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Multivalent Functions with Respect to Symmetric Conjugate Points

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Abstract. Using convolution, classes of \( p \)-valent functions with respect to symmetric conjugate points are introduced. Integral representation and closure properties under convolution of general classes with respect to \((2j, k)\) symmetric points are investigated.

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1. Introduction, Definitions and Preliminaries

Let \( A_p \) be the class of functions analytic in the open unit disc \( U = \{ z : |z| < 1 \} \) of the form

\[
    f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \geq 1).
\]

and let \( A = A_1 \).

We denote by \( S^*, C, K \) and \( C^* \) the familiar subclasses of \( A \) consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in \( U \). Our favorite references of the field are [4, 5] which covers most of the topics in a lucid and economical style.

For the functions \( f(z) \) of the form (1.1) and \( g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \), the Hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[
    (f \ast g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.
\]
Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $k$ be a positive integer and $j = 0, 1, 2, \ldots (k - 1)$. A domain $D$ is said to be $(j, k)$-fold symmetric if a rotation of $D$ about the origin through an angle $2\pi j/k$ carries $D$ onto itself. A function $f \in \mathcal{A}$ is said to be $(j, k)$-symmetrical if for each $z \in \mathcal{U}$

$$f(\epsilon z) = \epsilon^j f(z),$$

where $\epsilon = \exp(2\pi i/k)$. The family of $(j, k)$-symmetrical functions will be denoted by $\mathcal{F}_k^j$. For every function $f$ defined on a symmetrical subset $\mathcal{U}$ of $\mathbb{C}$, there exits a unique sequence of $(j, k)$-symmetrical functions $f_{j,k}(z)$, $j = 0, 1, \ldots, k - 1$ such that

$$f = \sum_{j=0}^{k-1} f_{j,k}.$$

Moreover,

$$f_{j,k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{f(\epsilon^\nu z)}{\epsilon^{\nu j k}}, \quad (f \in \mathcal{A}_p; \ k = 1, 2, \ldots; \ j = 0, 1, 2, \ldots (k-1)).$$

This decomposition is a generalization the well known fact that each function defined on a symmetrical subset $\mathcal{U}$ of $\mathbb{C}$ can be uniquely represented as the sum of an even function and an odd functions (see Theorem 1 of [6]). We observe that $\mathcal{F}_k^j$, $\mathcal{F}_k^j$ and $\mathcal{F}_k^j$ are well-known families of odd functions, even functions and $k$-symmetrical functions respectively.

Further, it is obvious that $f_{j,k}(z)$ is a linear operator from $\mathcal{U}$ into $\mathcal{U}$. The notion of $(j, k)$-symmetrical functions was first introduced and studied by P. Liczberski and J. Polubiński in [6].

The class of $(j, k)$-symmetrical functions was extended to the class $(j, k)$-symmetrical conjugate functions in [8]. For fixed positive integers $j$ and $k$, let $f_{2j,k}(z)$ be defined by the following equality

$$f_{2j,k}(z) = \frac{1}{2k} \sum_{\nu=0}^{k-1} \left[ e^{-\nu j k} f(\epsilon^\nu z) + e^{\nu j k} f(\epsilon^{-\nu} z) \right], \quad (f \in \mathcal{A}_p).$$

If $\nu$ is an integer, then the following identities follow directly from (1.4):

$$f'_{2j,k}(z) = \frac{1}{2k} \sum_{\nu=0}^{k-1} \left[ e^{-\nu j k + \nu} f'(\epsilon^\nu z) + e^{\nu j k - \nu} f'(\epsilon^{-\nu} z) \right],$$

$$f''_{2j,k}(z) = \frac{1}{2k} \sum_{\nu=0}^{k-1} \left[ e^{-\nu j k + 2\nu} f''(\epsilon^\nu z) + e^{\nu j k - 2\nu} f''(\epsilon^{-\nu} z) \right],$$

and

$$f_{2j,k}(\epsilon^\nu z) = e^{\nu j k} f_{2j,k}(z), \quad f_{2j,k}(\epsilon^{-\nu} z) = f_{2j,k}(z),$$

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Motivated by the concept introduced by Sakaguchi in [10], recently several subclasses of analytic functions with respect to $k$-symmetric points were introduced and studied by various authors (see [1, 2, 12, 13, 15, 16]). In this paper, using Hadamard product (or convolution) new classes of functions in $\mathcal{A}_p$ with respect to $(j, k)$-symmetric points are introduced. Throughout this paper, unless otherwise mentioned the function $h$ is a convex
univalent function with a positive real part satisfying $h(0) = 1$.

