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Some properties concerning lifting of Bishop formulas on tangent space TR^3

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Abstract. In this article, we study the vertical, horizontal and complete lifts of Bishop formulas given by (1. 1), the first acceleration pool centers and the Darboux vector defined on space R^3 to its tangent space $TR^3 = R^6$. In addition, we include all special cases of the first and second curvatures κ_1 and κ_2 of the Bishop formulas according to the vertical, horizontal and complete lifts on space R^3 to tangent space TR^3. As a result of this transformation on R^3 to tangent space TR^3 , it can be speak about the properties of Bishop formulas on space TR^3 by looking at the lifting of characteristics $(\kappa_1, \kappa_2, T, N_1, N_2)$ of the first curve on space R^3.

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1. INTRODUCTION

Lift method has an important role in differentiable gometry. Because, it can able to generalize it from the differentiable structures from any space (for example R^3) to the extended spaces (TR^3) using the lift function [3, 4, 12, 14, 15, 16, 19, 21]. So, it can be extended the following theorem given on R^3 to tangent space TR^3. Also the Riemannian manifolds and the tangent bundles studyed a lot of authors [1, 2, 5, 6, 10, 11, 12, 13, 17, 18] too.

Theorem 1.1. *For a unit speed curve $\alpha_0(t)$ with curvatures $\kappa_1, \kappa_2 \rangle 0$ on R^3, the derivatives of Bishop frame $\{T, N_1, N_2\}$ are given by* [8, 9, 19]

$$T^{'} = \kappa_1 N_1 + \kappa_2 N_2, \;\; N_1' = -\kappa_1 T, \;\; N_2' = -\kappa_2 T, \tag{1. 1}$$

where T, N_1, N_2 are the unit vectors of Bishop frame on any point of $\alpha_0(t)$ and κ_1, κ_2 are the first and second curvatures of the curve $\alpha_0(t)$.

Definition 1.2. *Let $\alpha_0(t)$ be a unit speed curve with curvatures $\kappa_1, \kappa_2\rangle 0$ on space R^3, and suppose that T, N_1, N_2 be unit vectors of Bishop frame on any point of $\alpha_0(t)$. Then, we call that triple $\{T, N_1, N_2\}$ is Bishop frame such that* [9, 19]

$$\begin{aligned} T.N_1 &= N_1.N_2 = N_2.T = 0, \\ T.T &= N_1.N_1 = N_2.N_2 = 1. \end{aligned} \tag{1. 2}$$

where ”.” is a dot (scalar) product.

The article is structured as follows: In second section, the vertical, horizontal and complete lifts of a vector field defined on any manifold M of dimension m and their lift properties will be extended to space TR^3. In the third section, the vertical lift of the Theorem 1.1 will be obtained. Then, smilar to the vertical lift, the horizontal and complete lifts analogues of the related theorem are given. Later, we get the first acceleration pool centers according to vertical and horizontal and complete lifts of the Bishop formulas on TR^3. Finally, Darboux vector with recpect to vertical, complete and horizontal lifts on TR^3 is defined.

In this study, all geometric objects will be assumed to be of class C^∞ and the sum is taken over repeated indices. Also, v, c and H denote the vertical, horizontal and complete lifts of any differentiable geometric structures defined on R^3 to tangent space TR^3, respectively.

2. LIFT OF THE VECTOR FIELD

The vertical lift of a vector field X on space R^3 to extended space $TR^3(= R^6)$ is vector field $X^v \in \chi(TR^3)$ given as [12, 21]:

$$X^v(f^c) = (Xf)^v,\ \forall f \in F(R^3)$$

The vector field $X^c \in \chi(TR^3)$ defined by

$$X^c(f^c) = (Xf)^c,\ \forall f \in F(R^3)$$

is called the complete lift of a vector field X on R^3 to its tangent space TR^3.

