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A Related Fixed Point Theorems using Contractive Mapping on Six Metric Spaces

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Abstract:The reason of this paper is to earn fixed point theorem on six metric spaces using contractive type mapping. This theorem generalizes the results given in [7].

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Key Words: Fixed point, Metric Space, Complete Metric Space.

1. Introduction

Related theorem on three metric spaces using fixed point have been introduced in a fixed point theorem for four metric spaces introduced by Jain et al. [6] on two metric space introduced by Gupta and Sharma [4]and such results are also studied by Kikina and Kikina [7]. Set valued mappings on three complete metric spaces are obtained by Jain and Fisher [5], Non expansive mappings, Hyperconvex metric spaces are prover by Kirk and Shahazad [8].

Quasiconformal and Quasiregular Harmonic mappings, Hyperbolic type metrics, Distance Ratio metrics introduced by Todorcevic [11]. The fixed point results for weak S-Contractions on partially ordered 2-metric spaces are developed by O.T.Omid, H.Koppelaar and S.Radenovoc[9].

Common fixed point result established by using φ weakly contractive mappings one step in development of the fixed point theory was given by A.H.Ansari[1] by the introduction of C-class function. Fixed point theorems in ordered metric spaces with two comparable metrics is proved by Shukla and Radenovic[10]. Common fixed point theorems for a pair of R-weakly commuting mappings of type(Ag) in modified intuitionistic Fuzzy metric spaces satisfying implicit relations proved by Sunny Chauhan,B.D.Pant and S.Radenovic[2]. Recent results on best approximation and fixed point theory in certain geodesic spaces these results are reladed to fundamental fixed point theorem this survey by Ciric[3].

Definition 1.1: A point that is fixed of a function F from a set S to itself and x is a point in S such that $F(x) = x$.

Definition 1.2: Let a non empty set X together with a distance function $d : X \times X \to R$ which satisfies the following conditions.

(1) $d(x, y) \geq 0, \forall x, y \in X$, (positivity)
(2) $d(x, y) = 0$ if and only if $x = y, \forall x, y \in X$,
(3) $d(x, y) = d(y, x), \forall x, y \in X$, (symmetry)
(4) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y \in X$. (triangle inequality)

The ordered pair (X, d) is said to be a metric space.

Definition 1.3: A metric space(X, d) is called complete if every Cauchy sequence converges to a point of X.

The next result is proved in[6]

Theorem 1.1[6]: Let$(X, d_1), (Y, d_2)$ and (Z, d_3) be complete metric spaces. Let T is map from X to Y, S is map from Y to Z, R is map from Z to X are satisfy the next inequalities:

$$d_1(RSy, RSTx) \leq \frac{cf_1(x,y)}{g_1(x,y)}$$

$$d_2(TRz, TRSy) \leq \frac{cf_2(y,z)}{g_2(y,z)}$$

$$d_3(STx, STRz) \leq \frac{cf_3(z,x)}{g_3(z,x)}$$

$\forall\ x \in X,\ y \in Y,\ z \in Z$ for which $g_3(z, x) \neq 0,\ g_2(y, z) \neq 0,\ g_1(x, y) \neq 0$,
where $1 > c \geq 0$ and
$f_1(x, y) = \max\{d_2(y, Tx), d_1(x, RSTx), d_3(Sy, STx),\ d_2(y, TRSy), d_1(x, RSy)d_1(x, RSTx)\}$
$f_2(y, z) = \max\{d_1(Rz, RSy), d_2(y, TRSy), d_2(y, TRSy),\ d_1(z, STRz), d_2(y, TRz), d_3(z, Sy)\}$
$f_3(z, x) = \max\{d_2(Tx, TRz), d_3(z, STRz), d_3(z, STRz),\ d_1(x, RSTx), d_3(z, STx), d_1(x, Rz)\}$
$g_1(x, y) = \max\{d_1(x, RSy), d_1(x, RSTx), d_2(Tx, TRSy)\}$
$g_2(y, z) = \max\{d_2(y, TRz), d_2(y, TRSy), d_3(Sy, STRz)\}$
$g_3(z, x) = \max\{d_3(z, STx), d_1(Rz, RSTx), d_3(z, STRz)\}$
After this $\alpha \in X$ is unique fixed point of RST,
$\beta \in Y$ is unique fixed point of TRS and $\gamma \in Z$ is unique fixed point of STR.
Similarly, $T\alpha = \beta,\ \gamma = S\beta,\ R\gamma = \alpha$.

