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Asymptotic Behavior of Linear Evolution Difference System

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Abstract. In this article we give some characterizations of exponential stability for a periodic discrete evolution family of bounded linear operators acting on a Banach space in terms of discrete evolution semigroups, acting on a special space of almost periodic sequences. As a result, a spectral mapping theorem is stated.

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1. Introduction

In the recent times Y. Wang et al. in [7] has proved that the discrete system $\lambda_{u+1} = \mathcal{A}_u\lambda_u$ is uniformly exponentially stable if and only if the unique solution of the initial value problem

$$\begin{cases} \lambda_{u+1} = \mathcal{A}_u\lambda_u + z(u+1), & u \in \mathbb{Z}_+, \\ \quad \lambda_0 = 0, \end{cases} \tag{$\mathcal{A}_u, 0$}$$

is bounded for any natural number u and any almost d-periodic sequence $z(u)$ with $z(0) = 0$. Here, $\mathcal{A}_u$ is a sequence of bounded linear operators on Banach space X. It is well known, see e.g. [1, 4, 6, 8] that if the initial value problem

$$\frac{d\lambda}{dt} = \mathcal{A}\lambda(t) + e^{i\beta t}\gamma,\ t \geq 0, \quad \lambda(0) = 0,$$

has a bounded solution on $\mathbb{R}_+$ for every $\beta \in \mathbb{R}$ and any $\gamma \in X$ then the homogenous system $\frac{d\lambda}{dt} = \mathcal{A}\lambda(t)$, is uniformly exponentially stable.

In case of discrete semigroups recently Zada et al. [10] proved that the system $\lambda_{u+1} = \mathcal{D}(1)\lambda_u$ is uniformly exponentially stable if and only if for each d-periodic bounded sequence $f(u)$ with $f(0) = 0$ the solution of the initial value problem

$$\begin{cases} \lambda_{u+1} = \mathcal{D}(1)\lambda_u + e^{i\beta(u+1)} f(u+1), \\ \lambda(0) = 0 \end{cases} \qquad (\mathcal{D}(1), \mu, 0)$$

is bounded, where $\mathcal{D}(1)$ is the algebraic generator of the discrete semigroup $T(u)$, $u \in \mathbb{Z}_+$.

In this article we extended the result of last quoted paper to space of almost periodic sequences denoted by $AP_1(\mathbb{Z}_+, X)$, for such spaces we recommend [5].

2. Notations and Preliminaries

We denote by $\|\cdot\|$ the norms of operators and vectors. Denote by $\mathbb{R}_+$ the set of real numbers and by $\mathbb{Z}_+$ the set of all non-negative integers.

Let $\mathcal{B}(\mathbb{Z}_+, X)$ be the space of X-valued bounded sequences with the supremum norm, and $\mathbb{P}^d(\mathbb{Z}_+, X)$ be the space of d-periodic (with $d \geq 2$) sequences $z(n)$. Then $\mathbb{P}^d(\mathbb{Z}_+, X)$ is a closed subspace of $\mathcal{B}(\mathbb{Z}_+, X)$.

Throughout this paper, $\mathcal{A} \in \mathcal{B}(X)$, $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$, and $r(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$ denotes the spectral radius of $\mathcal{A}$. It is well known that $r(\mathcal{A}) = \lim\limits_{n\to\infty} \|\mathcal{A}^n\|^{\frac{1}{n}}$. The resolvent set of $\mathcal{A}$ is defined as $\rho(\mathcal{A}) := \mathbb{C}\backslash\sigma(\mathcal{A})$, i.e., the set of all $\lambda \in \mathbb{C}$ for which $\mathcal{A} - \lambda I$ is an invertible operator in $\mathcal{B}(X)$.

We give some results in the framework of general Banach space and spaces of sequences as defined above.

Recall that $\mathcal{A}$ is power bounded if there exists a positive constant M such that $\|\mathcal{A}^n\| \leq M$ for all $n \in \mathbb{Z}_+$.