The vector field $X^H \in \chi(TR^3)$ determined by

$$X^H(f^v) = (Xf)^v,\ \forall f \in F(R^3).$$

The general features of vertical, horizontal and complete lifts of a vector field on R^3 as follows:

Proposition 2.1. [19, 20, 21]*Let be functions all $f, g \in F(R^3)$ and vector fields all $X, Y \in \chi(R^3)$. Then it is satisfied the following equalities.*

$$\begin{aligned} (X+Y)^c &= X^c + Y^c, (X+Y)^v = X^v + Y^v,\ (X+Y)^H = X^H + Y^H, \\ (fX)^c &= f^c X^v + f^v X^c, (fX)^v = f^v + X^v,\ X^v(f^v) = 0, (fg)^H = 0, \\ X^c(f^v) &= X^v(f^c) = (Xf)^v, X^H(f^v) = (Xf)^v, X^c(f^c) = (Xf)^c, \\ \chi(U) &= Sp\left\{\frac{\partial}{\partial x^\alpha}\right\},\ \chi(TU) = Sp\left\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}\right\}, \\ \left(\frac{\partial}{\partial x^\alpha}\right)^c &= \frac{\partial}{\partial x^\alpha}, \left(\frac{\partial}{\partial x^\alpha}\right)^v = \frac{\partial}{\partial y^\alpha},\ \left(\frac{\partial}{\partial x^\alpha}\right)^H = \frac{\partial}{\partial x^\alpha} - \chi\Gamma^\alpha_\beta \frac{\partial}{\partial y^\alpha}. \end{aligned}$$

where Γ^{α}_{β} are Christoper symbols, U and TU are respectively topolgical opens of R^3 and TR^3, $f^v, f^c \in F(TR^3)$, $X^v, Y^v, X^c, Y^c, X^H, Y^H \in \chi(TR^3), 1 \leq \alpha, \beta \leq 3$.

3. Lifting Bishop formulas

In this section, we compute vertical, complete, horizontal lifts of Bishop formulas given by T, N_1, N_2 unit vectors of Bishop frame on any point of unit speed curve $\alpha_0(t)$ with curvatures $\kappa_1, \kappa_2 \rangle 0$ on space R^3.

3.1. **The vertical lifting Bishop formulas.** Let T^v be vertical lift of tangent vector T on a unit speed curve $\alpha_0(t)$. Lenght of T^v is given as:

$$\|T^v\| = T^v T^v = (TT)^v = 1$$

according to product rule, it follows

$$0 = (T^v T^v)' = (T^v)' T^v + T^v (T^v)' = 2T^v (T^v)'.$$

Thus, $T^v (T^v)' = 0$ and $(T^v)'$ is found orthonormal to T^v. Therefore it can be said that $(T^v)'$ is normal to unit speed curve $\alpha_1(t) = (\alpha_0(t))^v$. Similarly, from (1. 2), we have

$$T^v.(N_1)^v = (N_1)^v.(N_2)^v = (N_2)^v.T^v = 0.$$

In this case, T^v, N_1^v and N_2^v are three orthonormal Bishop vectors on $\alpha_1(t) = (\alpha_0(t))^v$ in the $6-$dimensional space TR^3.

Theorem 3.2. *For a unit speed curve $\alpha_1(t)$ with curvatures $(\kappa_1)^v, (\kappa_2)^v \rangle 0$ on TR^3, the derivative's vertical lifts of the Bishop vectors are given as:*

$$\begin{aligned} (T')^v &= (\kappa_1)^v (N_1)^v + (\kappa_2)^v (N_2)^v, \\ (N_2')^v &= -(\kappa_2)^v (T)^v, \\ (N_1')^v &= -(\kappa_1)^v (T)^v, \end{aligned}$$

where $(\kappa_1)^v$ and $(\kappa_2)^v$ are the first and second curvatures of the curve $\alpha_1(t)$.

Proof. Let $(T')^v, (N_1')^v, (N_2')^v$ be vertical lifts of T', N_1', N_2' which are derivatives T, N_1, N_2, respectively. We already know

$$(T')^v = (\kappa_1)^v (N_1)^v + (\kappa_2)^v (N_2)^v$$

by definition of $(N_1)^v, (N_2)^v$, where the curvatures $(\kappa_1)^v, (\kappa_2)^v$ describes variation in direction of T^v. Also, we shall find $(N_1')^v$ and $(N_2')^v$. In particular, given

$$(N_2')^v = a_1 (T)^v + b_1 (N_1)^v + c_1 (N_2)^v.$$

If it can be identified $a_1, b_1, c_1, T^v, (N_1)^v$ and $(N_2)^v$ then it will be known $(N_2')^v$. Firstly, we have

$$\begin{aligned} T^v (N_2')^v &= a_1 T^v T^v + b_1 T^v (N_1)^v + c_1 T^v (N_2)^v \\ &= a_1 (TT)^v + b_1 (TN_1)^v + c_1 (TN_2)^v \\ &= a_1.1 + b_1.0 + c_1.0 \\ &= a_1 \end{aligned}$$