Theorem 1.2 [7] Let$(X, d_1), (Y, d_2), (Z, d_3)$ and (U, d_4)these are metric spaces it is complete. prevent T is mapping from X to Y,S is mapping from Y to Z,R is mapping from Z to U and Q is mapping from U to X be four mappings satisfactory the inequalities are given below:

$$d_1(QRSy, QRSTx) \leq \frac{cF_1(x,y)}{G_1(x,y)}$$

$$d_2(TQRz, TQRSy) \leq \frac{cF_2(y,z)}{G_2(y,z)}$$

$$d_3(STQu, STQRz) \leq \frac{cF_3(z,u)}{G_3(z,u)}$$

$$d_4(RSTx, RSTQu) \leq \frac{cF_4(u,x)}{G_4(u,x)}$$

Every $x, y, z \in X, Y, Z$,respectively and $u \in U$ hence there for and $G_1(x, y)$not equal to 0, $G_2(y, z)$ not equal to 0 and $G_3(z, u)$ not equal to 0, $G_4(u, x)$ not equal to 0 where $1 > c \geq 0$ and

$F_1(x, y) = \max\{d_1(x, QRSTx)d_3(Sy, STy); d_1(x, QRSTx)d_2(y, TQRSy);$
$d_1(x, QRSTx)d_4(RSy, RSTx); d_1(x, QRSTx)d_2(y, Tx)\}$
$F_2(y, z) = \max\{d_2(y, TQRSy)d_1(QRz, QRSy); d_4(Rz, RSy),d_2(y, TQRSy);$
$d_2(y, TQRSy),d_3(z, STQRz); d_2(y, TQRSy)d_3(z, Sy)\}$
$F_3(z, u) = \max\{d_3(z, STQRz)d_1(Qu, QRz); d_3(z, STQRz)d_2(TQu, TQRz);$
$d_3(z, STQRz)d_4(u, RSTQu);d_3(z, STQu)d_1(u, Rz)\}$
$F_4(u, x) = \max\{d_4(u, RSTQu)d_1(x, Qu);d_4(u, RSTQu)d_2(Tx, TQu)$
$d_4(u, RSTQu)d_3(STx, STQu);d_4(u, RSTQu)d_1(x, QRSTx);\}$
$G_1(x, y) = \max\{d_1(x, QRSTx), d_2(Tx, TQRSy), d_1(x, QRSy)\}$
$G_2(y, z) = \max\{d_2(y, TQRz), d_2(y, TQRSy), d_3(Sy, STQRz)\}$
$G_3(z, u) = \max\{d_4(Rz, RSTQu), d_3(z, STQu), d_3(z, STQRz),\}$
$G_4(u, x) = \max\{d_1(Qu, QRSTx), d_4(u, RSTQu), d_4(u, RSTx)\}$

After this QRST is only one point that is fixed $\alpha \in X$. TQRS is only one point that is fixed $\beta \in Y$. STQR is only one point that is fixed $\gamma \in Z$.
RSTQ is only one point that is fixed $\delta \in U$. Auxiliary,$\beta = T\alpha$,$\gamma = S\beta$,$\delta = R\gamma$ and $\alpha = Q\delta$.

2. MAIN RESULTS

Theorem 2.1: Let $(X, d_1), (Y, d_2), (Z, d_3), (U, d_4), (V, d_5)$ and (W, d_6) be complete metric spaces. Consider $S : X \to Y$, $R : Y \to Z$, $T : Z \to U$, $A : U \to V$, $B : V \to W$, and $C : W \to X$ be satisfy the following inequalities:

$$d_1(CBATRy, CBATRSx) \leq \frac{c\,F_1(x,y)}{G_1(x,y)} \tag{2.1}$$

$$d_2(SCBATz, SCBATRy) \leq \frac{c\,F_2(y,z)}{G_2(y,z)} \tag{2.2}$$

$$d_3(RSCBAu, RSCBATz) \leq \frac{c\,F_3(z,u)}{G_3(z,u)} \tag{2.3}$$

$$d_4(TRSCBv, TRSCBAu) \leq \frac{c\,F_4(u,v)}{G_4(u,v)} \tag{2.4}$$

$$d_5(ATRSCw, ATRSCBv) \leq \frac{c\,F_5(v,w)}{G_5(v,w)} \tag{2.5}$$

$$d_6(BATRSx, BATRSCw) \leq \frac{c\,F_6(w,x)}{G_6(w,x)} \tag{2.6}$$

$\forall\, x \in X,\ y \in Y,\ z \in Z, u \in U,\ v \in V$ and $w \in W$,
suppose that $G_1(x,y) \neq 0, G_2(y,z) \neq 0, G_3(z,u) \neq 0,\ G_4(u,v) \neq 0,\ G_5(v,x) \neq 0$ and $G_6(w,x) \neq 0$ where $1 > c \geq 0$ and
we have $F_1(x,y) = \max\{d_1(x, CBATRSx)d_6(BATRy, BATRSx);$
$d_1(x, CBATRSx)d_5(ATRy, ATRSx); d_1(x, CBATRSx)d_4(TRy, TRSx);$
$d_1(x, CBATRSx)d_3(Ry, RSx); d_1(x, CBATRSx)d_2(y, SCBATRy);$
$d_1(x, CBATRSx)d_1(y, Sx)\}.$