We define the following.

**Definition 1.** A function $f \in \mathcal{A}_p$ is said to be in the class $S_{p}^{j,k}(h)$ if and only if it satisfies the condition

$$\frac{1}{p} \frac{zf'(z)}{f_{2j,k}(z)} < h(z),$$

where $f_{2j,k}(z) \neq 0$ and is defined by the equality (1.4). Similarly, we call the class $C_{p}^{j,k}(h)$ of functions $f \in \mathcal{A}_p$ with $f'_{2j,k}(z) \neq 0$ satisfying the subordination condition

$$\frac{1}{p} \frac{(zf'(z))'}{f_{2j,k}(z)} < h(z).$$

**Remark 2.** Since $f \in \mathcal{A}_p$, the condition $f_{2j,k}(z) \neq 0$ in the Definition 1 is essential as $h(z)$ is assumed to be a function with positive real part.

It is interesting to note that several well known and new subclasses of analytic functions can be obtained as special cases of $S_{p}^{j,k}(h)$ and $C_{p}^{j,k}(h)$. Here we list a few of them.

1. If we let $p = j = 1$ in definition 1, then the classes $S_{p}^{j,k}(h)$ and $C_{p}^{j,k}(h)$ reduces to $S_{p}^{k}(h)$ and $C_{p}^{k}(h)$ respectively. The function classes $S_{p}^{k}(h)$ and $C_{p}^{k}(h)$ were introduced by Wang in [14].

2. If $p = j = k = 1$ and $h(z) = \frac{1 + \beta z}{1 - \alpha \beta z}$ in definition 1, then the classes $S_{p}^{j,k}(h)$ and $C_{p}^{j,k}(h)$ reduces to

$$S_{p}^{k} = \left\{ f : f \in \mathcal{A}, \left| \frac{zf'(z)}{f(z) + f'(\bar{z})} - 1 \right| < \beta \left| \frac{zf'(z)}{f(z) + f'(\bar{z})} + 1 \right|, z \in \mathcal{U} \right\},$$

and

$$C_{p}^{k} = \left\{ f : f \in \mathcal{A}, \left| \frac{(zf'(z))'}{f(z) + f'(\bar{z})} - 1 \right| < \beta \left| \frac{zf'(z)}{f(z) + f'(\bar{z})} + 1 \right|, z \in \mathcal{U} \right\}$$

respectively. The class $S_{p}^{k}$ was introduced by Sudharsan et. al. in [11].

3. If $p = j = k = 1$ and $h(z) = \frac{1 + \beta}{1 - \alpha \beta}$ in definition 1, then the class $S_{p}^{j,k}(h)$ reduces to the class $S_{p}^{k}$ investigated by EL Ashwa and Thomas in [3].

**Definition 3.** A function $f \in \mathcal{A}_p$ is said to be in the class $K_{p}^{k}(h)$ if and only if it satisfies the condition

$$\frac{1}{p} \frac{zf'(z)}{\phi_{2j,k}(z)} < h(z),$$

where $\phi_{2j,k}(z) \in S_{p}^{j,k}(h)$ with $\phi_{2j,k}(z) \neq 0$ in $\mathcal{U}$.