The family $\Omega := \{\xi(u, v) : u, v \in \mathbb{Z}_+, u \geq v\}$ of bounded linear operators is called d-periodic discrete evolution family, for a fixed integer $d \in \{2, 3, \ldots\}$, if it satisfies the following properties:

- $\xi(u, u) = I, \ \forall\, u \in \mathbb{Z}_+$.
- $\xi(u, v)\xi(v, r) = \xi(u, r), \ \forall\, u \geq v \geq r$, $u, v, r \in \mathbb{Z}_+$.
- $\xi(u + d, v + d) = \xi(u, v), \ \forall\, u \geq v$, $u, v \in \mathbb{Z}_+$.

It is well known that any d-periodic evolution family Ω is exponentially bounded, that is, there exist $\rho \in \mathbb{R}$ and $M_\rho \geq 0$ such that

$$\|\xi(u, v)\| \leq M_\rho e^{\rho(u-v)}, \ \forall\, u \geq v \in \mathbb{Z}_+. \qquad (2.\ 1)$$

When family Ω is exponentially bounded its growth bound, $\rho_0(\Omega)$, is the infimum of all $\rho \in \mathbb{R}$ for which there exists $M_\rho \geq 1$ such that the relation (2. 1) is fulfilled. It is known that

$$\begin{aligned} \rho_0(\Omega) &= \lim_{u\mapsto\infty} \frac{ln\|\xi(u, 0)\|}{u} && (2.\ 2) \\ &= \frac{1}{d} \ln(r(\xi(d, 0))). && (2.\ 3) \end{aligned}$$

In fact

$$\begin{aligned}
\rho_0(\Omega) : &= \lim_{u\mapsto\infty} \frac{\ln\|\xi(u,0)\|}{u} \\
&= \lim_{u\mapsto\infty} \frac{\ln\|\xi(ud,0)\|}{ud} \\
&= \frac{1}{d}\lim_{u\mapsto\infty} \ln\|\xi^u(d,0)\|^{\frac{1}{u}} \\
&= \frac{1}{d}\ln\lim_{u\mapsto\infty} \|\xi^u(d,0)\|^{\frac{1}{u}} \\
&= \frac{1}{d}\ln(r(\xi(d,0))).
\end{aligned}$$

A family Ω is uniformly exponentially stable if $\rho_0(\Omega)$ is negative, or equivalently, there exists $M \geq 1$ and $\rho \geq 0$ such that $\|\xi(u,v)\| \leq Me^{-\rho(u-v)}$, for all $u \geq v \in \mathbb{Z}_+$.
The following lemma is a consequence of (2.1).

Lemma 2.1. [9] *The discrete evolution family Ω is uniformly exponentially stable if and only if $r(\xi(d,0)) < 1$.*

The map $\xi(d,0)$ is also called the Poincare map or monodromy operator of the evolution family Ω.

Proposition 2.2. *Let $\Omega = \{\xi(u,v) : u,v \geq 0\}$ be a d-periodic discrete evolution family acting on the Banach space X. The following four statements are equivalent:*
(1) $\xi(u,v)$ is uniformly exponentially stable.
(2) There exists two positive constants M and w such that

$$\|\xi(u,v)\| \leq Me^{-w(u-v)},\ \forall\, u,v \geq 0.$$

(3) The spectral radius of $\xi(u,0)$ is less than one; i.e.,

$$r(\xi(u,0)) = \sup\{|\lambda| : \lambda \in \sigma(\xi(u,0))\} = \lim_{k\to\infty} \|\xi(u)^k\|^{\frac{1}{k}} < 1.$$

(4) For each $\mu \in \mathbb{R}$, one has

$$\sup_{u\geq 1} \|\sum_{k=1}^{u} e^{-i\mu k}\xi(u,k)^k\| = M(\mu) < \infty.$$

The proof of the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ is obvious from the definitions. The implication of $(4) \Rightarrow (1)$ can be found in Lemma (1) of [3].