$F_2(y,z) = \max\{d_2(y, SCBATRy)d_1(CBATz, CBATRy);$
$d_2(y, SCBATRy)d_6(BATz, BATRy); d_2(y, SCBATRy)d_5(ATz, ATRy);$
$d_2(y, SCBATRy)d_4(Tz, TRy); d_2(y, SCBATRy)d_3(z, CBATRSz);$
$d_2(y, SCBATz)d_3(z, Ry)\}.$

$F_3(z,u) = \max\{d_3(z, RSCBATz)d_2(SCBAu, SCBATz);$
$d_3(z, RSCBATz)d_1(CBAu, CBATz); d_3(z, RSCBATz)d_6(BAu, BATz);$
$d_3(z, RSCBATz)d_5(Au, ATz); d_3(z, RSCBATz)d_4(u, TRSCBAu);$
$d_3(z, RSCBAu)d_4(u, Tz)\}.$

$F_4(u,v) = \max\{d_4(u, TRSCBAu)d_3(RSCBv, RSCBAu);$
$d_4(u, TRSCBAu)d_2(SCBv, SCBAu); d_4(u, TRSCBAu)d_1(CBv, CBAu);$
$d_4(u, TRSCBAu)d_6(Bv, BAu); d_4(u, TRSCBAu)d_v(v, ATRSCBv);$
$d_4(u, TRSCBv)d_4(v, Au)\}.$

$F_5(v,w) = \max\{d_5(v, ATRSCBv)d_4(TRSCw, TRSCBv);$
$d_5(v, ATRSCBv)d_3(RSCw, RSCBv); d_5(v, ATRSCBv)d_2(SCw, SCBv);$
$d_5(v, ATRSCBv)d_1(Cw, CBv); d_5(v, ATRSCBv)d_6(w, BATRSCw);$
$d_5(v, ATRSCw)d_6(w, Bv)\}.$

$F_6(w,x) = \max\{d_6(w, BATRSCw)d_5(ATRSx, ATRSCw);$
$d_6(w, BATRSCw)d_4(TRSx, TRSCw); d_6(w, BATRSCw)d_3(RSx, RSCw);$
$d_6(w, BATRSCw)d_2(Sx, SCw); d_6(w, BATRSCw)d_1(x, BATRSCw);$
$d_6(w, BATRSx)d_1(x, Cw)\}.$

$G_1(x,y) = \max\{d_1(x, CBATRy), d_1(x, CBATRSx), d_2(Sx, SCBATRy)\}.$
$G_2(y,z) = \max\{d_2(y, SCBATz), d_2(y, SCBATRy), d_3(Ry, RSCBATz)\}.$
$G_3(z,u) = \max\{d_3(z, RSCBAu), d_3(z, RSCBATz), d_4(Tz, TRSCBAu)\}.$
$G_4(u,v) = \max\{d_4(u, TRSCBv), d_4(u, TRSCBAu), d_5(Au, ATRSCBv)\}.$

$G_5(v,w)$= max$\{d_5(v,ATRSCw)d_5(v,ATRSCBv)$,$d_6(Bv,BATRSCw)\}$.
$G_6(w,x)$= max$\{d_6(w,BATRSx)d_6(w,BATRSCw)d_1(Cw,CBATRSx)\}$.

Then $\alpha \in X$ is a unique fixed point of CBATRS,
$\beta \in Y$ is a unique fixed point of SCBATR ,
$\gamma \in Z$ is a unique fixed point of RSCBAT,
$\delta \in U$ is a unique fixed point of TRSCBA,
$\sigma \in V$ is a unique fixed point of ATRSCB,
$\rho \in W$ is a unique fixed point of BATRSC.
$S\alpha = \beta,\ R\beta = \gamma,\ T\gamma = \delta,\ A\delta = \sigma,\ B\sigma = \rho, C\rho = \alpha.$

Proof: Let $x_0 \in X$ is an any point and it is arbitrary first we define the six sequences $\{x_n\}$,$\{y_n\}$, $\{z_n\}$,$\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ in X,Y,Z,U,V and W respectively as below.
x_n=$(CBATRS)^n x_0$; y_n=Sx_{n-1}; Z_n=Ry_n;U_n=Tz_n; v_n=Au_n and
w_n=Bv_n; $\forall n \in N$.
We consider as $x_n \neq x_{n+1},\ y_n \neq$ to$y_n,\ \ z_{n+1} \neq z_n, u_n \neq u_{n+1},$
for otherwise x_n=x_{n+1} for few n, y_{n+}=$y_{n+2},\ z_{n+1} = z_{n+2}$,$u_{n+2} = u_{n+1},$
$v_{n+2} = v_{n+1}\ w_{n+2}$=$w_{n+1}$.
and we could put $x_n = \alpha y_{n+1} = \beta,\ z_{n+1} = \gamma,\ \delta = u_{n+1}$
$v_{n+1} = \sigma, w_{n+1}$=$\rho$. if y_n=y_{n+1}, then$z_n = z_{n+1},\ u_n = u_{n+1},\ v_n = v_{n+1},\ w_n = w_{n+1}$ and the later equalities imply that$x_n = x_{n+1}$ if $z_{n+1} = z_n$ after that $u_n = u_{n+1}$ and $v_n = v_{n+1}$ and the similarly if $x_n = x_{n+1}$ and $y_n = y_{n+1}$ in similar way if $v_n = v_{n+1}$ or $w_n = w_{n+1}$.
again $x_n = x_{n+1}$,
taking z_n= z_{n+1}, y= y_n in (2.2) we get