Similarly, the class $QC_{p}^{j,k}(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the subordination condition

$$\frac{1}{p} \frac{(zf'(z))'}{\phi_{2j,k}(z)} < h(z),$$

for some $\phi_{2j,k}(z) \in S_{p}^{j,k}(h)$ with $\phi_{2j,k}(z) \neq 0$.
The general classes $S^{j,k}_p(g,h)$, $C^{j,k}_p(g,h)$, $K^{j,k}_p(g,h)$ and $QC^{j,k}_p(g,h)$ consist of functions $f \in A_p$ for which $f \ast g$ respectively belongs to $S^{j,k}_p(h)$, $C^{j,k}_p(h)$, $K^{j,k}_p(h)$ and $QC^{j,k}_p(h)$.

For a choice of the fixed function $g(z) = z^p/(1 - z)$, then the classes $S^{j,k}_p(g,h)$, $C^{j,k}_p(g,h)$, $K^{j,k}_p(g,h)$ and $QC^{j,k}_p(g,h)$ reduces respectively to $S^{j,k}_p(h)$, $C^{j,k}_p(h)$, $K^{j,k}_p(h)$ and $QC^{j,k}_p(h)$.

For $\gamma < 1$, the class $R_\gamma$ of prestarlike functions of order $\gamma$ is defined by

$$R_\gamma = \left\{ f \in A : f \ast \frac{z}{(1 - z)^{2 - 2\gamma}} \in S^*(\gamma) \right\},$$

while $R_1$ consists of $f \in A$ satisfying $Re f(z)/z > 1/2$. The well-known result that the classes of starlike functions of order $\gamma$ and convex functions of order $\gamma$ are closed under convolution with prestarlike functions of order $\gamma$ is a consequence of the following:

**Lemma 4.** [9] Let $\gamma < 1$, $\phi \in R_\gamma$ and $f \in S^*(\gamma)$. Then

$$\frac{\phi \ast (Hf)}{\phi \ast f}(U) \subset \overline{\sigma}(H(U)),$$

for any analytic function $H \in H(U)$, where $\overline{\sigma}(H(U))$ denote the closed convex hull $H(U)$.

Using Lemma 4, we have the following result.

**Lemma 5.** If $\phi(z)/z^{p-1} \in R_\gamma$ and $f(z) \in S^*(\gamma)$. Then

$$\frac{\phi \ast (Hf)}{\phi \ast f}(U) \subset \overline{\sigma}(H(U)),$$

for any analytic function $H \in H(U)$.

2. INCLUSION RELATIONSHIP

**Theorem 6.** Let $h$ be a convex univalent function satisfying

$$Re h(z) > 1 - \frac{1 - \gamma}{p}, \quad (0 \leq \gamma < 1),$$

and $\phi \in A_p$ with $\phi/z^{p-1} \in R_\gamma$. If $f \in S^{j,k}_p(g,h)$ for a fixed function $g$ in $A_p$, then $\phi \ast f \in S^{j,k}_p(h)$.

**Proof.** From the definition of $S^{j,k}_p(h)$, then for any fixed $z \in U$ we have

$$\frac{1}{p} \frac{zf'(z)}{f_{2j,k}(z)} \in h(U). \quad (2.1)$$

If we replace $z$ by $\nu z$ in (2.1), then (2.1) will be of the form

$$\frac{1}{p} \frac{e^{\nu z}f'(e^{\nu z})}{f_{2j,k}(e^{\nu z})} \in h(U), \quad (z \in U; \nu = 0, 1, 2, \ldots, k - 1). \quad (2.2)$$

From (2.2), we have

$$\frac{1}{p} \frac{e^{\nu z}f'(e^{\nu z})}{f_{2j,k}(e^{\nu z})} \in h(U), \quad (z \in U; \nu = 0, 1, 2, \ldots, k - 1). \quad (2.3)$$

Using the equality (1.6), (2.2) and (2.3) can be rewritten as
\[
\frac{1}{p} \frac{e^{\nu-pj}z f'(e^{\nu}z)}{f_{2j,k}(z)} \in h(\mathcal{U}), \quad (z \in \mathcal{U}; \, \nu = 0, 1, 2, \ldots, k-1),
\]
and
\[
\frac{1}{p} \frac{e^{\nu pj-\nu} z f'(e^{\nu}\overline{z})}{f_{2j,k}(z)} \in h(\mathcal{U}), \quad (z \in \mathcal{U}; \, \nu = 0, 1, 2, \ldots, k-1).
\]