We recall the following result from [7] which is very helpful in the proof of our main result.

Theorem 2.3. [7] *Let $\Omega := \{\xi(u,v) : u,v \in \mathbb{Z}_+, u \geq v\}$ be a discrete evolution family on X. If the sequence*

$$\zeta_u = \sum_{k=0}^{u} e^{i\theta k}\xi(u,k)z(k)$$

is bounded for each real number θ and each d-periodic sequence $z(u) \in \mathcal{W}$, then Ω is uniformly exponentially stable.

3. Discrete Evolution Semigroup

Here we consider a space of X-valued sequences and define a discrete evolution semigroup acting on it. For this purpose, we need the following spaces:
$\mathcal{B}(\mathbb{Z}, X)$ which is the space of all X-valued bounded and uniformly convergent sequences defined on $\mathbb{Z}$, endowed with the norm $\|f\|_\infty = \sup_{u\in\mathbb{Z}} \|f(u)\|$.
$P^d(\mathbb{Z}, X)$ which is the subspace of $\mathcal{B}(\mathbb{Z}, X)$ consisting of all sequences F such that $\mathrm{F}(u+d) = \mathrm{F}(u)\ for\ all\ u \in \mathbb{Z}$.
$AP_1(\mathbb{Z}, \mathbf{X})$ which is the space of all X-valued sequences defined on $\mathbb{Z}$ representable in the form $\mathrm{f}(u) = \sum\limits_{k=-\infty}^{k=\infty} e^{i\mu_k u} c_k(\mathrm{f})\ for\ all\ u \in \mathbb{Z}$, where $\mu_k \in \mathbb{Z},\ c_k(\mathrm{f}) \in X$ and

$$\|\mathrm{f}\|_1 = \sum_{k=-\infty}^{k=\infty} \|c_k(\mathrm{f})\| < \infty.$$

For almost periodic sequences, see [2, 5].

For an arbitrary $u \geq 0$, we denote by $\hat{\mathbf{I}}_u$ the set of all X-valued sequences defined on $\mathbb{Z}$ such that there exists a sequence F in $P^d(\mathbb{Z}, X) \bigcap AP_1(\mathbb{Z}, X)$ with $\mathrm{F}(u) = 0,\ \mathrm{f} = \mathrm{F}_{|\{u,u+1,\ldots\}}$ and $\mathrm{f}(v) = 0$ for all $v < u$. Set $\hat{\mathbf{I}} = \{e^{i\mu} \otimes \mathrm{f} : \mu \in \mathbb{R}\ and\ \mathrm{f} \in \cup_{u\geq 0} \hat{\mathbf{I}}_u\}$ and let $\mathbf{E}(\mathbb{Z}, X) = span(\hat{\mathbf{I}})$. Consider the space $\widetilde{\mathbf{E}}(\mathbb{Z}, X) = \overline{span}(\hat{\mathbf{I}})$ which is a closed subspace of $\mathcal{B}(\mathbb{Z}, X)$ endowed with sup norm. The discrete evolution semigroup $\mathbf{Ţ} = \{Ţ(u)\}_{u\geq 0}$ associated to a q-periodic discrete family $\Omega = \{\xi(u, v)\}_{u,v\geq 0}$ on $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$ is defined as:

$$\big(Ţ(v)\widetilde{\mathrm{f}}\big)(u) = \begin{cases} \xi(u, u-v)\widetilde{\mathrm{f}}(u-v), & \text{if } u \geq v, \\ 0, & \text{if } u < v, \end{cases} \tag{3. 4}$$

for $\widetilde{\mathrm{f}} \in \widetilde{\mathbf{E}}(\mathbb{Z}, X)$.