$$d_2(y_n,y_{n+1}) = d_2(SCBATz_{n_1},SCBATRy_n) \leq \frac{cF_2(y_n,z_{n-1})}{G_2(y_n,z_{n-1})}$$

$$= \frac{c\,max\{d_2(y_n,y_{n+1})d_6(w_{n-1},w_n);d_2(y_n,y_{n+1})d_5(v_{n-1},v_n);d_2(y_n,y_{n+1})d_4(u_{n-1},u_n);}{max\{d_2(y_n,y_n)d_2(y_n,y_{n+1})d_3(z_n,z_{n+1})\}}$$

$$\frac{d_2(y_n,y_n)d_2(y_n,y_{n+1});d_2(y_n,y_{n+1})d_6(z_{n-1},z_n);d_3(z_{n-1},z_n)d_2(y_n,y_{n+1})\}}{max\{d_3(z_n,z_{n+1})d_2(y_n,y_n)d_2(y_n,y_{n+1})\}}$$

$$= \frac{c\,max\{d_2(y_n,y_{n+1})[d_6(w_{n-1},w_n);d_5(v_{n-1},v_n);d_3(z_{n-1},z_n);d_4(u_{n-1},u_n);d_1(x_{n-1},x_n)]\}}{d_2(y_n,y_{n+1})}$$

Thus, we have,
$d_2(y_n,y_{n+1}) \leq$ c max$\{d_3(z_{n-1},z_n),d_1(x_{n+1},x_n),d_4(u_{n-1},u_n),d_5(v_{n-1},v_n),d_6(w_{n-1},w_n)\}$(2.7)
Taking u=u_{n-1} and z=z_{n-1} in (2.3) we get
$d_2(z_n,z_{n+1})$= $d_3(RSCBAu_{n-1},RSCBATz_n) \leq \frac{cF_3(z_u,u_{n-1})}{G_3(z_n,u_{n-1})}$

$$= \frac{c\,max\{d_3(z_n,z_{n+1})d_6(w_{n-1},w_n);d_3(z_n,z_{n+1})d_5(v_{n-1},v_n);d_3(z_n,z_{n+1})d_4(u_{n-1},u_n);}{max\{d_3(z_n,z_n)d_3(z_n,z_{n+1})d_4(u_n,u_{n+1})\}}$$

$$\frac{d_2(y_{n-1},y_n)d_3(z_n,z_{n+1});d_4(u_{n-1},u_n)d_3(z_n,z_{n+1});d_3(z_n,z_{n+1})d_1(x_{n-1},x_n)\}}{max\{d_3(z_n,z_{n+1})d_4(u_n,u_{n+1})d_3(z_n,z_n)\}}$$

$$= \frac{c\, max\{d_3(z_n,z_{n+1})[d_6(w_{n-1},w_n);d_5(v_{n-1},v_n);d_4(u_{n-1},u_n);d_1(x_{n-1},x_n)d_2(y_{n-1},y_n);]\}}{d_2(y_n,y_{n+1})}$$

$d_3(z_n, z_{n+1}) \leq$ c max$\{d_1(x_{n-1}, x_n)$,$d_2(y_{n-1}, y_n)$,$d_4(u_{n-1}, u_n)$, $d_5(v_{n-1}, v_n)$,$d_6(w_{n-1}, w_n)\}$
Using (2.7)
$d_3(z_n, z_{n+1}) \leq$ c max$\{d_1(x_{n-1}, x_n)$,$d_2(y_{n-1}, y_n)$,$d_4(u_{n-1}, u_n)$, $d_5(v_{n-1}, v_n)$,$d_6(w_{n-1}, w_n)\}(2.8)$

Taking $v{=}v_{n-1}$ and $u{=}u_n$ in (2.4)

$$d_4(u_n, u_{n+1})\ d_4(TRSCBv_{n-1}, TRSCBAu_n) \leq \frac{cF_4(u_n,v_{n-1})}{G_4(u_n,v_{n-1})}$$

$$= \frac{c\, max\{d_4(u_n,u_{n+1})d_6(w_{n-1},w_n);d_1(x_{n-1},x_n)d_4(u_n,u_{n+1});d_4(u_n,u_{n+1})d_5(v_{n-1},v_n);}{max\{d_4(u_n,u_n)d_4(u_n,u_{n+1})d_5(v_n,v_n)\}}$$

$$\frac{d_4(u_n,u_{n+1}),d_3(z_{n-1},z_n);d_4(u_n,u_{n+1})d_2(y_{n-1},y_n);d_4(u_n,u_{n+1})d_5(v_{n-1},v_n)\}}{max\{d_4(u_n,u_n),d_4(u_n,u_{n+1}),d_5(v_n,v_n)\}}$$

$$= \frac{c\, max\{d_4(u_n,u_{n+1})[d_6(w_{n-1},w_n);d_5(v_{n-1},v_n);d_3(z_{n-1},z_n);d_2(y_{n-1},y_n);\, d_1(x_{n-1},x_n)]\}}{d_4(u_n,u_{n+1})}$$

