Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(10)(2020) pp. 1-23

Common Fixed Point of Multivalued Mappings in Ordered Dislocated Quasi G-Metric Spaces

Abdullah Shoaib
Department of Mathematics and Statistics,
Riphah International University, Islamabad, Pakistan,
Email: abdullahshoaib15@yahoo.com

Saiqa Mustafa
Department of Mathematics and Statistics,
Riphah International University, Islamabad, Pakistan,
Email: saiqamustufa88@gmail.com

Aqeel Shahzad
Department of Mathematics and Statistics,
Riphah International University, Islamabad, Pakistan,
Email: aqeel4all84@gmail.com

Received: 02 December, 2019 / Accepted: 22 June, 2020 / Published online: 10 October, 2020

Abstract.: The motivation of this activity is to introduce the notion of bi p-sequentially complete ordered dislocated quasi G-metric spaces and to obtain fixed point results for a pair of multivalued mappings satisfying generalized contractions on the intersection of an open ball and a sequence in these spaces. An example has been built to express the novelty of results. These results generalize and extend the results of Altun et.al (J. Funct. Spaces, Article ID 6759320, 2016).

AMS (MOS) Subject Classification Codes: 47H10; 54H25
Key Words: common fixed point; bi p-sequentially complete dislocated quasi G-metric spaces; open ball; multivalued mappings.

1. Introduction and Preliminaries

The field of fixed point theory covers both pure and applied mathematics. Fixed point theory is a special branch of functional analysis and its results are used to find the solution of different mathematical models. A multivalued mapping B from C to the subsets of C

has a fixed point $k \in C$, if $k \in Bk$. If we take elements of C instead of subsets of C, then B represents a singlevalued mappings from C to C. A singlevalued mapping $B : C \longrightarrow C$ has a fixed point $y \in C$, if $y = By$.

The notion of metric spaces in analysis plays an important role in applied and pure sciences such as biology, computer science and physics. One of the generalization of metric is G metric, which had been developed by Sims and Mustafa [20]. Karapınar et al. [18] and singh et al. [37] discussed fixed point results in G metric spaces, which distinguish G metric spaces from other spaces. Their results can not be established from the corresponding results in other spaces. For more results on G metric spaces see [1, 2, 9, 10, 11, 12, 15, 17, 19, 26, 35, 39]. Another remarkable generalization of metric is dislocated quasi metric. Several fixed point results appeared in dislocated quasi metric spaces or quasi-metric-like spaces (see [3, 4, 8, 13, 28, 24, 40, 38]). Recently, the idea about generalization of both G-metric spaces and dislocated metric spaces in terms of dislocated quasi G-metric space was introduced by Shoaib et al. (see [29, 31, 34, 36]).

Ran and Reurings [23] gave a fixed point result with an order and obtained solution to matrix equations as an application. Nieto et al. [21] gave an extension to the result in [23] for ordered mappings and used it to give a unique solution for ODE with periodic boundary conditions. Altun et al. [5] introduced a new approach to common fixed point of mappings, satisfying a generalized contraction endowed with a new restriction of order, in a complete ordered metric space. For more results endowed with order see [6, 7, 13, 14, 16, 22, 25, 27, 30].

Shoaib et al. (see [32, 33]) discussed some results on an intersection of a closed ball and a sequence. In this paper, we have obtained fixed point results for multivalued mappings satisfying generalized contractions on the intersection of an open ball and a sequence in bi p-sequentially complete ordered dislocated quasi G-metric spaces. An example has been built to express the novelty of results. These results generalize and extend the results of Altun et.al [5]. The following definitions will be needed in the sequel.

Definition 1.1 Let χ be a nonempty set and $D_q : \chi \times \chi \times \chi \to [0, \infty)$ be a function, called a dislocated quasi G-metric (or simply D_q-metric) if the following conditions hold for any $j, p, l, d \in \chi$:
i) $D_q(j, p, l) = D_q(p, j, l) = D_q(l, j, p) = D_q(j, l, p) = D_q(p, l, j) = D_q(l, j, p) = 0$, then $j = p = l$;
ii) $D_q(j, p, l) \leq D_q(j, d, d) + D_q(d, p, l)$;
iii) $D_q(j, p, l) \leq D_q(j, p, d) + D_q(d, d, l)$.
The pair (χ, D_q) is called a dislocated quasi G-metric space. It is clear that if $D_q(j, p, l) = D_q(p, j, l) = D_q(l, j, p) = D_q(j, l, p) = D_q(p, l, j) = D_q(l, j, p) = 0$
then from (i) $j = p = l$. But if $j = p = l$, then $D_q(j, p, l)$ may not be 0.
It is observed that if $D_q(j, p, l) = D_q(p, j, l) = D_q(l, j, p) = D_q(j, l, p) = D_q(p, l, j) = D_q(l, j, p)$ for all $j, p, l \in \chi$,
then (χ, D_q) becomes a dislocated G-metric space.

Definition 1.2 Let (χ, D_q) be a dislocated quasi G-metric space and for $j_0 \in \chi, r > 0$, then $B_{D_q}(j_0, r) = \{p \in \chi : D_q(j_0, p, p) < r$ and $D_q(p, p, j_0) < r\}$ and $\overline{B_{D_q}(j_0, r)} = \{p \in \chi : D_q(j_0, p, p) \leqslant r$ and $D_q(p, p, j_0) \leqslant r\}$ are bi open ball and bi closed ball with centre j_0 and radius r in (χ, D_q) respectively.

Shoaib et al. [34] introduced the notion of right p-Cauchy sequence and right p-sequentially completeness in dislocated quasi G_d-metric space. Now, we introduce bi p-Cauchy sequence, bi convergent sequence and bi p-sequentially completeness in dislocated quasi G-metric space.

Definition 1.3 Let (χ, D_q) be a dislocated quasi G- metric space.
i) A sequence $\{j_n\}$ in (χ, D_q) is called left (respectively right) p-Cauchy if for any $\epsilon > 0$, $\exists\, n_0 \in N$ such that $\forall\, n > s \geq n_0$, $D_q(j_s, j_n, j_n) < \epsilon$ (respectively $D_q(j_n, j_n, j_s) < \epsilon$). If a sequence is both left and right p-Cauchy then it is called bi p-Cauchy sequence.
ii) A sequence $\{j_n\}$ dislocated quasi bi G-converges (for short bi D_q -converges) to j if $\lim\limits_{n\to\infty} D_q(j_n, j, j) = \lim\limits_{n\to\infty} D_q(j, j, j_n) = 0$.
iii) (χ, D_q) is called bi p-sequentially complete if every bi p-Cauchy sequence in χ, bi D_q converges to a point $j \in \chi$ such that $D_q(j, j, j) = 0$.

Definition 1.4 Let (χ, D_q) be a dislocated quasi G- metric space. Let K be a nonempty subset of χ and let $j \in \chi$. An element $k_0 \in K$ is called a best approximation in K if

$$D_q(j, K, K) = D_q(j, k_0, k_0), \text{ where } D_q(j, K, K) = \inf_{k\in K} D_q(j, k, k)$$

and

$$D_q(K, K, j) = D_q(k_0, k_0, j), \text{ where } D_q(K, K, j) = \inf_{k\in K} D_q(k, k, j).$$

If each $j \in \chi$ has at least one best approximation in K, then K is called a proximinal set. We denote $P(\chi)$ be the set of all proximinal subsets of χ.

Definition 1.5 The function $H_{D_q} : P(\chi) \times P(\chi) \times P(\chi) \to \mathbb{R}^+ \cup \{0\}$, defined as

$$H_{D_q}(V, W, W) = \max\{\sup_{v\in V} D_q(v, W, W), \sup_{w\in W} D_q(V, w, w)\}.$$

is called dislocated quasi Hausdorff G-metric on $P(\chi)$. Also $(P(\chi), H_{D_q})$ is known as dislocated quasi Hausdorff G- metric space.

Lemma 1.6 Let (χ, D_q) be a dislocated quasi G- metric space. Let $(P(\chi), H_{D_q})$ be a dislocated quasi Hausdorff G-metric space on $P(\chi)$.Then, for all $V, W \in P(\chi)$ and for each $v \in V$, there exists $w_v \in W$, such that $H_{D_q}(V, W, W) \geq D_q(v, w_v, w_v)$ and $H_{D_q}(W, W, V) \geq D_q(w_v, w_v, v)$.

Lemma 1.7 Every closed ball $\hat{Y}$ in a bi p-sequentially complete dislocated quasi G- metric space χ is bi p-sequentially complete.

Definition 1.8 [5] Let $\psi \in \Psi$ and Ψ denotes the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions:
(Ψ^1) ψ is non-decreasing.
(Ψ^2) For all $t > 0$, we have

$$\mu_0(t) = \sum_{p=0}^{\infty} \psi^p(t) < \infty.$$

Where ψ^p is the p^{th} iterate of ψ. The function $\psi \in \Psi$ is called comparison function.

Lemma 1.9 [5] Let $\psi \in \Psi$. Then
i) $\psi(t) < t, \forall\, t > 0$.
ii) $\psi(0) = 0$.

Definition 1.10 [31] $(\chi, \preceq, D_q)$ is called an ordered dislocated quasi G- metric space, if

i) (χ, D_q) is dislocated quasi G- metric space.
ii) $\preceq$ is a partial order on χ.

2. MAIN RESULT

Let (χ, D_q) be a dislocated quasi G- metric space, $j_0 \in \chi$ and $T : \chi \to P(\chi)$ be a multivalued mapping on χ. As Tj_0 is a proximinal set, then there exists $j_1 \in Tj_0$ such that$D_q(j_0, Tj_0, Tj_0) = D_q(j_0, j_1, j_1)$ and $D_q(Tj_0, Tj_0, j_0) = D_q(j_1, j_1, j_0)$. Now, for $j_1 \in \chi$, there exist $j_2 \in Tj_1$ be such that $D_q(j_1, Tj_1 Tj_1,) = D_q(j_1, j_2, j_2)$ and $D_q(Tj_1, Tj_1, j_1) = D_q(j_2, j_2, j_1)$. Continuing this process, we construct a sequence j_n of points in χ such that $j_{n+1} \in Tj_n$, $D_q(j_n, Tj_n, Tj_n) = D_q(j_n, j_{n+1}, j_{n+1},)$ and $D_q(Tj_n, Tj_n, j_n) = D_q(j_{n+1}, j_{n+1}, j_n)$. We denote this iterative sequence $\{\chi T(j_n)\}$ and say that $\{\chi T(j_n)\}$ is a sequence in χ generated by j_0.