Proposition 3.1. *The space $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$ is invariant under the discrete evolution semigroup $Ţ$, defined in* (3.4).
Proof. *Let $\widetilde{\mathrm{f}}(u) = e^{i\mu u}\mathrm{f}(u)$, with $\mu \in \mathbb{R}$ and $\mathrm{f} \in \cup_{u\geq 0}\hat{\mathbf{I}}_u$. Then there exists $r \geq 0$ and a sequence $\mathrm{F}(u) \in P^d(\mathbb{Z}, X) \cap AP_1(\mathbb{Z}, X)$ such that $\mathrm{F}(r) = 0$, $\mathrm{f}(u) = \mathrm{F}(u)\ for\ u \geq r$ and $\mathrm{f}(u) = 0$ for $u < r$. Thus, for each $u \geq 0$ and $v \in \mathbb{Z}$, we have*

$$\big(Ţ(v)\widetilde{\mathrm{f}}\big)(u) = \begin{cases} e^{i\mu(u-v)}\xi(u-v)\mathrm{F}(u-v), & if\ u \geq v + r, \\ 0, & if\ 0 \leq u < v + r. \end{cases} \tag{3. 5}$$

The sequence $\mathbf{G}(u) = e^{-i\mu(u-v)}\xi(u-v)\mathrm{F}(u-v)$ is d-periodic and belongs to $AP_1(\mathbb{Z}, X)$. Moreover

$$\|\mathbf{G}(.)\|_1 \leq \|\xi(u-v)\| \Big\| \sum_{k=-\infty}^{k=\infty} e^{i\mu_k(u-v)}c_k(\mathrm{F})\Big\| \leq Me^{\nu u}\|\mathrm{F}(.)\|_1 < \infty$$

for some $M \geq 1$ and $w \in \mathbb{R}$. Thus $Ţ(u)\widetilde{\mathrm{f}} \in \hat{\mathbf{I}}$.

As an operator from $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$ to $\mathcal{B}(\mathbb{Z}, X)$, $Ţ(u)$ is linear. When $\widetilde{\mathrm{f}} = \alpha\widetilde{\mathrm{g}} + \beta\widetilde{\mathrm{h}} \in \mathbf{E}(\mathbb{Z}, X)$, with $\widetilde{\mathrm{g}}, \widetilde{\mathrm{h}} \in \hat{\mathbf{I}}$ and α, β are complex scalars, one has $Ţ(u)\widetilde{\mathrm{f}} = \alpha Ţ(u)\widetilde{\mathrm{g}} + \beta Ţ(u)\widetilde{\mathrm{h}}$. But $Ţ(u)\widetilde{\mathrm{g}}, Ţ(u)\widetilde{\mathrm{h}} \in \hat{\mathbf{I}}$ and therefore $Ţ(u)\widetilde{\mathrm{f}}$ belongs to $\mathbf{E}(\mathbb{Z}, X)$. Thus $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$ is invariant under the discrete evolution semigroup $\mathbf{Ţ}$.

4. RESULTS

Let $G = \mathfrak{T}(1) - I$, where $\mathfrak{T}(1)$ is called the algebraic generator of the discrete evolution semigroup $\mathbf{\mathfrak{T}}$. Thus, for discrete semigroups, the Taylor formula of order one is:

$$\mathfrak{T}(u)\widetilde{f} - \widetilde{f} = \sum_{k=0}^{u-1} \mathfrak{T}(k)G\widetilde{f}, \quad \text{for all } u \in \mathbb{Z}_+ \text{ with } u \geq 1, \tag{4. 6}$$

for all $\widetilde{f} \in X$.

Lemma 4.1. *Let* $\widetilde{\mathrm{f}}, \widetilde{y} \in \widetilde{\mathbf{E}}(\mathbb{Z}, X)$. *The following two statements are equivalent:*

- $G\widetilde{y} = -\widetilde{\mathrm{f}}$.
- $\widetilde{y}(u) = \sum\limits_{k=0}^{u} \xi(u,k)\widetilde{\mathrm{f}}(k)$ *for all* $n \in \mathbb{Z}_+$.