Similarly, $(N_1)^v.(N_2')^v = b_1$ and $(N_2)^v.(N_2')^v = c_1$. So, it follows

$$(N_2')^v = (T^v(N_2')^v)(T)^v + ((N_1)^v.(N_2')^v)(N_1)^v + ((N_2)^v.(N_2')^v)(N_2)^v.$$

Now, let's identify $T^v(N_2')^v$. we know $T^v.(N_2)^v = (T.N_2)^v = 0$, so that

$$0 = (T^v.(N_2)^v)^{'} = (T^{'})^v(N_2)^v + T^v(N_2')^v$$

by vertical lift properties and the product rule.

$$\begin{aligned} T^v(N_2')^v &= -(T^{'})^v(N_2)^v \\ &= -(\kappa_1)^v(N_1)^v(N_2)^v - (\kappa_2)^v(N_2)^v(N_2)^v \\ &= -(\kappa_1)^v(N_1.N_2)^v - (\kappa_2)^v(N_2N_2)^v \\ a_1 &= -(\kappa_2)^v. \end{aligned}$$

From $0 = ((N_1)^v.(N_2)^v)^{'} = (N_1^{'})^v.(N_2)^v + (N_1)^v.(N_2^{'})^v$, we get

$$\begin{aligned} (N_1)^v.(N_2^{'})^v &= -(N_1^{'})^v.(N_2)^v = -(\kappa_1)^v(T)^v(N_2)^v \\ &= -(\kappa_1)^v(TN_2)^v \\ b_1 &= 0 \end{aligned}$$

From $1 = (N_2)^v(N_2)^v = (N_2N_2)^v$, we have

$$\begin{aligned} 0 &= ((N_2^{'})^v.(N_2)^v)^{'} + (N_2)^v(N_2^{'})^v \\ &= 2(N_2)^v(N_2^{'})^v. \end{aligned}$$

Thus, we get $c_1 = (N_2)^v(N_2^{'})^v = 0$. From above, $(N_2^{'})^v$ is calculated as:

$$(N_2')^v = -(\kappa_2)^v(T)^v$$

Now, it will be obtained $(T^{'})^v$. Just as for $(N_2^{'})^v$, it follows

$$(T^{'})^v = (T^v(T')^v)(T)^v + ((N_1)^v.(T')^v)(N_1)^v + ((N_2)^v.(T')^v)(N_2)^v$$

From the same types of calculations, we get

$$T^v(T')^v = 0,\ (N_1)^v.(T')^v = (\kappa_1)^v,\ (N_2)^v.(T')^v = (\kappa_2)^v.$$

Hence, $(T')^v$ is computed to be

$$(T^{'})^v = (\kappa_1)^v(N_1)^v + (\kappa_2)^v(N_2)^v$$

Therfore, the proof is completed. □

Corollary 3.3. *The Bishop formulas on TR^3 is similar structure and apperance to R^3 with respect to vertical lifts.*

Example 3.4. *Let a curve $\alpha_0(t)$ on R^3 has constant curvatures κ_1 and κ_2. Then such curves are circles according to the bishop frame. Because of the fact that curvatures κ_1 and κ_2 are constant, we have $(\kappa_1)^v = \kappa_1$ and $(\kappa_2)^v = \kappa_2$. So, the curve $\alpha_1(t) = (\alpha_0(t))^v$ on TR^3 has the same κ_1 and κ_2 . Then, $\alpha_0(t)$ on R^3 has similar appearance with the curve $\alpha_1(t) = (\alpha_0(t))^v$ on TR^3.*