Using (2.7) and (2.8) inequalities we obtain that
$d_4(u_n, yu_{n+1}) \leq$ c max$\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_2(y_{n-1}, y_n),\ d_5(v_{n-1}, v_n), d_6(w_{n-1}, w_n)\}(2.9)$
taking $w{=}w_{n-1}$ and $v{=}v_n$ in (2.5)
$d_4(v_n, v_{n+1})\ d_5(ATRSCw_{n-1}, ATRSCBv_n) \leq \frac{cF_5(v_n,w_{n-1})}{G_5(v_n,w_{n-1})}$

$$= \frac{c\, max\{d_5(v_n,v_{n+1})d_6(w_{n-1},w_n);\, d_5(v_n,v_{n+1})d_1(x_{n-1},x_n);\, d_5(v_n,v_{n+1})d_4(u_{n-1},u_n);}{max\{d_5(v_n,v_n)d_5(v_n,v_{n+1})d_6(w_n,w_n)\}}$$

$$\frac{d_5(v_n,v_{n+1}),d_3(z_{n-1},z_n);\, d_5(v_n,v_{n+1})d_2(y_{n-1},y_n);\, d_5(v_n,v_{n+1})d_6(w_{n-1},w_n)\}}{max\{d_5(v_n,v_n),d_5(v_n,v_{n+1}),d_6(w_n,w_n)\}}$$

$$= \frac{c\, max\{d_5(v_n,v_{n+1})[d_6(w_{n-1},w_n);d_4(u_{n-1},u_n);d_2(y_{n-1},y_n);\, d_3(z_{n-1},z_n);d_1(x_{n-1},x_n)]\}}{d_5(v_n,v_{n+1})}$$

$=$ c max $\{d_6(w_{n-1}, w_n);d_4(u_{n-1}, u_n);\, d_2(y_{n-1}, y_n);d_1(x_{n-1}, x_n);\, d_3(z_{n-1}, z_n)\}(2.10)$
Using (2.7),(2.8) and (2.9) we obtain that
taking $w{=}w_n$ and $x{=}x_n$ in (2.6)

$$d_6(w_n, w_{n+1})\ d_6(BATRSx_{n-1}, BATRSCw_n) \leq \frac{cF_6(w_n,x_{n-1})}{G_6(w_n,x_{n-1})}$$

$$= \frac{c\, max\{d_6(w_n,w_{n+1})d_5(v_n,v_{n+1});d_6(w_n,w_{n+1})d_4(u_n,u_{n+1});d_6(w_n,w_{n+1})d_3(z_n,z_{n+1});}{max\{d_6(w_n,w_n),d_6(w_n,w_{n+1})d_1(x_n,x_n)\}}$$

$$\frac{d_6(w_n,w_{n+1}),d_2(y_n,z_{n+1});d_6(w_n,w_{n+1})d_1(x_n,x_{n+1});d_6(w_n,w_{n+1})d_1(x_{n-1},x_n)\}}{max\{d_6(w_n,w_n),d_6(w_n,w_{n+1})d_1(x_n,x_n)\}}$$

$$= \frac{c\,max\{d_6(w_n,w_{n+1})[d_5(v_n,v_{n+1});d_4(u_n,u_{n+1});d_2(y_n,y_{n+1});d_3(z_n,z_{n+1});d_1(x_{n-1},x_n)]\}}{d_6(w_n,w_{n+1})}$$

Using (2.7),(2.8),(2.9) and (2.10) we obtain that

$$d_6(w_n,w_{n+1}) \leq \text{c}\max\{d_4(u_{n-1},u_n),d_2(y_{n-1},y_n)d_3(z_{n-1},z_n),d_1(x_{n-1},x_n), d_6(w_{n-1},w_n)\} \qquad (2.11)$$