Adding (2.4) and (2.5), we get
\[
\frac{1}{p} \frac{z \left[ e^{\nu-pj} f'(e^{\nu}z) + e^{\nu pj-\nu} f'(e^{\nu}\overline{z}) \right]}{f_{2j,k}(z)} \in h(\mathcal{U}), \quad (z \in \mathcal{U}).
\]

Let \( \nu = 0, 1, 2, \ldots, k-1 \) in (2.6) respectively and summing them, we get
\[
\frac{1}{p} \frac{z \left[ \sum_{\nu=0}^{k-1} e^{\nu-pj} f'(e^{\nu}z) + e^{\nu pj-\nu} f'(e^{\nu}\overline{z}) \right]}{f_{2j,k}(z)} \in h(\mathcal{U}), \quad (z \in \mathcal{U}).
\]

Or equivalently,
\[
\frac{1}{p} \frac{zf_{2j,k}(z)}{f_{2j,k}(z)} \in h(\mathcal{U}), \quad (z \in \mathcal{U}),
\]
that is \( f_{2j,k}(z) \in \mathcal{S}^{j,k}_p(h) \).

Set \( H(z) \) and \( \psi(z) \) by
\[
H(z) = \frac{zf'(z)}{pf_{2j,k}(z)} \quad \text{and} \quad \psi_{2j,k}(z) = \frac{f_{2j,k}(z)}{zp^{1}}.
\]

Now \( \Re h(z) > 1 - \frac{1}{p} \) yields
\[
\Re \frac{z \psi'_{2j,k}(z)}{\psi_{2j,k}(z)} = \Re \frac{zf_{2j,k}(z)}{f_{2j,k}(z)} = (p-1) > \gamma.
\]

Inequality (2.7) shows that the function \( \psi_{2j,k}(z) \) is starlike of order \( \gamma \), which we denote by \( \mathcal{S}^{*}(\gamma) \). A simple computation shows that
\[
\frac{z (\phi * f)'(z)}{p(\phi * f)_{2j,k}(z)} = \frac{\left( \phi * \left( p^{-1}zf' \right) \right) (z)}{\left( \phi * f_{2j,k} \right) (z)} = \frac{(\phi * (H f_{2j,k})) (z)}{(\phi * f_{2j,k}) (z)}
\]

Since \( \phi/z^{p-1} \in \mathcal{R}_{\gamma} \) and \( \psi_{2j,k} \in \mathcal{S}^{*}(\gamma) \), Lemma 5 yields
\[
\frac{(\phi * (H f_{2j,k})) (z)}{(\phi * f_{2j,k}) (z)} \in \overline{\mathcal{R}}(H(\mathcal{U})).
\]

The subordination \( H \prec h \) implies
\[
\frac{z (\phi * f)'(z)}{p(\phi * f)_{2j,k}(z)} \prec h(z).
\]

Thus \( \phi * f \in \mathcal{S}^{j,k}_p(h) \). That is
\[
f \in \mathcal{S}^{j,k}_p(h) \implies f * g \in \mathcal{S}^{j,k}_p(h) \implies \phi * f \in \mathcal{S}^{j,k}_p(h),
\]
or equivalently \( \phi * f \in \mathcal{S}^{j,k}_p(g, h) \). \( \square \)
Remark 7. Using the condition (1.7) together with the result \( f_{j, k}(z) \in S_{p}^{j, k}(h) \) shows that the functions in \( S_{p}^{j, k}(h) \) are contained in \( K_{p}^{j, k}(h) \). In general, \( S_{p}^{j, k}(g, h) \subset K_{p}^{j, k}(g, h) \).

**Theorem 8.** Let \( h \) be a convex univalent function satisfying
\[
Re h(z) > 1 - \frac{1-\gamma}{p}, \quad (0 \leq \gamma < 1),
\]
and \( \phi \in A_{p} \), with \( \phi/z^{p-1} \in \mathcal{R}_{\gamma} \). If \( f \in C_{p}^{j, k}(g, h) \) for a fixed function \( g \) in \( A_{p} \), then \( \phi \ast f \in C_{p}^{j, k}(g, h) \).