3.5. **The complete lifting Bishop formulas.**

Theorem 3.6. *For a unit speed curve $\alpha_2(t) = (\alpha_0(t))^c$ with curvatures $(\kappa_1)^c, (\kappa_2)^c \rangle 0$ on tangent space TR^3, complete lifts of the derivatives of the Bishop frame are given by the following equalities:*

$$(T^{'})^c = (\kappa_1)^c (N_1)^c + (\kappa_2)^c (N_2)^c,\ (N_2')^c = -(\kappa_2)^c (T)^c,\ \ (N_1')^c = -(\kappa_1)^c (T)^c,$$

where $(\kappa_1)^c$ and $(\kappa_2)^c$ are the first and second curvatures of the curve $\alpha_2(t)$, respectively.

Proof. Similarly to vertical lifts, the theorem easily proved with respect to complete lift. □

Corollary 3.7. *Let the curvatures κ_1 and κ_2 of the curve $\alpha_0(t)$ on R^3 are non-constant functions, then the Bishop formulas on TR^3 are similar structure to R^3 with respect to complete lifts.*

Corollary 3.8. *Let the curvatures κ_1 and κ_2 of the curve $\alpha_0(t)$ on R^3 be constant functions. Then the curve $\alpha_2(t) = (\alpha_0(t))^c$ on TR^3 is line with respect to complete lifts.*

Proof. From the formulations of $\kappa = \sqrt{(\kappa_1)^2 + (\kappa_2)^2}$ and $\tau = -(\arctan(\frac{\kappa_2}{\kappa_1}))^{'}$ **[?]**, we get the following results: κ is a constant and $\tau = 0$. Then, the curve $\alpha_0(t)$ on R^3 is circle. Also, we get $(T^{'})^c = (N_1')^c = (N_2')^c = 0$. Then we say $\alpha_2(t) = (\alpha_0(t))^c$ on TR^3 is line. □

Theorem 3.9. *All curves $\alpha_0(t)$ on R^3 is line on TR^3 with respect to horizontal lifts.*

Proof. Let the curvatures κ_1 and κ_2 of the curve $\alpha_0(t)$ be constant or non-constant functions on R^3. For all functions on R^3, we write $f^H = 0$ with respect to horizontal lifts. So, $(\kappa_1)^H = 0 = (\kappa_2)^H$ and $(T^{'})^H = (N_1')^H = (N_2')^H = 0$ on TR^3.Consequently, $\alpha_3(t) = (\alpha_0(t))^H$ on TR^3 is line. □

3.10. **The first acceleration pool centers of the Bishop formulas on TR^3.**

Definition 3.11. *The first acceleration pool centers of the Bishop formulas on R^3 are given by the following equalities* [9]*:*

$$\begin{aligned} T^{''} &= -(\kappa_1^2 + \kappa_2^2)T + \kappa_1^{'} N_1 + \kappa_2^{'} N_2, \\ N_1^{''} &= -\kappa_1^{'} T - \kappa_1^2 N_1 - \kappa_1.\kappa_2 N_2, \\ N_2^{''} &= -\kappa_2^{'} T - \kappa_2^2 N_2 - \kappa_1.\kappa_2 N_1, \end{aligned}$$

where T, N_1, N_2 are unit vectors of Bishop frame on any point of $\alpha_0(t)$ and κ_1, κ_2 are the first and second curvatures of curve $\alpha_0(t)$.

It is possible to generalize the first acceleration pool centers with respect to vertical lifts of the Bishop formulas on space R^3 to its tangent space TR^3 using lift function [12, 14, 19, 21].

Theorem 3.12. *For a unit speed curve $\alpha_1(t)$ with curvatures $(\kappa_1)^v, (\kappa_2)^v \rangle 0$ on TR^3, the first acceleration pool centers with respect to vertical lifts of the Bishop formulas on TR^3*

are given as:

$$\begin{aligned}(T^{''})^v &= -((\kappa_1^2)^v + (\kappa_2^2)^v)T^v + (\kappa_1^{'})^v(N_1)^v + (\kappa_2^{'})^v(N_2)^v,\\ (N_1^{''})^v &= -(\kappa_1^{'})^v T^v - (\kappa_1^2)^v(N_1)^v - (\kappa_1)^v.(\kappa_2)^v(N_2)^v,\\ (N_2^{''})^v &= -(\kappa_2^{'})^v T^v - (\kappa_2^2)^v(N_2)^v - (\kappa_1)^v.(\kappa_2)^v(N_1)^v,\end{aligned}$$

where $(\kappa_1)^v$ *and* $(\kappa_2)^v$ *are the first and second curvatures of the curve* $\alpha_1(t)$ *on* TR^3.