Theorem 2.1 Let $(\chi, \preceq, D_q)$ be an ordered bi p-sequentially complete dislocated quasi G metric space and $S, T : \chi \to P(\chi)$ be the multivalued mappings. Suppose that the following assertions hold:

(i) There exists a function $\mu \in \Psi$, $j_0 \in \chi$ and $r > 0$ such that

$$\max\{H_{D_q}(Tj, Tp, Tp), H_{D_q}(Tp, Tp, Tj)\} \leq \mu(W(j, p, p)),$$

for all $j, p \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}$ with $j \succeq Sj$, $p \preceq Sp$, where

$$W(j, p, p) = \max\{D_q(j, p, p), D_q(j, Tj, Tj), D_q(p, Tp, Tp)\}.$$

(ii) If $j \in B_{D_q}(j_0, r)$, $D_q(j, Tj, Tj) = D_q(j, p, p)$ and $D_q(Tj, Tj, j) = D_q(p, p, j)$, then

$$(a) \text{ If } j \preceq Sj, \text{ then } p \succeq Sp \quad (b) \text{ If } j \succeq Sj, \text{ then } p \preceq Sp.$$

(iii)The set $C(S) = \left\{ j : j \preceq Sj \text{ and } j \in B_{D_q}(j_0, r) \right\}$ is closed and contains j_0.

(iv)

$$\sum_{i=0}^{n} \max\{\mu^i(D_q(j_1, j_1, j_0), \mu^i(D_q(j_0, j_1, j_1)) < r. \text{ for all } n \in \mathbb{N}.$$

Then the subsequence $\{j_{2n}\}$ of $\{\chi T(j_n)\}$ is a sequence in $C(S)$ and $\{j_{2n}\} \to j^* \in C(S)$ and $D_q(j^*, j^*, j^*) = 0$. Also, if the inequality (i) holds for j^*. Then S and T have a common fixed point j^* in $B_{D_q}(j_0, r)$.

Proof. As j_0 be an element of $C(S)$, from condition (iii) $j_0 \preceq Sj_0$. Consider the sequence $\{\chi T(j_n)\}$, then there exists $j_1 \in Tj_0$ such that

$$D_q\,(j_0,\ Tj_0, Tj_0) = D_q\,(j_0,\ j_1,\ j_1) \text{ and } D_q\,(Tj_0, Tj_0, j_0) = D_q(j_1, j_1, j_0).$$

From condition (ii) $j_1 \succeq Sj_1$. From condition (iv), we have

$$\begin{aligned} & \max\{D_q(j_1, j_1, j_0), D_q(j_0, j_1, j_1)\} \\ \leq \quad & \sum_{i=0}^{j} \max\{\mu^i(D_q(j_1, j_1, j_0), \mu^i(D_q(j_0, j_1, j_1))\} < r. \end{aligned}$$

It follows that, $D_q(j_0, j_1, j_1) < r$ and $D_q(j_1, j_1, j_0) < r$. So, we have $j_1 \in B_{D_q}(j_0, r)$. Also,

$$D_q(j_1, Tj_1 Tj_1,) = D_q(j_1, j_2, j_2) \text{ and } D_q(Tj_1, Tj_1, j_1) = D_q(j_2, j_2, j_1).$$

As $j_1 \succeq Sj_1$, so from condition (ii), we have $j_2 \preceq Sj_2$. By triangular inequality, we have

$$D_q(j_0, j_2, j_2) \leq D_q(j_0, j_1, j_1) + D_q(j_1, j_2, j_2). \tag{2.1}$$

Now, by Lemma 1.6, we have

$$\begin{aligned} D_q(j_1, j_2, j_2) &\leq H_{Gd_q}(Tj_0, Tj_1, Tj_1) \\ &\leq \max\{H_{D_q}(Tj_0, Tj_1, Tj_1), H_{D_q}(Tj_1, Tj_1, Tj_0)\}. \end{aligned}$$

As $j_0, j_1 \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}$, $j_1 \succeq Sj_1$ and $j_0 \preceq Sj_0$, then by (i), we have

$$\begin{aligned} D_q(j_1, j_2, j_2) &\leq \mu(W(j_1, j_1, j_0)) \\ &= \mu(\max\{D_q(j_1, j_1, j_0), D_q(j_1, Tj_1, Tj_1), D_q(j_0, Tj_0, Tj_0)\}) \\ &\leq \mu(\max\{D_q(j_1, j_1, j_0), D_q(j_1, j_2, j_2), D_q(j_0, j_1, j_1)\}). \end{aligned}$$

If $\max\{D_q(j_1, j_1, j_0), D_q(j_1, j_2, j_2), D_q(j_0, j_1, j_1)\} = D_q(j_1, j_2, j_2)$ then a contradiction arise by the fact $\mu(\check{t}) < \check{t}$, so we have

$$D_q(j_1, j_2, j_2) \leq \mu(\max\{D_q(j_1, j_1, j_0), D_q(j_0, j_1, j_1)\}). \tag{2.2}$$

Now, inequality (2.1) implies

$$\begin{aligned} D_q(j_0, j_2, j_2) &\leq D_q(j_0, j_1, j_1) + \mu(\max\{D_q(j_1, j_1, j_0), D_q(j_0, j_1, j_1)\}) \\ &\leq \max\{D_q(j_1, j_1, j_0), D_q(j_0, j_1, j_1)\} \\ &\quad + \mu(\max\{D_q(j_1, j_1, j_0), D_q(j_0, j_1, j_1)\}) \\ &= \max\{D_q(j_1, j_1, j_0), D_q(j_0, j_1, j_1)\} \\ &\quad + \max\{\mu(D_q(j_1, j_1, j_0)), \mu(D_q(j_0, j_1, j_1))\} \\ &= \sum_{i=0}^{1} \max\{\mu^i(D_q(j_1, j_1, j_0), \mu^i(D_q(j_0, j_1, j_1))\} < r. \end{aligned}$$

By using (iv),we have

$$D_q(j_0, j_2, j_2) < r. \tag{2.3}$$

Now, by triangular inequality, we have

$$D_q(j_2, j_2, j_0) \leq D_q(j_2, j_2, j_1) + D_q(j_1, j_1, j_0). \tag{2.4}$$

Now, by Lemma 1.6, we have

$$\begin{aligned} D_q(j_2, j_2, j_1) &\leq H_{D_q}(Tj_1, Tj_1, Tj_0) \\ &\leq \max\{H_{D_q}(Tj_1, Tj_1, Tj_0), H_{D_q}(Tj_0, Tj_1, Tj_1)\}. \end{aligned}$$

As $j_1, j_0 \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}$, $j_1 \succeq Sj_1$ and $j_0 \preceq Sj_0$, then by (i), we have

$$\begin{aligned} D_q(j_2, j_2, j_1) &\leq \mu(W(j_1, j_1, j_0)) \\ &= \mu(\max\{D_q(j_1, j_1, j_0), D_q(j_1, Tj_1, Tj_1), D_q(j_0, Tj_0, Tj_0)\}) \\ &\leq \mu(\max\{D_q(j_1, j_1, j_0), D_q(j_1, j_2, j_2), D_q(j_0, j_1, j_1)\}). \end{aligned}$$

If $\max\{D_q(j_1, j_1, j_0), D_q(j_1, j_2, j_2), D_q(j_0, j_1, j_1)\} = D_q(j_1, j_2, j_2)$, then by (2.2), we have

$$D_q(j_2, j_2, j_1) \leq \mu(\max\left\{\begin{array}{c} D_q(j_1, j_1, j_0), \\ \mu(\max\{D_q(j_1, j_1, j_0), D_q(j_0, j_1, j_1)\}), D_q(j_0, j_1, j_1) \end{array}\right\}).$$

If $\max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1)\} = D_q(j_0, j_{1,}j_1)$, then, we have

$$\begin{aligned} D_q(j_2, j_2, j_1) &\leq \mu(\max\{D_q(j_1, j_{1,}j_0), \mu(D_q(j_0, j_{1,}j_1)), D_q(j_0, j_{1,}j_1)\}) \\ &\leq \mu(\max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1), D_q(j_0, j_{1,}j_1)\}) \\ &= \mu(\max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1)\}). \end{aligned}$$

Similarly, if $\max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1)\} = D_q(j_1, j_{1,}j_0)$, then, we have

$$D_q(j_2, j_2, j_1) \leq \mu(\max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1)\}).$$

Now, inequality (2.4) implies

$$\begin{aligned} D_q(j_2, j_{2,}j_0) &\leq D_q(j_1, j_1, j_0) + \mu(\max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1)\}) \\ &\leq \max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1)\} + \\ &\quad \mu(\max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1)\}) \\ &= \max\{D_q(j_1, j_{1,}j_0), D_q(j_0, j_{1,}j_1)\} + \\ &\quad \max\{\mu(D_q(j_1, j_{1,}j_0)), \mu(D_q(j_0, j_{1,}j_1))\} \\ &= \sum_{i=0}^{1} \max\{\mu^i(D_q(j_1, j_{1,}j_0), \mu^i(D_q(j_0, j_{1,}j_1))\} < r. \end{aligned}$$

It follows that, $D_q(j_2, j_2, j_0) < r$. By (2.3) $D_q(j_0, j_2, j_2) < r$. So, $j_2 \in B_{D_q}(j_0, r)$. Also,

$$D_q(j_2, Tj_2, Tj_2) = D_q(j_2, j_3, j_3) \text{ and } D_q(Tj_2, Tj_2, j_2) = D_q(j_3, j_3, j_2).$$

As $j_2 \preceq Sj_2$, so from condition (ii), we have $j_3 \succeq Sj_3$. Let $j_3, \ldots, j_{2i} \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}$, $j_{j'} \preceq Sj_j$ and $j_{j'-1} \succeq S_{j'-1}$for some $j' \in \mathbb{N}$, where $j' = 2i$, $i = 2, 3 \ldots \frac{j'}{2}$. Now, by Lemma 1.6, we obtain

$$\begin{aligned} D_q(j_{2i}, j_{2i+1}, j_{2i+1}) &\leq H_{D_q}(Tj_{2i-1,}Tj_{2i}, Tj_{2i}) \\ &\leq \max\{H_{D_q}(Tj_{2i-1}, Tj_{2i}, Tj_{2i}), H_{D_q}(Tj_{2i}, Tj_{2i}, Tj_{2i-1})\}. \end{aligned}$$