Again taking $x = x_n$ and $y = y_n$ in (2.1) we get

$$d_1(x_n,x_{n+1}) = d_1(CBATRy_n, CBATRSx_n) \leq \frac{cF_1(x_n,y_n)}{G_1(x_n,y_n)}$$

$$= \frac{c\,max\{d_1(x_n,x_{n+1})d_6(w_n,w_{n+1});d_1(x_n,x_{n+1})d_5(v_n,v_{n+1});d_1(x_n,x_{n+1})d_4(u_n,u_{n+1});}{max\{d_2(y_n,y_{n+1})d_1(x_n,x_n),d_1(x_n,x_{n+1})\}}$$

$$\frac{d_1(x_n,x_{n+1}),d_3(z_n,z_{n+1});d_1(x_n,x_{n+1})d_2(y_n,y_{n+1});d_1(x_n,x_n)d_2(y_n,y_{n+1})\}}{max\{d_1(x_n,x_n),d_1(x_n,x_{n+1})d_2(y_n,y_{n+1})\}}$$

$$= \frac{c\,max\{d_1(x_n,x_{n+1})[d_6(w_n,w_{n+1});d_5(v_n,v_{n+1});d_2(y_n,y_{n+1});d_3(z_n,z_{n+1});d_4(u_n,x_{n+1})]\}}{d_1(x_n,x_{n+1})}$$

$$= \text{c}\max\{d_6(w_n,w_{n+1}),d_5(v_n,v_{n+1}),d_4(u_n,u_{n+1}),d_3(z_n,z_{n+1}),d_6(y_n,y_{n+1})\} \qquad (2.12)$$

Continuing this process by induction on inequalities (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12) we obtain the following inequalities.

$$d_1(x_n,x_{n+1}) \leq c^{n-1}\{d_3(z_1,z_2), d_4(u_1,u_2), d_1(x_1,x_2), d_5(5_1,v_2), d_6(w_1,w_2)\}$$

$$d_2(y_n,y_{n+1}) \leq c^{n-1}\{d_3(z_1,z_2), d_4(u_1,u_2), d_5(5_1,v_2), d_1(x_1,x_2), d_6(w_1,w_2)\}$$

$$d_3(z_n,z_{n+1}) \leq c^{n-1}\{d_3(z_1,z_2), d_4(u_1,u_2), d_5(5_1,v_2), d_1(x_1,x_2), d_6(w_1,w_2)\}$$

$$d_4(u_n,u_{n+1}) \leq c^{n-1}\{d_3(z_1,z_2), d_4(u_1,u_2), d_5(5_1,v_2), d_1(x_1,x_2), d_6(w_1,w_2)\}$$

$$d_5(v_n,v_{n+1}) \leq c^{n-1}\{d_1(x_1,x_2),d_3(z_1,z_2),d_4(u_1,u_2),d_5(5_1,v_2),d_6(w_1,w_2)\}$$

$$d_6(w_n,w_{n+1}) \leq c^{n-1}\{d_3(z_1,z_2),d_5(5_1,v_2),d_4(u_1,u_2),d_1(x_1,x_2),d_6(w_1,w_2)\}$$

Since $0 \leq c < 1$, the sequences$\{x_n\}$,$\{y_n\}$,$\{z_n\}$,$\{u_n\}$, $\{v_n\}$ and $\{w_n\}$. are cauchy sequences.

Again since(X,d_1), (Y,d_2), (Z,d_3), (U,d_4), (v,d_5) and (W,d_6) are complete metric spaces,
we get,
$\lim_{n\to\infty} x_n$=α∈X, $\lim_{n\to\infty} y_n$=β∈Y, $\lim_{n\to\infty} z_n$=γ∈Z, $\lim_{n\to\infty} u_n$=δ∈U,
$\lim_{n\to\infty} v_n$=σ∈V, $\lim_{n\to\infty} w_n$=ρ∈W.
taking x=x_n and y=β in (2.1) we get

$$d_1(CBATR\beta, x_{n+1})\ d_1(CBATR\beta, CBATRSx_n) \leq \frac{cF_1(x_n,\beta)}{G_1(x_n,\beta)}$$

$$=\frac{c\,max\{d_1(x_n,x_{n+1})d_6(BATR\beta,w_{n+1});\,d_1(x_n,x_{n+1})d_5(ATR\beta,v_{n+1});\,d_1(x_n,x_{n+1})d_4(TR\beta,u_{n+1});}{max\{d_1(x_n,CBATR\beta),d_1(X_n,x_{n+1})d_2(y_{n+1},SCBATR\beta)\}}$$

$$\frac{d_1(x_n,x_{n+1}),d_3(R\beta,z_{n+1});d_1(x_n,x_{n+1})d_2(\beta,y_{n+1});d_1(x_n,CBATR\beta)d_2(\beta,SCBATR\beta)\}}{max\{d_1(x_C BATR\beta),d_2(y_{n+1},d_1(x_n,x_{n+1})SCBATR\beta)\}}$$

Letting $n\to\infty$ we get $d_1(CBATR\beta)\leq 0$
From which it follows that

CBATR$\beta=\alpha$, As same as , we using the inequalities (2.2), (2.3), (2.4), (2.5) and (2.6) we will show that $BATRS\alpha{=}\rho$, $ATRSC\rho=\sigma$, $TRSCB\sigma=\delta$, $RSCBA\delta=\gamma$, $SCBAT\gamma=\beta$. Taking z $=R\beta$ and y $=y_n$ in (2.2) we get
$d_2(SCBAT\beta,y_{n+1})$

