdot w = uu \cdot v = vu \cdot u = vu \cdot w.$$

Thus $uv \cdot w = vu \cdot w$. Thus S is LC-LA-semigroup.

Hence the result proved. □

Next, we prove that a non-associative structure LU-LA-semigroup becomes commutative and thus lead to an associative structure if it holds the any of the additional properties T^1 or T_b^4 or T_l^3 or it simply contains a left identity element.

Theorem 3. *An LU-LA-semigroup S is commutative semigroup if it satisfies any of the properties:*

(i) S is T^1.
(ii) S is T_b^4.
(iii) S is T_l^3.
(iv) S has a left identity element.

Proof. Let S be an LU-LA-semigroup, and let $a, b, c \in S$. Then

(i) Let S also be T^1 then by Equation (2.4), T^1, we have

$$ab = ac \Rightarrow ba = ca = cb \Rightarrow ab = bc = ba.$$

Thus $ab = ba$.

(ii) Assume, S also be T_b^4, then

$$ab = ac \Rightarrow ca = ba \Rightarrow ca = bb \Rightarrow bc = ab \Rightarrow ba = ab.$$

Hence $ab = ba$.

(iii) Let S also be T_l^3 then, we have

$$ab = aa \Rightarrow ba = aa \Rightarrow ba = ab.$$

Thus $ab = ba$.

(iv) Assume S has a left identity element e, then, by Equation (2.4) and repeated use of Equation (1.1), we have

$$ab = (e \cdot a)b = ba \cdot e = bb \cdot e = eb \cdot b = eb \cdot a = ba.$$

Thus $ab = ba$.

Therefore, for all the above conditions S is commutative. Since in LA-semigroup commutativity implies associativity therefore in each case, S is a commutative semigroup. □

Example (3) shows that LU-LA-semigroup is not T_b^4-LA-semigroup and is not T_l^3-LA-semigroup.

Example 3. *LU-LA-semigroup that is neither a T_l^3-LA-semigroup nor a T_b^4-LA-semigroup.*

$*$	u	v	w
u	u	u	u
v	u	u	u
w	v	v	v

Theorem 4. *An LU-LA-semigroup S is a semigroup if it also holds any of the following:*

(i) S is AG^*-groupoid.
(ii) S is flexible.
(iii) S is right alternative.
(iv) S is LA- band.

Proof. Let S be a LU-LA-semigroup, and let $x, y, z \in S$.

(i) Assume that S is also an AG^*-groupoid. Then by Equations (1.1), (2.4) and by the definition of AG^*-groupoid, we have

$$xy \cdot z = zy \cdot x = zx \cdot x = zx \cdot y = yx \cdot z = x \cdot yz.$$

Thus $xy \cdot z = x \cdot yz$.

(ii) Assume that S is also flexible. Then by Equation (2.4) and by definition of flexible LA-semigroup we have

$$xy \cdot z = xy \cdot x = x \cdot yx = x \cdot yz.$$

Thus $xy \cdot z = x \cdot yz$.

(iii) Assume that S is right alternative, then by Equation (2.4) and by the assumption, we have

$$x \cdot yz = x \cdot yy = xy \cdot y = xy \cdot z.$$

Thus $x \cdot yz = xy \cdot z$.

(iv) Assume that S is also an LA- band. Then by Equations (1.1, 2.4) and by the assumption, we have

$$\begin{aligned} xy \cdot z = xx \cdot z = xx \cdot zz = xz \cdot xz &= xz \cdot xy = xx \cdot zy = x(zz \cdot y) \\ = x(yz \cdot z) = x(yy \cdot z) &= x \cdot yz. \end{aligned}$$

Thus $xy \cdot z = x \cdot yz$.

Hence in each case an LU-LA-semigroup S becomes a semigroup. □

2.3. Unar LA-semigroup.

Definition 4. *An LA-semigroup S is called unar LA-semigroup if it is both LU-LA-semigroup and right unar LA-semigrop.*

We prove that non-associative unar LA-semigroups does not exist.

Theorem 5. *Every unar LA-semigroup is commutative semigroup.*

Proof. Let S be a unar LA-semigroup, and let $u, v, w \in S$. So by Equations (2.1) and (2.4)

$$uv = uw = vw = vu \quad \Rightarrow \quad uv = vu.$$

Thus S is commutative, but in any LA-semigroup commutativity implies associativity, therefore S is a commutative semigroup. □

3. Enumeration of left unar LA-semigroups

Enumeration of semigroups and monoids have been done up to order 9 and 10 respectively by constraint satisfaction techniques implemented in the Minion constraint solver with bespoke symmetry breaking provided by the computer algebra system GAP [9]. M. Shah [18] in his PhD thesis, in collaboration with A. Distler (the author of [5, 6, 7]) enumerated LA-semigroups up to isomorphism that has been published in [8] using a similar techniques developed for semigroups and monoids by A. Distler.

It is to mention also that the data presented in [8] has been verified by one of the reviewers of the such article with the help of Mace-4 and Isofilter as has been mentioned in the acknowledgment of the mentioned article. Using the same technique and relevant data of [8] we enumerate our newly introduced subclass of LU-LA-semigroup up to order 6 using the following coding in GAP:

Algorithm 1. ***GAP Function for Counting left unar LA-semigroups***

```
InstallMethod( IsLU-LA-semigroupTable, "for matrix",
[IsMatrix]
function( ls ),
local i, j;
if not IsAGGroupoidTable( ls ) then
return false;
fi;
for i in [1..Length(ls)] do
for j in [1..Length(ls)] do
```

for k in [1..Length(ls)] do
if ls[i][j] <> ls[i][k] then
return false;
fi;
od;
od;
od;
return true;
end);

Table 1 below, presents the enumeration of LU-LA-semigroups of order 3 to 6. Note that no non-associative LA-semigroups of order 2 and 1 exist.

Order	3	4	5	6
Total LA-semigroups	20	331	31913	40179867
Non Associative LA-semigroups	8	269	31467	40097003
Associative LA-semigroups	12	62	446	82864
Total LU-LA-semigroups	2	3	5	7
Non Associative LU-LA-semigroups	1	2	4	6
Associative LU-LA-semigroups	1	1	1	1

Table 1. Enumeration of left unar LA-semigroups up to order 6.

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