As $j_{2i-1}, j_{2i} \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}, j_{2i-1} \succeq Sj_{2i-1}$, $j_{2i} \preceq Sj_{2i}$, then by (i), we have

$$\begin{aligned} D_q(j_{2i}, j_{2i+1}, j_{2i+1}) &\leq \mu(W(j_{2i-1,}j_{2i}, j_{2i})) \\ D_q(j_{2i}, j_{2i+1}, j_{2i+1}) &= \mu\left(\max\left\{\begin{array}{c} D_q(j_{2i-1,}j_{2i}, j_{2i}), D_q(j_{2i-1}, Tx_{2i-1}, Tx_{2i-1}), \\ D_q(j_{2i}, Tx_{2i}, Tx_{2i}). \end{array}\right\}\right) \\ &\leq \mu\left(\max\left\{\begin{array}{c} D_q(j_{2i-1,}j_{2i}, j_{2i}), D_q(j_{2i-1}, j_{2i}, j_{2i}), \\ D_q(j_{2i}, j_{2i+1}, j_{2i+1}). \end{array}\right\}\right) \\ &\leq \mu(\max\{D_q(j_{2i-1,}j_{2i}, j_{2i}), D_q(j_{2i}, j_{2i+1}, j_{2i+1})\}). \end{aligned}$$

If

$$\max\{D_q(j_{2i-1,}j_{2i}, j_{2i}), D_q(j_{2i}, j_{2i+1}, j_{2i+1})\} = D_q(j_{2i}, j_{2i+1}, j_{2i+1}).$$

then $D_q(j_{2i}, j_{2i+1}, j_{2i+1}) \leq \mu(D_q(j_{2i}, j_{2i+1}, j_{2i+1}))$, which is contradiction to the fact $\mu(\check{t}) < \check{t}$. Therefore

$$\max\{D_q(j_{2i-1,}j_{2i}, j_{2i}), D_q(j_{2i}, j_{2i+1}, j_{2i+1})\} = D_q(j_{2i-1}, j_{2i}, j_{2i}).$$

Then, we have

$$D_q(j_{2i}, j_{2i+1}, j_{2i+1}) \leq \mu(D_q(j_{2i-1}, j_{2i}, j_{2i})). \tag{2.5}$$

which implies that

$$D_q(j_{2i}, j_{2i+1}, j_{2i+1}) \leq \max\{\mu(D_q(j_{2i-1}, j_{2i}, j_{2i}), \mu(D_q(j_{2i}, j_{2i}, j_{2i-1})\}. \tag{2.6}$$

Now, by Lemma 1.6

$$\begin{aligned} D_q(j_{2i-1}, j_{2i}, j_{2i}) &\leq H_{D_q}(Tx_{2i-2}, Tx_{2i-1}, Tx_{2i-1}) \\ &\leq \max\left\{ \begin{array}{c} H_{D_q}(Tx_{2i-2}, Tx_{2i-1}, Tx_{2i-1}), \\ H_{D_q}(Tx_{2i-1}, Tx_{2i-1}, Tx_{2i-2}) \end{array} \right\}. \end{aligned}$$

As $j_{2i-1}, j_{2i-2} \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}$, $j_{2i-1} \succeq Sj_{2i-1}$ and $j_{2i-2} \preceq Sj_{2i-2}$, then by (i), we have

$$\begin{aligned} D_q(j_{2i-1}, j_{2i}, j_{2i}) &\leq \mu(W(j_{2i-1}, j_{2i-1}, j_{2i-2})) \\ &= \mu(\max\{D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}), (D_q(j_{2i-1}, j_{2i}, j_{2i}), \\ &\quad D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})\}). \end{aligned}$$

If $\max\{D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}), (D_q(j_{2i-1}, j_{2i}, j_{2i}), D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})\} = D_q(j_{2i-1}, j_{2i}, j_{2i})$, then contradiction arise to the fact $\mu(\check{t}) < \check{t}$. Now

$$D_q(j_{2i-1}, j_{2i}, j_{2i}) \leq \mu \max\{D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}), D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})\}.$$

Applying μ on both side. As μ is non decreasing function, so

$$\mu(D_q(j_{2i-1}, j_{2i}, j_{2i})) \leq \mu^2(\max\{D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}), D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})\}).$$

$$\mu(D_q(j_{2i-1}, j_{2i}, j_{2i})) = \max\left\{\mu^2(D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}), \mu^2(D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}))\right\}. \tag{2.7}$$

Now, by using (2.7) in (2.5), we have

$$D_q(j_{2i}, j_{2i+1}, j_{2i+1})) \leq \max\left\{\mu^2(D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}), \mu^2(D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}))\right\}. \tag{2.8}$$

Now, by Lemma 1.6

$$\begin{aligned} D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}) &\leq H_{D_q}(Tx_{2i-3}, Tx_{2i-2}, Tx_{2i-2}) \\ &\leq \max\left\{ \begin{array}{c} H_{D_q}(Tx_{2i-3}, Tx_{2i-2}, Tx_{2i-2}), \\ H_{D_q}(Tx_{2i-2}, Tx_{2i-2}, Tx_{2i-3}) \end{array} \right\}. \end{aligned}$$

As $j_{2i-3}, j_{2i-2} \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}$, $j_{2i-3} \succeq Sj_{2i-3}$ and $j_{2i-2} \preceq Sj_{2i-2}$, then, we have

$$\begin{aligned} D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}) &\leq \mu(W(j_{2i-3}, j_{2i-2}, j_{2i-2})) \\ &\leq \mu(\max\left\{ \begin{array}{c} D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), D_q(j_{2i-3}, Tx_{2i-3}, Tx_{2i-3}), \\ D_q(j_{2i-2}, Tx_{2i-2}, Tx_{2i-2}) \end{array} \right\} \\ &\leq \mu(\max\left\{ \begin{array}{c} D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), \\ D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}) \end{array} \right\} \\ &= \mu(\max\{D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})\}. \end{aligned}$$

If $\max\{D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})\} = D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})$, then contradiction arise to the fact $\mu(\check{t}) < \check{t}$. Therefore

$$D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}) \leq \mu(D_q(j_{2i-3}, j_{2i-2}, j_{2i-2})) \tag{2.9}$$

$$\leq \mu(\max\{D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})\}.$$

$$\mu^2 D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}) \leq \mu^3 \left(\max \left\{ \begin{array}{c} D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), \\ D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}) \end{array} \right\} \right). \tag{2.10}$$

Now, by Lemma 1.6

$$\begin{aligned} D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}) &\leq H_{D_q}(Tx_{2i-2}, Tx_{2i-2}, Tx_{2i-3}) \\ &\leq \max \left\{ \begin{array}{c} H_{D_q}(Tx_{2i-3}, Tx_{2i-2}, Tx_{2i-2}), \\ H_{D_q}(Tx_{2i-2}, Tx_{2i-2}, Tx_{2i-3}) \end{array} \right\}. \end{aligned}$$

As $j_{2i-3}, j_{2i-2} \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}$, $j_{2i-3} \succeq Sj_{2i-3}$ and $j_{2i-2} \preceq Sj_{2i-2}$, then by (i), we have

$$\begin{aligned} D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}) &\leq \mu(W(j_{2i-3}, j_{2i-2}, j_{2i-2})) \\ &\leq \mu \left(\max \left\{ \begin{array}{c} D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), \\ D_q(j_{2i-2}, j_{2i-1}, j_{2i-1}) \end{array} \right\} \right) \\ &= \mu(\max \{D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), D_q(j_{2i-2}, j_{2i-1}, j_{2i-1})\}). \end{aligned}$$

By using inequality (2.9), we have

$$D_q(j_{2i-1}, j_{2i-1}, j_{2i-2}) \leq \mu(\max \{D_q(j_{2i-2}, j_{2i-2}, j_{2i-3}), \mu(D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}))\}.$$

As we know that

$$\mu(D_q(j_{2i-3}, j_{2i-2}, j_{2i-2})) < D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}).$$

By the fact $\mu(\check{t}) < \check{t}$, which implies that

$$\mu^2(D_q(j_{2i-1}, j_{2i-1}, j_{2i-2})) \leq \mu^2(\mu(\max \{D_q(j_{2i-3}, j_{2i-2}, j_{2i-2}), D_q(j_{2i-2}, j_{2i-2}, j_{2i-3})\}. \tag{2.11}$$

Combining inequalities (2.8) , (2.10) and (2.11), we have

$$D_q(j_{2i}, j_{2i+1}, j_{2i+1}) \leq \max \left\{ \mu^3(D_q(j_{2i-3}, j_{2i-2}, j_{2i-2})), \mu^3(D_q(j_{2i-2}, j_{2i-2}, j_{2i-3})) \right\}. \tag{2.12}$$

Following the patterns of inequalities (2.6), (2.8) and (2.12), we get

$$D_q(j_{2i}, j_{2i+1}, j_{2i+1}) \leq \max \left\{ \mu^{2i}(D_q(j_0, j_1, j_1)), \mu^{2i}(D_q(j_1, j_1, j_0)) \right\}.$$

Similarly, we have

$$D_q(j_{2i-1}, j_{2i-1}, j_{2i}) \leq \max \left\{ \mu^{2i-1}(D_q(j_0, j_1, j_1)), \mu^{2i-1}(D_q(j_1, j_1, j_0)) \right\}.$$

Combining the above two inequalities, we have

$$D_q(j_{j'}, j_{j'+1}, j_{j'+1}) \leq \max \left\{ \mu^{j'}(D_q(j_0, j_1, j_1)), \mu^{j'}(D_q(j_1, j_1, j_0)) \right\}. \tag{2.13}$$

Now by Lemma 1.6

$$\begin{aligned} D_q(j_{2i+1}, j_{2i+1}, j_{2i}) &\leq H_{D_q}(Tj_{2i}, Tj_{2i}, Tj_{2i-1}) \\ &\leq \max \left\{ H_{D_q}(Tj_{2i-1}, Tj_{2i}, Tj_{2i}), H_{D_q}(Tj_{2i}, Tj_{2i}, Tj_{2i-1}) \right\}. \end{aligned}$$

As $j_{2i-1}, j_{2i} \in B_{D_q}(j_0, r) \cap \{\chi T(j_n)\}$, $j_{2i-1} \succeq Sj_{2i-1}$ and $j_{2i} \preceq Sj_{2i}$ then by (i), we have

$$D_q(j_{2i+1}, j_{2i+1}, j_{2i}) \leq \mu(W(j_{2i-1}, j_{2i}, j_{2i})).$$

After simplification, we have

$$D_q(j_{2i+1}, j_{2i+1}, j_{2i}) \leq \mu(\max \{D_q(j_{2i-1}, j_{2i}, j_{2i}), D_q(j_{2i}, j_{2i+1}, j_{2i+1})\}). \tag{2.14}$$