$d_2(SCBAT\beta,SCBATRy_n)\leq\frac{cF_2(y_n,R\beta)}{G_2(y_n,R\beta)}$

$$=\frac{c\,max\{d_2(y_n,y_{n+1})d_6(CBATR\beta,w_n);d_2(y_n,y_{n+1})d_5(BATR\beta,v_n);d_2(y_n,y_{n+1})d_4(ATR\beta,u_n);}{max\{d_2(y_n,SCBATR\beta),d_2(y_n,y_{n+1})d_3(z_n,RSCBATR\beta)\}}$$

$$\frac{d_2(y_n,y_{n+1}),d_3(TR\beta,z_{n+1});d_2(y_n,y_{n+1})d_1(CBATR\beta,x_n);d_2(y_n,SCBATR\beta)d_3(R\beta,z_n)\}}{max\{d_2(y_n,SCBATR\beta),d_2(y_n,y_{n+1})d_1(z_n,RSCBATR\beta)\}}$$

Letting $n\to\infty$ since $d_1(CBATR\beta,\beta){=}\alpha$ we get,

$$d_2(y_n,SCBATR\beta)\leq\frac{cd_2(\beta,SCBATR\beta),d_3(R\beta,\gamma);}{max\{d_2(\beta,SCBATR\beta),)d_3(\gamma,RS\alpha)\}}$$

Here two cases arises:
Case(1): If max$\{d_2(SCBATR\beta)$,$d_3(\gamma,RS\alpha)\}$ we have

$$d_2(SCBATR\beta)=d_2(S\alpha,\beta)\leq\frac{cd_2(\beta,SCBATR\beta),d_3(R\beta,\gamma);}{max\{d_2(\beta,SCBATR\beta)\}}$$

$$=c\,d_3(R\beta,\gamma)$$

Case(2): If max$\{d_2(\beta,SCBATR\beta)=d_3(\gamma,RS\alpha)\}$
$d_3(\gamma,RS\alpha)$ and $d_2(\beta,SCBATR\beta)\neq 0$ we obtain that

$$d_2(SCBATR\beta,\beta)=d_2(S\alpha,\beta)\leq\frac{cd_2(\beta,SCBATR\beta),d_3(R\beta,\gamma);}{max\{d_3(\gamma,RS\alpha)\}}$$

$$=c\,d_3(R\beta,\gamma)$$

Thus in both cases we obtain that
$d_2(SCBATR\beta,\beta)=d_2(S\alpha,\beta)$

$= c\, d_3(R\beta, \gamma)$

Taking z=z_n and u$=T\gamma$ in (2.3) we get
$d_3(RSCBAT\gamma, z_{n+1}) = c\, d_3(RSCBAT\gamma, RSCBATz_n)$

$\leq \frac{cF_3(z_n,T\gamma)}{G_3(z_n,T\gamma)}$

$=\frac{c\, max\{d_3(z_n,z_{n+1})d_6(BAT\gamma,w_n);d_3(z_n,z_{n+1})d_5(AT\gamma,v_n);d_3(z_n,z_{n+1})d_4(T\gamma,u_n);}{max\{d_3(z_n,RSCBAT\gamma),d_3(z_n,z_{n+1})d_4(u_n,TRSCBAT\gamma)\}}$

$\frac{d_3(z_n,z_{n+1}),d_1(BAT\gamma,x_n);d_3(z_n,z_{n+1})d_6(T\gamma,u_n)\}}{max\{d_3(z_n,RSCBAT\gamma),d_3(z_n,z_{n+1}),d_4(u_n,TRSCBAT\gamma)\}}$

Letting $n \to \infty$ and since $(SCBAT\gamma)$=β we have

$d_3(RSCBAT\gamma, \gamma) = d_3(R\beta, \gamma)$

$\leq \frac{c\, d_3(\gamma,RSCBAT\gamma),d_4(T\gamma,\delta)\}}{max\{d_3(\gamma,RSCBAT\gamma),d_4(\delta,TR\beta)\}}$

$=$ c $d_4(T\gamma, \delta)$

Thus as above inequalities we obtain the following inequality
$d_3(RSCBAT\gamma, \gamma) = d_3(R\beta, \gamma)$
$=$ c $d_4(T\gamma, \delta)$ (2.13)

In the similar way using inequalities (2.4) (2.5) (2.6) and (2.1) we get,
$d_4(TSCBA\delta, \delta) = d_4(T\gamma, \delta)$

$\leq$ c $d_5(A\delta, \sigma)$ (2.14)

$d_5(ATSCB\sigma, \sigma) = d_5(A\delta, \sigma)$

$\leq$ c $d_6(B\sigma, \rho)$ (2.15)

$d_6(BATSC\rho, \rho) = d_6(B\sigma, \rho)$

$\leq$ c $d_1(C\rho, \alpha)$ (2.16)