**Proof.** From the identity
\[
\frac{(z(g * f)'(z))'}{p(g * f)'_{2j, k}(z)} = \frac{z(g * p^{-1}z f)'(z)}{p(g * p^{-1}z f)'_{2j, k}(z)},
\]
we have \( f \in C_{p}^{j, k}(g, h) \) if and only if \( \frac{z^{p}}{p} \in S_{p}^{j, k}(g, h) \) and by Theorem 6 it follows that \( \phi \ast \frac{z^{p}}{p} \in S_{p}^{j, k}(g, h) \). Hence \( \phi \ast f \in C_{p}^{j, k}(g, h) \). \( \square \)

**Remark 9.** Analogous to the result in Theorem 6, it can be proved that \( f_{j, k}(z) \in C_{p}^{j, k}(h) \). Using this result together with condition (1.7) shows that the functions in \( C_{p}^{j, k}(h) \) are contained in \( QC_{p}^{j, k}(h) \). In general, \( C_{p}^{j, k}(g, h) \subset QC_{p}^{j, k}(g, h) \).

Using the arguments similar to those detailed in Theorem 6 and Theorem 8, we can prove the following two Theorems. We therefore, choose to omit the details involved.

**Theorem 10.** Let \( h \) be a convex univalent function satisfying
\[
Re h(z) > 1 - \frac{1-\gamma}{p}, \quad (0 \leq \gamma < 1),
\]
and \( \phi \in A_{p} \) with \( \phi(z)/z^{p-1} \in \mathcal{R}_{\gamma} \). If \( f \in K_{p}^{j, k}(g, h) \), then \( \phi \ast f \in K_{p}^{j, k}(g, h) \).

**Theorem 11.** Let \( h \) be a convex univalent function satisfying
\[
Re h(z) > 1 - \frac{1-\gamma}{p}, \quad (0 \leq \gamma < 1),
\]
and \( \phi \in A_{p} \) with \( \phi(z)/z^{p-1} \in \mathcal{R}_{\gamma} \). If \( f \in QC_{p}^{j, k}(g, h) \), then \( \phi \ast f \in QC_{p}^{j, k}(g, h) \).

3. INTEGRAL REPRESENTATION

**Theorem 12.** Let \( f \in S_{p}^{j, k}(g, h) \), then we have
\[
s_{2j, k}(z) = z^{p} \exp \left\{ \frac{p}{2k} \sum_{\nu=0}^{k-1} \int_{0}^{1} \frac{1}{\zeta} \left[ \phi(w(\zeta)) + \phi(w(\zeta)) - 2 \right] d\zeta \right\}, \quad (3.1)
\]
where \( s_{2j, k}(z) = (f * g)_{j, k}(z) \), and \( w(z) \) is analytic in \( U \) with \( w(0) = 0, |w(z)| < 1 \).

**Proof.** From the definition of \( S_{p}^{j, k}(g, h) \), we have
\[
\frac{z(f * g)'(z)}{p s_{2j, k}(z)} = \phi(w(z)), \quad (3.2)
\]
where \( w(z) \) is analytic in \( U \) and \( w(0) = 0, |w(z)| < 1 \). Substituting \( z \) by \( \zeta \) in the equality (3.2) respectively \( (\nu = 0, 1, 2, \ldots, k-1, \zeta = 1) \), we have
\[
\frac{\zeta(f * g)'(\zeta)}{p s_{2j, k}(\zeta)} = \phi(w(\zeta)) \quad (3.3)
\]
On simple computation, we get
\[ \frac{z^{p\nu} (f \ast g)(\epsilon^{\nu} \zeta)}{p \, s_{2j, k}(\epsilon^{\nu} \zeta)} = \phi(w(\epsilon^{\nu} \zeta)). \] (3.4)