Proof. From the derivatives of the Theorem 3.2, we get the following results:

$$\begin{aligned}(T^{''})^v &= ((\kappa_1)^v)^{'}(N_1)^v + (\kappa_1)^v((N_1)^v)^{'} + ((\kappa_2)^v)^{'}(N_2)^v + (\kappa_2)^v((N_2)^v)^{'}\\ &= (\kappa_1^{'})^v(N_1)^v + (\kappa_2^{'})^v(N_2)^v + (\kappa_1)^v(-(\kappa_1)^v T^v) + (\kappa_2)^v(-(\kappa_2)^v T^v)\\ &= -((\kappa_1^2)^v + (\kappa_2^2)^v)T^v + (\kappa_1^{'})^v(N_1)^v + (\kappa_2^{'})^v(N_2)^v.\end{aligned}$$

$$\begin{aligned}(N_1^{''})^v &= -(\kappa_1^{'})^v T^v - (\kappa_1)^v(T^{'})^v\\ &= -(\kappa_1^{'})^v T^v - (\kappa_1)^v((\kappa_1)^v(N_1)^v + (\kappa_2)^v(N_2)^v)\\ &= -(\kappa_1^{'})^v T^v - (\kappa_1^2)^v(N_1)^v - (\kappa_1)^v.(\kappa_2)^v(N_2)^v\end{aligned}$$

$$\begin{aligned}(N_2^{''})^v &= -(\kappa_2^{'})^v T^v - (\kappa_2)^v(T^{'})^v\\ &= -(\kappa_2^{'})^v T^v - (\kappa_2)^v((\kappa_1)^v(N_1)^v + (\kappa_2)^v(N_2)^v)\\ &= -(\kappa_2^{'})^v T^v - (\kappa_2^2)^v(N_2)^v - (\kappa_1)^v.(\kappa_2)^v(N_1)^v\end{aligned}$$

Therfore, the proof is finished. □

Similarly, we can easily prove the following theorem of the first acceleration pool centers with respect to complete lifts of the Bishop formulas on TR^3.

Theorem 3.13. *Let* $(\kappa_1)^c$ *and* $(\kappa_2)^c$ *be the first and second curvatures of the curve* $\alpha_2(t) = (\alpha_0(t))^c$ *on* TR^3*. The first acceleration pool centers according to complete lifts of the Bishop formulas on* TR^3 *are given as:*

$$\begin{aligned}(T^{''})^c &= -((\kappa_1^2)^c + (\kappa_2^2)^c)T^c + (\kappa_1^{'})^c(N_1)^c + (\kappa_2^{'})^c(N_2)^c,\\ (N_1^{''})^c &= -(\kappa_1^{'})^c T^c - (\kappa_1^2)^c(N_1)^c - (\kappa_1)^c.(\kappa_2)^c(N_2)^c,\\ (N_2^{''})^c &= -(\kappa_2^{'})^c T^c - (\kappa_2^2)^c(N_2)^c - (\kappa_1)^c.(\kappa_2)^c(N_1)^c,\end{aligned}$$

where $\alpha_2(t) = (\alpha_0(t))^c$ *a unit speed curve with curvatures* $(\kappa_1)^c, (\kappa_2)^c\rangle 0$ *on* TR^3.