By inequality (2.5), we have

$$D_q(j_{2i+1},j_{2i+1},j_{2i}) \leq \mu(\max\{D_q(j_{2i-1},j_{2i},j_{2i}),\mu(D_q(j_{2i-1},j_{2i},j_{2i})\}).$$

As $\mu(\check{t}) < \check{t}$

$$D_q(j_{2i+1},j_{2i+1},j_{2i}) \leq \mu(D_q(j_{2i-1},j_{2i},j_{2i})). \tag{2.15}$$

Now,

$$D_q(j_{2i+1},j_{2i+1},j_{2i}) \leq \max\{\mu\left(D_q(j_{2i-1},j_{2i},j_{2i})\right),\mu(D_q(j_{2i},j_{2i},j_{2i-1})\}. \tag{2.16}$$

Now, by using (2.7) in (2.15), we have

$$D_q(j_{2i+1},j_{2i+1},j_{2i}) \leq \max\left\{\mu^2\left(D_q(j_{2i-1},j_{2i-1},j_{2i-2})\right),\mu^2\left(D_q(j_{2i-2},j_{2i-1},j_{2i-1})\right)\right\}. \tag{2.17}$$

Combining inequalities (2.10), (2.11) and(2.17), we have

$$D_q(j_{2i+1},j_{2i+1},j_{2i}) \leq \max\left\{\mu^3\left(D_q(j_{2i-3},j_{2i-3},j_{2i-2})\right),\mu^3\left(D_q(j_{2i-2},j_{2i-3},j_{2i-3})\right)\right\}. \tag{2.18}$$

Following the patterns of inequalities (2.16), (2.11) and (2.18), we have

$$D_q(j_{2i+1},j_{2i+1},j_{2i}) \leq \max\left\{\mu^{2i}\left(D_q(j_1,j_1,j_0)\right),\mu^{2i}\left(D_q(j_0,j_1,j_1)\right)\right\}.$$

Similarly, we have

$$D_q(j_{2i},j_{2i},j_{2i-1}) \leq \max\left\{\mu^{2i-1}\left(D_q(j_1,j_1,j_0)\right),\mu^{2i-1}\left(D_q(j_0,j_1,j_1)\right)\right\}.$$

Combining the above two inequalities, we have

$$D_q(j_{j'+1},j_{j'+1},j_{j'}) \leq \max\left\{\mu^{j'}\left(D_q(j_1,j_1,j_0)\right),\mu^{j'}\left(D_q(j_0,j_1,j_1)\right)\right\}. \tag{2.19}$$

By using inequalities (2.13), (iv) and triangle inequality,we have

$$\begin{aligned} D_q(j_0,j_{j'+1},j_{j'+1}) &\leq D_q(j_0,j_1,j_1)+\ldots+D_q(j_{j'},j_{j'+1},j_{j'+1}) \\ &\leq \max\{D_q(j_1,j_1,j_0)),(D_q(j_0,j_1,j_1))\}+ \\ &\quad \ldots+\max\left\{\mu^{2i}(D_q(j_1,j_1,j_0)),\mu^{2i}(D_q(j_0,j_1,j_1))\right\} \\ &\leq \sum_{i=0}^{j'}\max\{\mu^i(D_q(j_1,j_1,j_0),\mu^i(D_q(j_0,j_1,j_1))\}. \end{aligned}$$

$$D_q(j_0,j_{j'+1},j_{j'+1}) < r. \tag{2.20}$$

Similarly,by using inequalities (2.19), (iv) and triangle inequality, we have

$$D_q(j_{j'+1},j_{j'+1},j_0) \leq \sum_{i=0}^{j'}\max\{\mu^i(D_q(j_1,j_1,j_0),\mu^i(D_q(j_0,j_1,j_1))\} < r. \tag{2.21}$$

By Inequality (2.20) and (2.21), we have $j_{j'+1} \in B_{D_q}(j_0,r)$. Also

$$D_q\left(j_{j'+1},Tj_{j'+1},Tj_{j'+1}\right) = D_q\left(j_{j'+1},j_{j'+2},j_{j'+2}\right)$$

and

$$D_q\left(Tj_{j'+1},Tj_{j'+1},j_{j'+1}\right) = D_q\left(j_{j'+2},j_{j'+2},j_{j'+1}\right).$$

As $j_{j'+1} \succeq Sj_{j'+1}$, so from condition (ii), we have $j_{j'+2} \preceq Sj_{j'+2}$. Similarly we get

$$D_q(j_{j'+1}, j_{j'+2}, j_{j'+2}) \leq \max\left\{\mu^{j'+1}(D_q(j_1, j_1, j_0))), \mu^{j'+1}(D_q(j_0, j_1, j_1))\right\}. \quad (2.22)$$

and

$$D_q(j_{j'+2}, j_{j'+2}, j_{j'+1}) \leq \max\left\{\mu^{j'+1}(D_q(j_1, j_1, j_0))), \mu^{j'+1}(D_q(j_0, j_1, j_1))\right\}. \quad (2.23)$$

Also

$$D_q(j_0, j_{j'+2}, j_{j'+2}) < r \text{ and } D_q(j_{j'+2}, j_{j'+2}, j_0) < r.$$

It follows that $j_{j'+2} \in B_{D_q}(j_0, r)$. Also

$$D_q\left(j_{j'+2}, Tj_{j'+2}, Tj_{j'+2}\right) = D_q\left(j_{j'+2}, j_{j'+3}, j_{j'+3}\right).$$

and

$$D_q\left(Tj_{j'+2}, Tj_{j'+2}, j_{j'+2}\right) = D_q\left(j_{j'+3}, j_{j'+3}, j_{j'+2}\right).$$

As $j_{j'+2} \preceq Sj_{j'+2}$, so from condition (ii), we have $j_{j'+3} \succeq Sj_{j'+3}$. Hence by mathematical induction $j_n \in B_{D_q}(j_0, r)$, $j_{2n} \preceq Sj_{2n}$ and $j_{2n+1} \succeq Sj_{2n+1}$ for all $n \in \mathbb{N}$. Also $j_{2n} \in C(S)$. Now inequalities (2.13) and (2.19),(2.22) and (2.23) can be written as for all $n \in \mathbb{N}$

$$D_q(j_n, j_{n+1}, j_{n+1}) \leq \max\{\mu^n(D_q(j_1, j_1, j_0)), \mu^n(D_q(j_0, j_1, j_1))\}. \quad (2.24)$$

$$D_q(j_{n+1}, j_{n+1}, j_n) \leq \max\{\mu^n(D_q(j_1, j_1, j_0)), \mu^n(D_q(j_0, j_1, j_1))\}. \quad (2.25)$$

Fix $\varepsilon > 0$ and let $k_1(\varepsilon) \in \mathbb{N}$ such that

$$\sum \max\{\mu^{k'}(D_q(j_1, j_1, j_0)), \mu^{k'}(D_q(j_0, j_1, j_1))\} < \varepsilon.$$

Let $n, m' \in \mathbb{N}$ with $m' > n > k_1(\varepsilon)$, then

$$\begin{aligned} D_q(j_n, j_{m'}, j_{m'}) &\leq \sum_{k'=n}^{m'-1} D_q(j_{k'}, j_{k'+1}, j_{k'+1}) \\ &\leq \sum_{k'=n}^{m'-1} \max\{\mu^{k'}(D_q(j_1, j_1, j_0)), \mu^{k'}(D_q(j_0, j_1, j_1))\}, \text{ by (2.24)} \\ D_q(j_n, j_{m'}, j_{m'}) &\leq \sum_{k' \geq k_1(\varepsilon)} \max\{\mu^n(D_q(j_1, j_1, j_0)), \mu^n(D_q(j_0, j_1, j_1))\} < \varepsilon. \end{aligned}$$

Thus, $\{\chi T(j_n)\}$ is a left p-Cauchy sequence in $(B_{D_q}(j_0, r), D_q)$. Similarly, by using (2.25), we have

$$D_q(j_{m'}, j_{m'}, j_n) \leq \sum_{k'=n}^{m'-1} D_q(j_{k'+1}, j_{k'+1}, j_{k'}) < \varepsilon.$$

So, $\{\chi T(j_n)\}$ is a right p-Cauchy sequence in $(B_{D_q}(j_0, r), D_q)$. Thus we proved that $\{\chi T(j_n)\}$ is a bi p-Cauchy sequence in $(B_{D_q}(j_0, r), D_q)$. As every closed set in bi p-sequentially complete dislocated quasi G- metric space is bi p-sequentially complete and $C(S)$ is closed set, so $C(S)$ is bi p-sequentially complete. As the subsequence $\{j_{2n}\}$

of $\{\chi T(j_n)\}$ is a bi p-Cauchy sequence in $C(S)$, so there exists $j^* \in C(S)$ such that $\{j_{2n}\} \to j^*$, that is

$$\lim_{n\to\infty} D_q(j_{2n}, j^*, j^*) = \lim_{n\to\infty} D_q(j^*, j^*, j_{2n}) = 0. \tag{2.26}$$

Also

$$j^* \preceq Sj^*. \tag{2.27}$$

Now

$$D_q(j^*, j^*, j^*) \leq D_q(j^*, j^*, j_{2n}) + D_q(j_{2n}, j^*, j^*).$$

This implies $D_q(j^*, j^*, j^*) = 0$ as $n \to \infty$. Now

$$\begin{aligned} D_q(j^*, Tj^*, Tj^*) &\leq D_q(j^*, j_{2n+2}, j_{2n+2}) + D_q(j_{2n+2}, Tj^*, Tj^*) \\ &\leq D_q(j^*, j_{2n+2}, j_{2n+2}) + H_{D_q}(Tj_{2n+1}, Tj^*, Tj^*), \quad \text{(by Lemma 1.6)} \\ &\leq D_q(j^*, j_{2n+2}, j_{2n+2}) \\ &\quad + \max\left\{H_{D_q}(Tj_{2n+1}, Tj^*, Tj^*), H_{D_q}(Tj^*, Tj^*, Tj_{2n+1})\right\} \end{aligned}$$

By assumption, inequality (i) holds for j^*. Also $j_{2n+1} \succeq Sj_{2n+1}$ and $j^* \preceq Sj^*$, so

$$\begin{aligned} D_q(j^*, Tj^*, Tj^*) &\leq D_q(j^*, j_{2n+2}, j_{2n+2}) + \mu(W(j_{2n+1}, j^*, j^*)) \\ &= D_q(j^*, j_{2n+2}, j_{2n+2}) \\ &\quad + \mu\left(\max\left\{\begin{array}{c} D_q(j_{2n+1}, j^*, j^*), D_q(j_{2n+1}, Tj_{2n+1}, Tj_{2n+1}), \\ D_q(j^*, Tj^*, Tj^*) \end{array}\right\}\right) \\ &\leq D_q(j^*, j_{2n+2}, j_{2n+2}) \\ &\quad + \mu\left(\max\left\{\begin{array}{c} D_q(j_{2n+1}, j^*, j^*), D_q(j_{2n+1}, j_{2n}, j_{2n}), \\ D_q(j^*, Tj^*, Tj^*) \end{array}\right\}\right). \end{aligned}$$