Proof. $(\mathbf{1} \Rightarrow \mathbf{2})$: Using the Taylor formula (4.6), we have

$$\mathfrak{T}(u)\widetilde{y} - \widetilde{y} = \sum_{k=0}^{u-1} \mathfrak{T}(k)G\widetilde{y} = -\sum_{k=0}^{u-1} \mathfrak{T}(k)\widetilde{f}.$$

Hence, for every $u \in \mathbb{Z}_+$,

$$\begin{aligned} \widetilde{y}(u) &= (\mathfrak{T}(u)\widetilde{y})(u) + \sum_{k=0}^{u-1} (\mathfrak{T}(k)\widetilde{f})(u) \\ &= \xi(u,0)\widetilde{y}(0) + \sum_{k=0}^{u-1} \xi(u, u-k)\widetilde{f}(u-k) \\ &= \sum_{k=0}^{u} \xi(u,k)\widetilde{f}(k). \end{aligned}$$

$(\mathbf{2} \Rightarrow \mathbf{1})$: For the converse implication as $G = \mathfrak{T}(1) - I$, thus

$$\begin{aligned} G\widetilde{y}(u) &= (\mathfrak{T}(1) - I)\widetilde{y}(u) \\ &= \mathfrak{T}(1)\widetilde{y}(u) - \widetilde{y}(u) \\ &= \xi(u, u-1)\widetilde{y}(u-1) - \widetilde{y}(u) \\ &= \xi(u, u-1)\sum_{k=0}^{u-1} \xi(u-1, k)\widetilde{f}(k) - \widetilde{y}(u) \\ &= \sum_{k=0}^{u-1} \xi(u,k)\widetilde{f}(k) - \sum_{k=0}^{u} \xi(u,k)\widetilde{f}(k) \\ &= -\widetilde{f}(u) \end{aligned}$$

The proof is complete.

In the next theorem we give our main result.

Theorem 4.2. *Let* $\mathcal{U}$ *be a q-periodic discrete evolution family acting on a Banach space* X *and let* $\mathbf{\mathfrak{T}}$ *be its associated discrete evolution semigroup on* $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$. *Denote by* G *the operator* $\mathfrak{T}(1) - I$, *where* $\mathfrak{T}(1)$ *is the algebraic generator of* $\mathbf{\mathfrak{T}}$. *The following are equivalent.*
(1) Ω *is uniformly exponentially stable.*
(2) $\mathbf{\mathfrak{T}}$ *is uniformly exponentially stable.*

(3) G is invertible.

(4) For each $\widetilde{\mathrm{f}} \in \widetilde{\mathbf{E}}(\mathbb{Z}, X)$, the series $\sum_{k=0}^{u} \xi(u-k,0)\widetilde{\mathrm{f}}(k)$ belongs to $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$.

(5) For each $\mathrm{f} \in P^d(\mathbb{Z}, X)$, the series $\sum_{k=0}^{u} \xi(u-k,0)\mathrm{f}(k)$ is bounded on $\mathbb{Z}_+$.

Proof. $(1) \Rightarrow (2)$. *Let N and v be two positive constants such that $\|\xi(u,v)\| \le Ne^{-\nu(u-v)}$ for all $u \ge v$. Then, for all $u \ge 0$ and any $\widetilde{\mathrm{f}}$ $belonging\ to$ $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$, one has*

$$\begin{aligned}\|\mathcal{T}(v)\widetilde{\mathrm{f}}\|_{\widetilde{\mathbf{E}}(\mathbb{Z},X)} &= \sup_{u\ge v}\|\xi(u-v,0)\widetilde{\mathrm{f}}(u-v)\| \\ &\le Ne^{-\nu(u-v)}\sup_{u\ge v}\|\widetilde{\mathrm{f}}(u-v)\| \\ &= Ne^{-\nu(u-v)}\|\widetilde{\mathrm{f}}\|_{\widetilde{\mathbf{E}}(\mathbb{Z},X)}.\end{aligned}$$