Corollary 3.14. *Because of the Theorem 3.9, we get* $(T^{''})^H = (N_2^{''})^H = (N_1^{''})^H = 0.$

3.15. **The Darboux vector with recpect to vertical, horizontal and complete lifts on TR^3.**

Definition 3.16. *The Darboux vector* ω *on* R^3 *defined as* [7, 9]*:*

$$\omega = (0, -\kappa_2, \kappa_1) = -\kappa_2 N_1 + \kappa_1 N_2$$

ω is a vector in the plane (N_1, N_2) and perpendicular to the tangent vector of the curve. ω vector field has the following properties:

$$\begin{aligned} \omega.T &= 0,\ \omega.N_1 = -\kappa_2,\ \omega.N_2 = \kappa_1 \\ \omega\Lambda T &= 0,\ \omega\Lambda N_1 = N_1^{'},\ \omega\Lambda N_2 = N_2^{'} \end{aligned}$$

Theorem 3.17. *Let $\alpha_1(t)$ be a unit speed curve with curvatures $(\kappa_1)^v, (\kappa_2)^v\rangle 0$ on TR^3, The ω^v Darboux vector with respect to vertical lifts on TR^3 defined as:*

$$\omega^v = (0, -(\kappa_2)^v, (\kappa_1)^v) = -(\kappa_2)^v(N_1)^v + (\kappa_1)^v(N_2)^v$$

ω^v vector field has the following properties:

$$\begin{aligned} \omega^v.T^v &= 0,\ \omega^v.(N_1)^v = -(\kappa_2)^v,\ \omega^v.(N_2)^v = (\kappa_1)^v \\ \omega^v\Lambda T^v &= 0,\ \omega^v\Lambda(N_1)^v = (N_1^{'})^v,\ \omega^v\Lambda(N_2)^v = (N_2^{'})^v \end{aligned}$$

Proof. From Proposition 1 and Definition 3, we get the following results:

$$\begin{aligned} \omega^v.T^v &= (-(\kappa_2)^v(N_1)^v + (\kappa_1)^v(N_2)^v).T^v \\ &= -(\kappa_2)^v(N_1.T)^v + (\kappa_1)^v(N_2.T)^v \\ &= -(\kappa_2)^v.0 + (\kappa_1)^v.0 \\ &= 0 \\ \omega^v.(N_1)^v &= (-(\kappa_2)^v(N_1)^v + (\kappa_1)^v(N_2)^v).(N_1)^v \\ &= -(\kappa_2)^v(N_1.N_1)^v) + (\kappa_1)^v(N_2.N_1)^v \\ &= -(\kappa_2)^v \end{aligned}$$

$$\begin{aligned} \omega^v.(N_2)^v &= (-(\kappa_2)^v(N_1)^v + (\kappa_1)^v(N_2)^v).(N_2)^v \\ &= -(\kappa_2)^v(N_1.N_2)^v + (\kappa_1)^v(N_2.N_2)^v \\ &= (\kappa_1)^v \end{aligned}$$

□

Corollary 3.18. *If we defined ω^c Darboux vector with respect to complete lifts on TR^3, then we get $\omega^c = (0, -(\kappa_2)^c, (\kappa_1)^c) = -(\kappa_2)^c(N_1)^c + (\kappa_1)^c(N_2)^c$. From (1. 2) and Proposition 1, we get*

$$\omega^c.T^c = \omega^c.(N_1)^c = \omega^c.(N_2)^c = 0.$$

Corollary 3.19. *Let the curvatures κ_1 and κ_2 be constant. Then, we get $(\kappa_1)^c = 0$ and $(\kappa_2)^c = 0$. So, $\omega^c = 0$. Consequently, the Darboux vector ω^c with respect to complete lifts on TR^3 is point.*

Corollary 3.20. *Let the curvatures κ_1 and κ_2 of the curve $\alpha_0(t)$ on R^3 be non-constant and constant functions, respectively. Then, we get $\omega^c = (\kappa_1)^c(N_2)^c$ (the Darboux vector ω^c linear dependency $(N_2)^c$ on TR^3).*

Corollary 3.21. *Let the curvatures κ_1 and κ_2 of the curve $\alpha_0(t)$ on R^3 be constant and non-constant functions, respectively. Then, we get $\omega^c = -(\kappa_2)^c(N_1)^c$ (the Darboux vector ω^c linear dependency $(N_1)^c$ on TR^3).*

Theorem 3.22. *Darboux vector ω^H with respect to horizontal lifts on TR^3 is a point everytime .*

Proof. From Theorem 3.9, we get $(\kappa_1)^H = (\kappa_2)^H = 0$. So, $\omega^H = 0$ on TR^3 with respect to horizontal lifts. The theorem is proved. □

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