Letting $n^{'} \longrightarrow \infty$ and by using inequalities (2.24) and (2.26), we obtain

$$D_q(j^*, Tj^*, Tj^*) \leq \mu(D_q(j^*, Tj^*, Tj^*)).$$

This implies that

$$D_q(j^*, Tj^*, Tj^*) = 0. \tag{2.28}$$

Now,

$$\begin{aligned} D_q(Tj^*, Tj^*, j^*) &\leq D_q(Tj^*, Tj^*, j_{2n+2}) + D_q(j_{2n+2}, j_{2n+2}, j^*) \\ &\leq H_{D_q}(Tj^*, Tj^*, Tj_{2n+1}) + D_q(j_{2n+2}, j_{2n+2}, j^*) \\ &\leq \max\left\{H_{D_q}(Tj_{2n+1}, Tj^*, Tj^*), H_{D_q}(Tj^*, Tj^*, Tj_{2n+1})\right\} + \\ &\quad D_q(j_{2n+2}, j_{2n+2}, j^*). \end{aligned}$$

As inequality (i) holds for j^*, $j^* \preceq Sj^*$ and $j_{2n+1} \succeq Sj_{2n+1}$, then, we obtain

$$\begin{aligned} D_q(Tj^*, Tj^*, j^*) &\leq \mu(W(j_{2n+1}, j^*, j^*)) + D_q(j_{2n+2}, j_{2n+2}, j^*) \\ &= \mu\left(\max\left\{\begin{array}{c} D_q(j_{2n+1}, j^*, j^*), D_q(j_{2n+1}, Tj_{2n+1}, Tj_{2n+1}), \\ D_q(j^*, Tj^*, Tj^*) \end{array}\right\}\right) + \\ &\quad D_q(j_{2n+2}, j_{2n+2}, j^*) \\ &\leq \mu(\max\{D_q(j_{2n+1}, j^*, j^*), D_q(j_{2n+1}, j_{2n}, j_{2n}), D_q(j^*, Tj^*, Tj^*)\}) + \\ &\quad D_q(j_{2n+2}, j_{2n+2}, j^*). \end{aligned}$$

Taking $n^{'} \longrightarrow \infty$ and by using inequalities (2.24), (2.26) and (2.28), we have

$$D_q(Tj^*, Tj^*, j^*) = 0. \tag{2.29}$$

From inequalities (2.28) and (2.29), we have $j^* \in Tj^*$. As $j^* \preceq Sj^*$ and $D_q(j^*, Tj^*, Tj^*) = D_q(Tj^*, Tj^*, j^*) = 0 = D_q(j^*, j^*, j^*)$, then from (ii)

$$j^* \succeq Sj^*. \tag{2.30}$$

From (2.27) and (2.30), we have $j^* \preceq Sj^* \preceq j^*$. This implies $j^* \preceq p \preceq j^*$, for all $p \in Sj^*$. Therefore $j^* = p$, for all $p \in Sj^*$ or $Sj^* = \{j^*\}$. Hence, j^* is a common fixed point for S and T.

□

Example 2.2 Let $\chi = [0, \infty)$ and

$$D_q(j, p, l) = j + 2p + 2l, \ (j, p, l) \in \chi \times \chi \times \chi.$$

Then, $(\chi, \preceq, D_{q_g})$ be an ordered bi p-sequentially complete dislocated quasi G-metric space. Let $\mathcal{R}$ be the binary relation on χ defined by

$$\begin{aligned} \mathcal{R} &= \{(j, j) : j \in \chi\} \cup \left\{(j, \frac{j}{5}) : j \in \{0, 1, \frac{1}{25}, \frac{1}{625}, \frac{1}{15625}, ...\}\right\} \\ &\cup \left\{(\frac{j}{5}, j) : j \in \{0, \frac{1}{5}, \frac{1}{125}, \frac{1}{3125}, ...\}\right\}. \end{aligned}$$

Consider the partial order on χ defined by

$$(j, p) \in \chi \times \chi, \ j \preceq p \text{ if and only } (j, p) \in \mathcal{R}.$$

Define the pair of mapping $T, S : \chi \to \chi$ by

$$Tj = \left[\frac{j}{5}, \frac{j}{4}\right], \ Sj = \begin{cases} \{\frac{j}{5}\} : j \in [0, 1] \\ \{j + 5\} : j \geq 1 \end{cases}.$$

Observe that in this case, we have

$$\begin{aligned} A &= \{j : j \preceq Sj\} = \left\{0, 1, \frac{1}{25}, \frac{1}{625}, \frac{1}{15625}, ...\right\}, \\ B &= \{p : p \succeq Sp\} = \left\{0, \frac{1}{5}, \frac{1}{125}, \frac{1}{3125}, ...\right\}. \end{aligned}$$

Now, let $\mu(t) = \frac{5}{8}t, r > 0, j_0 = 1$, then

$$\begin{aligned} B_{D_q}(1, 7) &= \{p : D_q(1, p, p) < 7 \wedge D_q(p, p, 1) < 7\} \\ &= \{p : 1 + 2p + 2p < 7 \wedge p + 2p + 2(1) < 7\} \\ &= \{p : 4p < 6 \wedge 3p < 5\} \\ &= \left\{p : p < \frac{3}{2} \wedge p < \frac{5}{3}\right\} \\ B_{D_q}(1, 7) &= (\frac{3}{2}, \frac{5}{3}) \subset [0, \frac{3}{2}). \end{aligned}$$

Then,

$$\begin{aligned} C(S) &= \{j : j \preceq Sj \text{ and } j \in B_{D_q}(j_o,r)\} \\ &= \left\{0, 1, \frac{1}{25}, \frac{1}{625}, \frac{1}{15625}, ...\right\} \cap [0, \frac{3}{2}) \\ &= \left\{0, 1, \frac{1}{25}, \frac{1}{625}, \frac{1}{15625}, ...\right\}. \end{aligned}$$

Now, as $\frac{1}{5^{n-1}} \in B_{D_q}(j_o, r)$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} D_q(\frac{1}{5^{n-1}}, T\frac{1}{5^{n-1}}, T\frac{1}{5^{n-1}}) &= D_q(\frac{1}{5^{n-1}}, [\frac{1}{5.5^{n-1}}, \frac{1}{4.5^{n-1}}], [\frac{1}{5.5^{n-1}}, \frac{1}{4.5^{n-1}}]) \\ &= D_q(\frac{1}{5^{n-1}}, \frac{1}{5.5^{n-1}}, \frac{1}{5.5^{n-1}}) \end{aligned}$$

and

$$D_q(T\frac{1}{5^{n-1}}, T\frac{1}{5^{n-1}}, \frac{1}{5^{n-1}}) = D_q(\frac{1}{5.5^{n-1}}, \frac{1}{5.5^{n-1}}, \frac{1}{5^{n-1}}).$$

Also, $(\frac{1}{5^{n-1}}, \frac{1}{5.5^{n-1}}) \in \mathcal{R}$ for all $n \in \{1, 3, 5, ...\}$, so $\frac{1}{5^{n-1}} \preceq S\frac{1}{5^{n-1}}$. As $(\frac{1}{25.5^{n-1}}, \frac{1}{5.5^{n-1}}) \in \mathcal{R}$, so $\frac{1}{5.5^{n-1}} \succeq S\frac{1}{5.5^{n-1}}$ for all $n \in \{1, 3, 5, ...\}$. Now, as $(\frac{1}{5.5^{n-1}}, \frac{1}{5^{n-1}}) \in \mathcal{R}$ for all $n \in \{2, 4, 6, ...\}$, so $\frac{1}{5^{n-1}} \succeq S\frac{1}{5^{n-1}}$ for all $n \in \{2, 4, 6...\}$. Also, $\frac{1}{5.25^{n-1}} \preceq S\frac{1}{5.25^{n-1}}$, then $\frac{1}{25^{n-1}} \preceq S\frac{1}{25^{n-1}}$.

i) For all $j, p \in B_{D_q}(j_o, r) \cap \{\chi T(j_n)\}$ with $j \succeq Sj$, $p \preceq Sp$

$$B_{D_q}(j_o, r) \cap \{\chi T(j_n)\} = \left\{1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \frac{1}{625}, \frac{1}{3125}, \frac{1}{15625}, ...\right\}.$$

Case I: In general for $n \leq m'$ and

$$j = \frac{1}{5.25^{m'-1}},\ p = \frac{1}{25^{n-1}}.$$

We have

$$\begin{aligned} H_{D_q}(Tj, Tp, Tp) &= \max\left(\sup_{a\in Tj} D_q(a, Tp, Tp),\ \sup_{b\in Tp} D_q(Tj, b, b)\right) \\ &= \max\left\{\sup_{a\in Tj} D_q(a, \frac{1}{5\times 25^{n-1}}, \frac{1}{5\times 25^{n-1}}),\ \sup_{b\in Tp} D_q(\frac{1}{5\times 25^{m'-1}}, b, b)\right\} \\ &= \max\left\{\begin{array}{c} D_q(\frac{1}{20\times 25^{m'-1}}, \frac{1}{5\times 25^{n-1}}, \frac{1}{5\times 25^{n-1}}), \\ D_q(\frac{1}{5\times 25^{m'-1}}, \frac{1}{4\times 25^{n-1}}, \frac{1}{4\times 25^{n-1}}) \end{array}\right\}. \end{aligned}$$

$$H_{D_q}(Tj, Tp, Tp) = \max\left\{\begin{array}{c} (\frac{1}{20\times 25^{m'-1}} + \frac{2}{5\times 25^{n-1}} + \frac{2}{5\times 25^{n-1}}), \\ (\frac{1}{5\times 25^{m'-1}} + \frac{2}{4\times 25^{n-1}} + \frac{2}{4\times 25^{n-1}}) \end{array}\right\}. \tag{2.31}$$

$$H_{D_q}(Tj,Tp,Tp) = \max\left\{\begin{array}{c}(\frac{1}{20\times 25^{m'-1}}+\frac{2}{5\times 25^{n-1}}+\frac{2}{5\times 25^{n-1}}),\\ (\frac{1}{5\times 25^{m'-1}}+\frac{1}{2\times 25^{n-1}}+\frac{1}{2\times 25^{n-1}})\end{array}\right\}$$
$$= \max\left\{\begin{array}{c}\frac{40\times 25^{m'-n}+40\times 25^{m'-n}+5}{100.25^{m'-1}},\\ \frac{25\times 25^{m'-n}+25\times 25^{m'-n}+2}{50.25^{m'-1}}\end{array}\right\}$$
$$= \max\left\{\frac{80\times 25^{m'-n}+5}{100\times 25^{m'-1}},\frac{50\times 25^{m'-n}+2}{50\times 25^{m'-1}}\right\}$$
$$= \frac{100\times 25^{m'-n}+4}{100.25^{m'-1}}.$$