Proceeding as in Theorem 6, we have
\[ \frac{z s'_{2j, k}(z)}{s_{2j, k}(z)} = \frac{1}{2k} \left[ \frac{1}{z} \phi(w(\epsilon^{\nu} z)) + \phi(\bar{w}(\epsilon^{\nu} z)) \right], \]
which can be rewritten as
\[ \frac{s_{2j, k}(z)}{s_{2j, k}(z)} = \frac{p}{z} \left[ \frac{1}{z} \phi(w(\epsilon^{\nu} z)) + \phi(\bar{w}(\epsilon^{\nu} z)) - 2 \right]. \]
Integrating this equality, we get
\[ \log \left\{ \frac{s_{2j, k}(z)}{z^p} \right\} = \frac{p}{2k} \sum_{\nu=0}^{k-1} \int_0^z \frac{1}{\zeta} \left[ \phi(w(\epsilon^{\nu} \zeta)) + \phi(\bar{w}(\epsilon^{\nu} \zeta)) - 2 \right] d\zeta, \]
or equivalently,
\[ s_{2j, k}(z) = z^p \exp \left\{ \frac{p}{2k} \sum_{\nu=0}^{k-1} \int_0^z \frac{1}{\zeta} \left[ \phi(w(\epsilon^{\nu} \zeta)) + \phi(\bar{w}(\epsilon^{\nu} \zeta)) - 2 \right] d\zeta \right\}. \]
This completes the proof of Theorem 12.

**Theorem 13.** Let \( f \in S^{j, k}_p(g, h) \), then we have
\[ s(z) = \int_0^z p z^{p-1} \exp \left\{ \frac{p}{2k} \sum_{\nu=0}^{k-1} \int_0^z \frac{1}{\zeta} \left[ \phi(w(\epsilon^{\nu} \zeta)) + \phi(\bar{w}(\epsilon^{\nu} \zeta)) - 2 \right] d\zeta \right\} \cdot \phi(w(z)) dz \]
where \( s(z) = (f \ast g)(z) \) and \( w(z) \) is analytic in \( \mathcal{U} \) with \( w(0) = 0, \ |w(z)| < 1 \).

**Proof.** Let \( f \in S^{j, k}_p(g, h) \). Then from the definition, we have
\[ s'(z) = \frac{p \, s_{2j, k}(z)}{z} \cdot \phi(w(z)) \]
\[ = p z^{p-1} \exp \left\{ \frac{p}{2k} \sum_{\nu=0}^{k-1} \int_0^z \frac{1}{\zeta} \left[ \phi(w(\epsilon^{\nu} \zeta)) + \phi(\bar{w}(\epsilon^{\nu} \zeta)) - 2 \right] d\zeta \right\} \cdot \phi(w(z)). \]
Integrating the above equality will prove the assertions of the theorem.

**Theorem 14.** Let \( f \in C^{j, k}_p(g, h) \), then we have
\[ s_{2j, k}(z) = \int_0^z z^{p-1} \exp \left\{ \frac{p}{2k} \sum_{\nu=0}^{k-1} \int_0^z \frac{1}{\zeta} \left[ \phi(w(\epsilon^{\nu} \zeta)) + \phi(\bar{w}(\epsilon^{\nu} \zeta)) - 2 \right] d\zeta \right\} dz, \]
where \( s_{2j, k}(z) = (f \ast g)_{2j, k}(z) \), and \( w(z) \) is analytic in \( \mathcal{U} \) with \( w(0) = 0, \ |w(z)| < 1 \).

**Theorem 15.** Let \( f \in C^{j, k}_p(g, h) \), then we have
\[ s(z) = \int_0^z p \int_0^z z^{p-1} \exp \left\{ \frac{p}{2k} \sum_{\nu=0}^{k-1} \int_0^z \frac{1}{\zeta} \left[ \phi(w(\epsilon^{\nu} \zeta)) + \phi(\bar{w}(\epsilon^{\nu} \zeta)) - 2 \right] d\zeta \right\} dz \, dy, \]
where \( s(z) = (f \ast g)(z) \) and \( w(z) \) is analytic in \( \mathcal{U} \) with \( w(0) = 0, \ |w(z)| < 1 \).
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