$(2) \Rightarrow (3)$. *It is well known that the evolution semigroup $\mathcal{T}$ is uniformly exponentially stable if and only if $r(\mathcal{T}(1)) < 1$. It means that 1 is not an eigenvalue of $\mathcal{T}(1)$ i.e. $1 \in \rho(\mathcal{T}(1))$ and so $G = \mathcal{T}(1) - I$ is invertible.*

$(3) \Rightarrow (4)$. *As G is invertible. Thus for every $\widetilde{\mathrm{f}} \in \widetilde{\mathbf{E}}(\mathbb{Z}, X)$ there exists $\widetilde{y} \in \widetilde{\mathbf{E}}(\mathbb{Z}, X)$ such that $G\widetilde{y} = -\widetilde{\mathrm{f}}$. Thus by Lemma 4.1 we get $\widetilde{y}(u) = \sum_{k=0}^{n} \xi(u-k,0)\widetilde{\mathrm{f}}(k)$ and by Lemma 3.1 $\sum_{k=0}^{u} \xi(u-k,0)\widetilde{\mathrm{f}}(k)$ belongs to $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$.*

$(4) \Rightarrow (5)$. *The series $\sum_{k=0}^{u} \mathbb{U}(u-k,0)\widetilde{\mathrm{f}}(k)$ is bounded because it belongs to $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$ which is a subset of $\mathcal{B}(\mathbb{Z}, X)$.*

$(5) \Rightarrow (1)$. *It can be seen as a direct consequence of Theorem 2.3.*

In terms of initial value problems, the result contained in Theorem 4.2 may be read as follows.

Corollary 4.3. *Ω is uniformly exponentially stable if and only if for each $\widetilde{\mathrm{f}} \in \widetilde{\mathbf{E}}(\mathbb{Z}, X)$, the solution of the problem*

$$\begin{aligned}\lambda_{u+1} &= \mathcal{A}(u)\lambda_u + \widetilde{\mathrm{f}}(u+1), \quad u \in \mathbb{Z}_+ \\ \lambda_0 &= 0,\end{aligned}$$

is bounded on $\mathbb{Z}_+$.

5. APPLICATIONS

An immediate consequence of Theorem 4.2 is the spectral mapping theorem for the discrete evolution semigroup $\mathcal{T}$ on $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$.

Theorem 5.1. *Let Ω be a d-periodic discrete evolution family acting on X and let $\mathcal{T}$ be its associated discrete evolution semigroup on $\widetilde{\mathbf{E}}(\mathbb{Z}, X)$. Denote by G the operator $\mathcal{T}(1) - I$, where $\mathcal{T}(1)$ is the algebraic generator of $\mathcal{T}$. Then*

$$\sigma(G) = \{z \in \mathbb{T} : Re(z) \le r(G)\}.$$

Proof. *It is well-known that $\rho(G) \supseteq \{z \in \mathbb{T} : Re(z) > r(G)\}$. To establish the converse inclusion, let $\alpha \in \rho(G)$ and $\mu \in \mathbb{T}$ with $Re(\mu) \ge Re(\alpha)$. We prove that $\mu \in \rho(G)$. Consider the discrete evolution family $\mathbb{U}_\alpha(u,v) = e^{-\alpha(u-v)}\mathbb{U}(u,v)$, where $u \ge v \ge 0$, whose associated discrete evolution semigroup is $\mathcal{T}_\alpha(u) = e^{-\alpha u}\mathcal{T}(n)$. Obviously, $\alpha I - G$ is the* *infinitesimal generator of* $\mathcal{T}_\alpha$. *Because* $\alpha I - G$ *is invertible and applying Theorem 4.2,* $\mathcal{Ç}_\alpha$*(and then* $\mathcal{T}_\mu$*) is uniformly exponentially stable. Therefore, by applying again Theorem 4.2,* $\mu \in \rho(G)$.

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