Now

$$H_{D_q}(Tp,Tp,Tj) = \max\left(\sup_{a\in Tp} D_q(a,a,Tj),\ \sup_{b\in Tj} D_q(Tp,Tp,b)\right)$$
$$= \max\left\{\sup_{a\in Tj} D_q(a,a,\frac{1}{25\times 25^{m'-1}}),\ \sup_{b\in Tp} D_q(\frac{1}{5\times 25^{n-1}},\frac{1}{5\times 25^{n-1}},b)\right\}$$
$$= \max\left\{\begin{array}{c}D_q(\frac{1}{4\times 25^{n-1}},\frac{1}{4\times 25^{n-1}},\frac{1}{25\times 25^{m'-1}}),\\ D_q(\frac{1}{5\times 25^{n-1}},\frac{1}{5\times 25^{n-1}},\frac{1}{20\times 25^{m'-1}})\end{array}\right\}.$$

$$H_{D_q}(Tp,Tp,Tj) = \max\left\{\begin{array}{c}(\frac{1}{4\times 25^{n-1}}+\frac{2}{4\times 25^{n-1}}+\frac{2}{25\times 25^{m'-1}}),\\ (\frac{1}{5\times 25^{n-1}}+\frac{2}{5\times 25^{n-1}}+\frac{2}{20\times 25^{m'-1}})\end{array}\right\}. \tag{2.32}$$

$$H_{D_q}(Tp,Tp,Tj) = \max\left\{\begin{array}{c}(\frac{1}{4\times 25^{n-1}}+\frac{1}{2\times 25^{n-1}}+\frac{2}{25\times 25^{m'-1}}),\\ (\frac{1}{5\times 25^{n-1}}+\frac{2}{5\times 25^{n-1}}+\frac{1}{10\times 25^{m'-1}})\end{array}\right\}$$
$$= \max\left\{\begin{array}{c}\frac{25\times 25^{m'-n}+50\times 25^{m'-n}+8}{100.25^{m'-1}},\\ \frac{10\times 25^{m'-n}+20\times 25^{m'-n}+5}{50.25^{m'-1}}\end{array}\right\}$$
$$= \max\left\{\frac{75\times 25^{m'-n}+8}{100\times 25^{m'-1}},\frac{30\times 25^{m'-n}+5}{50\times 25^{m'-1}}\right\}$$
$$= \frac{75\times 25^{m'-n}+8}{100\times 25^{m'-1}}.$$

Also

$$\max\left\{H_{D_q}(Tj,Tp,Tp),H_{D_q}(Tp,Tp,Tj)\right\} = \max\left\{\frac{100\times 25^{m'-n}+4}{100.25^{m'-1}},\frac{75\times 25^{m'-n}+8}{100\times 25^{m'-1}}\right\}$$
$$= \frac{100\times 25^{m'-n}+4}{100.25^{m'-1}}.$$

Now, for $j \succeq Sj$, $p \preceq Sp$, we have

$$\begin{aligned}
W(j,p,p) &= \max\left\{D_q\left(j,p,p\right), D_q\left(j,Tj,Tj\right), D_q\left(p,Tp,Tp\right)\right\} \\
&= \max\left\{\begin{array}{c} D_q(\frac{1}{5\times 25^{m'-1}}, \frac{1}{25^{n-1}}, \frac{1}{25^{n-1}}), D_q\left(\frac{1}{5\times 25^{m'-1}}, T\frac{1}{5\times 25^{m'-1}}, T\frac{1}{5\times 25^{m'-1}}\right), \\ D_q\left(\frac{1}{25^{n-1}}, T\frac{1}{25^{n-1}}, T\frac{1}{25^{n-1}}\right) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} (\frac{1}{5\times 25^{m'-1}} + \frac{2}{25^{n-1}} + \frac{2}{25^{n-1}}), (\frac{1}{5\times 25^{m'-1}} + \frac{2}{25\times 25^{m'-1}} + \frac{2}{25.25^{m'-1}}), \\ (\frac{1}{25^{n-1}} + \frac{2}{5\times 25^{n-1}} + \frac{2}{5\times 25^{n-1}}) \end{array}\right\}
\end{aligned}$$

$$W(j,p,p) = \max\left\{\frac{1+10\times 25^{m'-n}+10\times 25^{m'-n}}{5\times 25^{m'-1}}, \frac{5+2+2}{25\times 25^{m'-1}}, \frac{5+2+2}{5\times 25^{n-1}}\right\}. \tag{2.33}$$

$$\begin{aligned}
W(j,p,p) &= \max\left\{\frac{1+20\times 25^{m'-n}}{5\times 25^{m'-1}}, \frac{5+2+2}{25\times 25^{m'-1}}, \frac{5+2+2}{5\times 25^{n-1}}\right\} \\
&= \frac{1+20\times 25^{m'-n}}{5\times 25^{m'-1}}.
\end{aligned}$$

As

$$\begin{aligned}
\frac{100\times 25^{m'-n}+4}{100.25^{m'-1}} &\leq \frac{1+20\times 25^{m'-n}}{5\times 25^{m'-1}} \\
\frac{100\times 25^{m'-n}+4}{100.25^{m'-1}} &\leq \frac{5}{8}\left(\frac{1+20\times 25^{m'-n}}{5\times 25^{m'-1}}\right) \\
\max\left\{H_{D_q}\left(Tj,Tp,Tp\right), H_{D_q}\left(Tp,Tp,Tj\right)\right\} &\leq \mu\left(W(j,p,p)\right).
\end{aligned}$$

Case II: For $n > m'$ and

$$j = \frac{1}{5.25^{m'-1}},\ p = \frac{1}{25^{n-1}}.$$

By using (2.31), we have

$$\begin{aligned}
H_{D_q}\left(Tj,Tp,Tp\right) &= \max\left\{\frac{80+5\times 25^{n-m'}}{100.25^{n-1}}, \frac{50+2\times 25^{n-m'}}{100\times 25^{n-1}}\right\} \\
&= \frac{80+5\times 25^{n-m'}}{100.25^{n-1}}.
\end{aligned}$$

Similarly, by using (2.32), we have

$$\begin{aligned}
H_{D_q}(Tp,Tp,Tj) &= \max\left\{\frac{75+8\times 25^{n-m'}}{100.25^{n-1}}, \frac{5\times 25^{n-m'}+30}{50.25^{n-1}}\right\} \\
&= \frac{10\times 25^{n-m'}+60}{100.25^{n-1}}.
\end{aligned}$$

L.H.S.

$$\begin{aligned}\max\left\{H_{D_q}\left(Tj,Tp,Tp\right),H_{D_q}\left(Tp,Tp,Tj\right)\right\} &= \max\left\{\frac{80+5\times 25^{n-m'}}{100.25^{n-1}},\frac{10\times 25^{n-m'}+60}{100.25^{n-1}}\right\}\\ &= \frac{10\times 25^{n-m'}+60}{100.25^{n-1}}.\end{aligned}$$

Now, by (2.33), R.H.S. for $j\succeq Sj$, $p\preceq Sp$,

$$\begin{aligned}W(j,p,p) &= \max\left\{\frac{25^{n-m'}+20}{5.25^{n-1}},\frac{5+2+2}{25.25^{m'-1}},\frac{5+2+2}{5.25^{n-1}}\right\}\\ &= \frac{25^{n-m'}+20}{5.25^{n-1}}.\end{aligned}$$

As

$$\begin{aligned}\frac{10\times 25^{n-m'}+60}{100.25^{n-1}} &\leq \frac{25^{n-m'}+20}{5.25^{n-1}}\\ \text{or}\qquad \frac{10\times 25^{n-m'}+60}{100.25^{n-1}} &\leq \frac{5}{8}\left(\frac{25^{n-m'}+20}{5.25^{n-1}}\right)\\ \text{or}\ \max\left\{H_{D_q}\left(Tj,Tp\right)Tp,H_{D_q}\left(Tp,Tp,Tj\right)\right\} &\leq \mu\left(D_q(j,p,p)\right).\end{aligned}$$

Case III: For

$$j=0,\ p=\frac{1}{25^{n-1}}.$$

We have

$$\begin{aligned}H_{D_q}(Tj,Tp,Tp) &= \max\left(\sup_{a\in Tj}D_q(a,Tp,Tp),\ \sup_{b\in Tp}D_q(Tj,b,b)\right)\\ &= \max\left\{\sup_{a\in Tj}D_q(a,\frac{1}{5\times 25^{n-1}},\frac{1}{5\times 25^{n-1}}),\ \sup_{b\in Tp}D_q(0,b,b)\right\}\\ &= \max\left\{D_q(0,\frac{1}{5\times 25^{n-1}},\frac{1}{5\times 25^{n-1}}),\ D_q(0,\frac{1}{4\times 25^{n-1}},\frac{1}{4\times 25^{n-1}})\right\}\\ &= \max\left\{(0+\frac{2}{5\times 25^{n-1}}+\frac{2}{5\times 25^{n-1}}),\ (0+\frac{2}{4\times 25^{n-1}}+\frac{2}{4\times 25^{n-1}})\right\}\\ &\quad \max\left\{\frac{2}{5\times 25^{n-1}}+\frac{2}{5\times 25^{n-1}},\frac{1}{2\times 25^{n-1}}+\frac{1}{2\times 25^{n-1}}\right\}\\ &\quad \max\left\{\frac{4}{5\times 25^{m'-1}},\frac{2}{2\times 25^{n-1}}\right\}=\frac{2}{2\times 25^{n-1}}.\end{aligned}$$

Now,

$$\begin{aligned}
H_{D_q}(Tp,Tp,Tj) &= \max\left(\sup_{a\in Tp} D_q(a,a,Tj),\ \sup_{b\in Tj} D_q(Tp,Tp,b)\right)\\
&= \max\left\{\sup_{a\in Tp} D_q(a,a,0),\ \sup_{b\in Tj} D_q(\frac{1}{5\times 25^{n-1}},\frac{1}{5\times 25^{n-1}},b)\right\}\\
&= \max\left\{D_q(\frac{1}{4\times 25^{n-1}},\frac{1}{4\times 25^{n-1}},0),\ D_q(\frac{1}{5\times 25^{n-1}},\frac{1}{5\times 25^{n-1}},0)\right\}\\
&= \max\left\{(\frac{1}{4\times 25^{n-1}}+\frac{2}{4\times 25^{n-1}}+2(0)),\ (\frac{1}{5\times 25^{n-1}}+\frac{2}{5\times 25^{n-1}}+2(0))\right\}\\
H_{D_q}(Tj,Tp,Tp) &= \max\left\{(\frac{1}{4\times 25^{n-1}}+\frac{1}{2\times 25^{n-1}}),\ (\frac{1}{5\times 25^{n-1}}+\frac{2}{5\times 25^{n-1}})\right\}\\
& \max\left\{\frac{3}{4\times 25^{n-1}},\frac{3}{5\times 25^{n-1}}\right\}\\
&= \frac{3}{4\times 25^{n-1}}.
\end{aligned}$$

L.H.S.

$$\begin{aligned}
\max\left\{H_{D_q}(Tj,Tp,Tp),H_q(Tp,Tp,Tj)\right\} &= \max\left\{\frac{2}{2\times 25^{n-1}},\frac{3}{4\times 25^{n-1}}\right\}.\\
&= \frac{2}{2\times 25^{n-1}}.
\end{aligned}$$

Now, R.H.S. for $j\succeq Sj, p\preceq Sp$

$$\begin{aligned}
W(j,p,p) &= \max\left\{D_q(j,p,p),D_q(j,Tj,Tj),D_q(p,Tp,Tp)\right\}\\
&= \max\left\{D_q(0,\frac{1}{25^{n-1}},\frac{1}{25^{n-1}}),D_q(0,T(0),T(0)),D_q\left(\frac{1}{25^{n-1}},T\frac{1}{25^{n-1}},T\frac{1}{25^{n-1}}\right)\right\}\\
&= \max\left\{(0+\frac{2}{25^{n-1}}+\frac{2}{25^{n-1}}),(0+2(0)+2(0)),(\frac{1}{25^{n-1}}+\frac{2}{5\times 25^{n-1}}+\frac{2}{5\times 25^{n-1}})\right\}\\
&= \max\left\{\frac{4}{25^{n-1}},0,\frac{9}{5\times 25^{n-1}}\right\}=\frac{4}{25^{n-1}}.
\end{aligned}$$

As

$$\begin{aligned}
\frac{2}{2\times 25^{n-1}} &\leq \frac{4}{25^{n-1}}\\
\text{or } \frac{2}{2\times 25^{n-1}} &\leq \frac{5}{8}\left(\frac{4}{25^{n-1}}\right)\\
\max\left\{H_{D_q}(Tj,Tp,Tp),H_{D_q}(Tp,Tp,Tj)\right\} &\leq \mu\left(W(j,p,p)\right).
\end{aligned}$$

Case IV: For

$$j=\frac{1}{25^{n-1}},\ p=0$$

We have

$$\begin{aligned}
H_{D_q}(Tj,Tp,Tp) &= \max\left(\sup_{a\in Tj} D_q(a,Tp,Tp),\ \sup_{b\in Tp} D_q(Tj,b,b)\right)\\
&= \max\left\{\sup_{a\in Tj} D_q(a,0,0),\ \sup_{b\in Tp} D_q(\frac{1}{25\times 25^{n-1}},b,b)\right\}\\
&= \max\left\{D_q(\frac{1}{20\times 25^{n-1}},0,0),\ D_q(\frac{1}{25\times 25^{n-1}},0,0)\right\}\\
&= \max\left\{\frac{1}{20\times 25^{n-1}},\ \frac{1}{25\times 25^{n-1}}\right\}\\
&= \frac{1}{20\times 25^{n-1}}.
\end{aligned}$$

Now

$$\begin{aligned}
H_{D_q}(Tp,Tp,Tj) &= \max\left(\sup_{a\in Tp} D_q(a,a,Tj),\ \sup_{b\in Tj} D_q(Tp,Tp,b)\right)\\
&= \max\left\{\sup_{a\in Tp} D_q(a,a,\frac{1}{25\times 25^{n-1}}),\ \sup_{b\in Tj} D_q(0,0,b)\right\}\\
&= \max\left\{D_q(0,0,\frac{1}{25\times 25^{n-1}}),\ D_q(0,0,\frac{1}{20\times 25^{n-1}})\right\}\\
&= \max\left\{\frac{2}{25\times 25^{n-1}},\ \frac{2}{20\times 25^{n-1}}\right\}\\
& \quad \max\left\{\frac{2}{25\times 25^{n-1}},\ \frac{1}{10\times 25^{n-1}}\right\}\\
&= \frac{1}{10\times 25^{n-1}}.
\end{aligned}$$

L.H.S.

$$\begin{aligned}
\max\left\{H_{D_q}\left(Tj,Tp,Tp\right), H_{D_q}\left(Tp,Tp,Tj\right)\right\} &= \max\left\{\frac{1}{20\times 25^{n-1}}, \frac{1}{10\times 25^{n-1}}\right\}\\
&= \frac{1}{10\times 25^{n-1}}.
\end{aligned}$$

Now, R.H.S. for $j \succeq Sj, p \preceq Sp$

$$\begin{aligned} W(j,p,p) &= \max\{D_q(j,p,p), D_q(j,Tj,Tj), D_q(p,Tp,Tp)\} \\ &= \max\left\{ \begin{array}{c} D_q(\frac{1}{5\times 25^{n-1}},0,0), D_q\left(\frac{1}{5\times 25^{n-1}}, \frac{1}{25\times 25^{n-1}}, \frac{1}{25\times 25^{n-1}}\right), \\ D_q(0,0,0) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} (\frac{1}{5\times 25^{n-1}}+0+0), (\frac{1}{5\times 25^{n-1}}+\frac{2}{25\times 25^{n-1}}+\frac{2}{25\times 25^{n-1}}), \\ (0+0+0) \end{array} \right\} \\ &= \max\left\{\frac{1}{5\times 25^{n-1}}, \frac{7}{25\times 25^{n-1}}, 0\right\} \\ &= \frac{7}{25\times 25^{n-1}}. \end{aligned}$$

As

$$\begin{aligned} \frac{1}{10\times 25^{n-1}} &\leq \frac{7}{25\times 25^{n-1}} \\ \text{or} \quad \frac{1}{10\times 25^{n-1}} &\leq \frac{5}{8}\left(\frac{7}{25\times 25^{n-1}}\right) \\ \text{or } \max\left\{H_{D_q}(Tj,Tp,Tp), H_{D_q}(Tp,Tp,Tj)\right\} &\leq \mu(W(j,p,p)). \end{aligned}$$

Case V: The result trivially holds for $j = 0$ and $p = 0$. As $j_0 = 1, j_1 = \frac{1}{5}$.

$$\begin{aligned} &\sum_{i=0}^{n} \max\{\mu^i(D_q(j_1,j_1,j_0), \mu^i(D_q(j_0,j_1,j_1)) \\ &= \sum_{i=0}^{n} \max\{\mu^i(\frac{1}{5},\frac{1}{5},1), \mu^i(1,\frac{1}{5},\frac{1}{5})\} \\ &= \sum_{i=0}^{n} \max\{\mu^i(\frac{1}{5}+\frac{2}{5}+2), \mu^i(1+\frac{2}{5}+\frac{2}{5})\} \\ &= \sum_{i=0}^{n} \max\{\mu^i(\frac{13}{5}), \mu^i(\frac{9}{5})\} \end{aligned}$$

$$= \max\{(\frac{13}{5}),(\frac{9}{5})\} + \max\{\mu^1(\frac{13}{5}),\mu^1(\frac{9}{5})\} + ... + \max\{\mu^n(\frac{13}{5}),\mu^n(\frac{9}{5})\}$$

$$\frac{13}{5} \times \sum_{i=0}^{n}(\frac{5}{8})^i = 6.93 < 7 = r$$

So,

$$\sum_{i=0}^{n} \max\{\mu^i(D_q(j_1,j_1,j_0), \mu^i(D_q(j_0,j_1,j_1)) < r. \text{ for all } n \in \mathbb{N}.$$

Then the subsequence $\{j_{2n}\}$ of $\{j_n\}$ is a sequence in $C(S)$ and $\{j_{2n}\} \to j^* = 0 \in C(S)$ and $D_q(j^*,j^*,j^*) = 0$. Also, inequality (i) holds for j^*. Hence S and T have a common fixed point j^* in $B_{D_q}(j_0, r)$.

By excluding ball, we obtain the following new result.

Theorem 2.3 Let $(\chi, \preceq, D_q)$ be an ordered bi p-sequentially complete dislocated quasi G metric space and $S, T : \chi \to P(\chi)$ be the multivalued mappings. Suppose that the following assertions hold:

(i) There exists a function $\mu \in \Psi$ such that for every $j, p \in \chi$ with $j \succeq Sj$, $p \preceq Sp$, we have

$$\max\{H_{D_q}(Tj, Tp, Tp), H_{D_q}(Tp, Tp, Tj)\} \leq \mu(W(j, p, p)),$$

where

$$W(j, p, p) = \max\{D_q(j, p, p), D_q(j, Tj, Tj), D_q(p, Tp, Tp)\}.$$

(ii) If $D_q(j, Tj, Tj) = D_q(j, p, p)$ and $D_q(Tj, Tj, j) = D_q(p, p, j)$, then

$$(a) \text{ If } j \preceq Sj, \text{ then } p \succeq Sp \ \ (b) \text{ If } j \succeq Sj, \text{ then } p \preceq Sp.$$

(iii)The set $C(S) = \{j : j \preceq Sj\}$ is closed and contains j_0.

Then the subsequence $\{j_{2n}\}$ of $\{\chi T(j_n)\}$ is a sequence in $C(S)$ and $\{j_{2n}\} \to j^* \in C(S)$ and $D_q(j^*, j^*, j^*) = 0$. Also, if the inequality (i) holds for j^*. Then S and T have a common fixed point j^* in χ.

By excluding order and taking closed ball instead of open ball, we obtain the following new result.

Theorem 2.4 Let (χ, D_q) be a bi p-sequentially complete dislocated quasi G metric space and $S, T : \chi \to P(\chi)$ be the multivalued mappings. Suppose that the following assertions hold:

(i) There exists a function $\mu \in \Psi$, $j_0 \in \chi$ and $r > 0$ such that for every $j, p \in \overline{B_{D_q}(j_0, r)} \cap \{\chi T(j_n)\}$, we have

$$\max\{H_{D_q}(Tj, Tp, Tp), H_{D_q}(Tp, Tp, Tj)\} \leq \mu(W(j, p, p)),$$

where

$$W(j, p, p) = \max\{D_q(j, p, p), D_q(j, Tj, Tj), D_q(p, Tp, Tp)\}.$$

(ii)

$$\sum_{i=0}^{n} \max\{\mu^i(D_q(j_1, j_1, j_0), \mu^i(D_q(j_0, j_1, j_1)) < r. \text{ for all } n \in \mathbb{N}.$$

Then the subsequence $\{j_{2n}\}$ of $\{\chi T(j_n)\}$ is a sequence in $\overline{B_{D_q}(j_0, r)}$ and $\{j_{2n}\} \to j^* \in \overline{B_{D_q}(j_0, r)}$ and $D_q(j^*, j^*, j^*) = 0$. Also, if the inequality (i) holds for j^*. Then S and T have a common fixed point j^* in $\overline{B_{D_q}(j_0, r)}$.

By excluding order and ball, we obtain the following new result.

Theorem 2.5 Let (χ, D_q) be a bi p-sequentially complete dislocated quasi G metric space and $T : \chi \to P(\chi)$ be a multivalued mapping. Suppose that there exists a function $\mu \in \Psi$ such that for every $j, p \in \chi$, we have

$$\max\{H_{D_q}(Tj, Tp, Tp), H_{D_q}(Tp, Tp, Tj)\} \leq \mu(W(j, p, p)),$$

where

$$W(j, p, p) = \max\{D_q(j, p, p), D_q(j, Tj, Tj), D_q(p, Tp, Tp)\}.$$

Then S and T have a common fixed point j^* in χ and $D_q(j^*, j^*, j^*) = 0$.

Remark 2.6 Corresponding fixed point results in metric spaces, quasi metric spaces, dislocated quasi metric spaces, partial quasi metric spaces, partial metric spaces, dislocated G-metric spaces and G- metric spaces can be obtained which will be still new in literature.

Remark 2.7 By taking six improper subsets of the set $W(j, p, p)$, we can obtain different new corollaries.

Remark 2.8 By taking self mapping instead of multivalued mappings, we can obtain different new corollaries.

Conflict of Interests

The authors declare that they have no competing interests.

Authors Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgement

The authors would like to thank the Editor and the anonymous referees for sparing their valuable time for reviewing this article. The thoughtful comments of reviewers are very useful to improve and modify this article.

REFERENCES

[1] R. Agarwal and E.Karapinar, *Remarks on some coupled fixed point theorems in G-metric spaces, Fixed Point Theory Appl.*, (**2013**), Art. No. 2 (2013).

[2] A. E. Al-Mazrooei, A. Shoaib, J. Ahmad, *Unique Fixed-Point Results for β-Admissible Mapping under (β-$\breve{\psi}$)-Contraction in Complete Dislocated G_d-Metric Space,* Mathematics **8**, No. 9, 1584(2020).

[3] B. Alqahtani, A. Fulga, E. Karapınar, P. S. Kumari, *Sehgal Type Contractions on Dislocated Spaces,* Mathematics **7**, No. 2, 153(2019).

[4] S. S. Alshoraify, A. Shoaib, M. Arshad, *New Types of Contraction for Multivalued Mappings and Related Fixed Point Results in Abstract Spaces*, Journal of Function Spaces, **2019**, Article ID 1812461(2019).

[5] I. Altun, N. A. Arifi, M. Jleli, A. Lashin and B. Samet, *A New Approach for the Approximations of Solutions to a Common Fixed Point Problem in Metric Fixed Point Theory,* Journal of Function Spaces, Vol. (**2016**) 1-5.

[6] I. Altun and H. Simsek, *Some Fixed Point Theorems on Ordered Metric Spaces and Application*, Fixed Point Theory Appl., **2010**, Art. No. 621469 (2010).

[7] M. Arshad, A. Shoaib and I. Beg, *Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered dislocated metric space,* Fixed Point Theory and Appl., **2013**, No.115, 1-15(2013).

[8] M. Arshad, Z. Kadelburg, S. Radenović, A. Shoaib and S. Shukla, *Fixed Points of α-Dominated Mappings on Dislocated Quasi Metric Spaces*, Filomat **31**, No.11 (2017) 3041–3056.

[9] M. Asadi, E. Karapinar and P. Salimi, *A new approach to G-metric and related fixed point theorems,* J. Inequal. Appl. (**2013**), 1-12.

[10] H. Aydi, N. Bilgili and E. Karapınar, *Common fixed point result from quasi metric space to G-metric,* J. Egyptian Math .Soc. **23**, (2015).

[11] H. Aydi, W. Shatanawi and C.Vetro, *On generalized weak G-contraction mappings in G-metric space,* Comput. Math. Appl. **2011**, No.62, 4222–4229.

[12] A. Azam and N. Mehmood, *Fixed point theorems for multivalued mappings in G-cone metric space,* J. Inequal. Appl., Vol.**(2013**) 1-14.

[13] I. Beg, M. Arshad and A. Shoaib, *Fixed point in a closed ball in ordered dislocated quasi metric space,* Fixed Point Theory **16**, No.2 (2015).

[14] T. G. Bhaskar, and V. Laskhmikantham, *Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis* **65**, No.7 (2006) 1379-1393.

[15] R. Chugh,T. Kadian, A. Rani and B. E. Rhoades, *Property P in G-metric spaces,* Fixed Point Theory and App.,Vol. **2010** 1-12.

[16] J. Harjani and K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Analysis,* **72**, No.3 (2010) 1188-1197.

[17] M. M. M. Jaradat, Z. Mustafa, S. Ullah Khan, M. Arshad, J. Ahmad, *Some fixed point results on G-metric and G_b-metric spaces,* Demonstratio Mathematica **50,** (2017), 190–207.

[18] E. Karapınar and R.P Agarwal, *Further fixed point results on G-metric spaces,* Fixed point theory and Appl., **2013**, No.154 (2013).

[19] S. K. Mohanta and S. Mohanta, *Some fixed point results for mappings in G-metric spaces,* Demonstratio Math. **47,** (2014) 1-15.

[20] Z. Mustafa, H.Obiedat and F. Awawdeh *some fixed point theorem for mappings on a complete G-metric spaces,* Fixed Point Theory Appl.2008.

[21] J. J. Nieto, and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22,** No.3 (2005) 223-239.

[22] V. Parvaneh, A. Razani and J. R. Roshan, *Common fixed points of six mappings in partially ordered G-metric spaces,* Mathematical Sciences, **7**:18, (2013).

[23] A. C. M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to metrix equations,* Proc. Amer. Math. Soc., **132,** No.5 (2004) 1435-1443.

[24] T. Rasham, A. Shoaib, C. Park, M. Arshad, *Fixed Point Results for a Pair of Multi Dominated Mappings on a Smallest Subset in K-Sequentially Dislocated quasi Metric Space with Application,* J. Comput. Anal. Appl. **25,** No.5 (2018) 975–986.

[25] A. Razani and V. Parvaneh, On generalized weakly G-contractive mappings in partially ordered G-metric spaces, Abstract and Applied Analysis, Article ID 701910, 2012.

[26] B. Samet, C. Vetro and F. Vetro, *Remarks on G-metric spaces,* Int. J. Anal. **2013,** (2013) 1-6.

[27] A. Shoaib, Fixed Point Results for α_*-ψ-multivalued Mappings, Bulletin of Mathematical Analysis and Applications **8**, No.4 (2016) 43-55.

[28] A. Shoaib, T. Rasham and A. Al-Rawashdeh, *DQF- contraction and Related Fixed Point Results in DQM Spaces with Applications*, Journal of Mathematical Analysis, Article in press.

[29] A. Shoaib, A. H. Ansari, Q. Mahmood and A. Shahzad, *Fixed Point Results for Complete Dislocated G_d-Metric Space via C-Class Functions, Bulletin of Mathematical Analysis and Applications* **9**, No.4 (2017) 1-11.

[30] A. Shoaib, M. Arshad and M. A. Kutbi, *Common fixed points of a pair of Hardy Rogers Type Mappings on a Closed Ball in Ordered Partial Metric Spaces*, J. Comput. Anal. Appl **17**, No.2 (2014) 255-264.

[31] A. Shoaib, M. Arshad, T. Rasham and M. Abbas, *Unique fixed points results on closed ball for dislocated quasi G-metric spaces,* Transaction of A. Razmadze Mathematical Instiute **30**, No.1 (2017) 1-10.

[32] A. Shoaib, A. Azam, A. Shahzad, *Common fixed point results for the family of multivalued mappings satisfying contraction on a sequence in Hausdorff fuzzy metric space*, J. Comp. Anal. Appl, **24** No.4 (2018) 692-699.

[33] A. Shoaib, A. Hussain, M. Arshad, and A. Azam, *Fixed point results for α_*-ψ-Ciric type multivalued mappings on an intersection of a closed ball and a sequence with graph,* J. Math. Anal. **7,** No.3 (2016) 41-50.

[34] A. Shoaib, Fahimuddin, M. Arshad, M. U. Ali, *Common Fixed Point results for α-ψ-locally contractive type mappings in right complete dislocated quasi G-metric spaces,* Thai J. Math. **17**, No.3 (2017) 627-638.

[35] A. Shoaib, M. Arshad and S. H. Kazmi, *Fixed Point Results for Hardy Roger Type Contraction in Ordered Complete Dislocated G_d Metric Space,* Turkish Journal of Analysis and Number Theory **5,** No.1 (2017) 5-12.

[36] A. Shoaib, Z. Nisar, A. Hussain , O. Ozer and M. Arshad, *Modified Banach Fixed Point results for locally contractive Mappings in complete G_d metric like space,* Electronic journal of mathematical analysis and applications **6,** No.1 (2018) 144-155.

[37] D. Singh, V. Joshi, M. Imdad, P. Kumam, *Fixed point theorems via generalized F-contractions with applications to functional equations occurring in dynamic programming*, J. Fixed Point Theory Appl., **19,** No.2 (2017) 1453–1479.

[38] F. M. Zeyada, G. H. Hassan and M. A. Ahmed, *A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces,* Arabian J. Science and Engineering **31,**No.1 (2006) 111–114.
[39] S. Zhou and F. Gu, *Some new fixed points in G-metric spaces,* J. Hangzhou Normal University **11,** (2010) 47-50.
[40] L. Zhu, C. Zhu, C. Chen and Z. Stojanović, *Multidimensional fixed points for generalized Ψ-quasi-contractions in quasi-metric-like spaces*, J. Inequalities and Appl., 2014, (2014):27.