$d_1(CBATS\alpha, \alpha) = d_1(C\rho, \alpha)$

$\leq$ c $d_2(R\alpha, \beta)$ (2.17)

Using (2.12) (2.13) (2.14) (2.15) (2.16) and (2.17) we get

$d_2(SCBATR\beta, \beta) = d_2(S\alpha, \beta) \leq c d_3(R\beta, \gamma) \leq c^2 d_4(T\gamma, \delta)$

$$\leq c^3 d_5(A\delta,\sigma) \leq c^4 d_6(B\sigma,\rho) \leq c^5 d_1(C\rho,\alpha) \leq c^6 d_2(R\alpha,\beta)$$

$$\leq c^6 d_2(SCBATR\beta,\beta)$$

from which it follows that

$SCBATR\beta=\beta; S\alpha=\beta, R\beta=\gamma, T\gamma=\delta, A\delta=\sigma B\sigma=\rho\ C\rho=\alpha$, since 0≤c<1

Similarly we can show that
$RSCBAT\gamma=\gamma; TRSCBA=\delta; ATRSCB=\sigma;$
$BATRSC=\rho; CBATRS=\alpha; SCBATR=\beta$

i.e.α,β,γ,δ,σ,ρ are fixed points CBATRS,SCBATR,RSCBAT,TRSCBA,ATRSCB and BATRSC respectively.

Now we show that these are unique fixed points let us postulate that α be the other fixed point of CBATRS.
Using inequality (2.1) for y $=S\alpha$ and x $=\alpha'$ we get

$$d_1(CBATRS\alpha, CBATRS\alpha') \leq \frac{cF_1(\alpha',S\alpha)}{G_1(\alpha',S\alpha)}$$

$$= \frac{c\,max\{d_1(\alpha'\alpha')d_6(BATRS\alpha,BATRS\alpha');d_5(ATRS\alpha,ATRS\alpha');d_1(\alpha'\alpha')d_4(TRS\alpha,TRS\alpha');d_1(\alpha'\alpha')}{max\{d_1(\alpha'\alpha),d_1(\alpha'\alpha')d_2(S\alpha',S\alpha)\}}$$

$$\frac{d_1(\alpha'\alpha'),d_2(S\alpha,SCBATRS\alpha);d_1(\alpha'\alpha')d_2(S\alpha',S\alpha\}}{max\{d_1(\alpha'\alpha),d_1(\alpha'\alpha'),d_2(S\alpha',S\alpha)\}}$$

Here two cases arise:
Case(a): If max$\{d_1(\alpha',\alpha)$,$d_2(S\alpha',S\alpha)\}$

$=d_2(S\alpha',S\alpha)$ then we get

$d_1(\alpha,\alpha') \leq$ c $d_1(\alpha'\alpha)$ which gives α'=α

Case(b): If max$\{d_1(\alpha',\alpha)$,$d_2(S\alpha',S\alpha)\}$
$=d_1(\alpha',\alpha)$ then we get

$d_1(\alpha,\alpha')$≤c$d_2 S\alpha', S\alpha$

Now taking z $=RSC\alpha$ and y $=S\alpha'$ in equation (2.2) we get

$$d_2(S\alpha,S\alpha')=d_2(SCBATR\alpha, SCBATRS\alpha')\leq \frac{cF_2(S\alpha',RS\alpha)}{G_2(S\alpha',RS\alpha)}$$

$$= \frac{c\,max\{d_2(S\alpha',S\alpha')d_1(\alpha',\alpha');d_2(S\alpha',S\alpha')d_6(CBATRS\alpha,CBATRS\alpha');d_2(S\alpha',S\alpha')d_5(BATRS\alpha,BATRS\alpha');}{max\{d_2(S\alpha',S\alpha),d_2(S\alpha',S\alpha)d_3(RS\alpha',RS\alpha)\}}$$

$\frac{d_2(S\alpha',S\alpha'),d_3(TRS\alpha,TRS\alpha');d_2(S\alpha',S\alpha')d_3(RS\alpha,RS\alpha'))\}}{max\{d_2(S\alpha',S\alpha),d_2(S\alpha',S\alpha)d_3(RS\alpha',RS\alpha)\}}$

$= \frac{cmax\{d_2(S\alpha',S\alpha);d_3(RS\alpha,RS\alpha')}{max\{d_2(S\alpha',S\alpha),d_3(RS\alpha',RS\alpha)\}}$

As discussed above we get

$$d_2(S\alpha, S\alpha') \leq \text{c}\, d_3(RS\alpha, RS\alpha') \quad (2.18)$$

Similarly we taking u $= TRS\alpha$ and z $= RS\alpha'$ in equation (2.3) we get

$$d_3(RS\alpha, RS\alpha') \leq \text{c}\, d_4(TRS\alpha, TRS\alpha'). \quad (2.19)$$

Taking v $= ATRS\alpha$ and u $= TRS\alpha'$ in equation (2.4) we get

$$d_4(TRS\alpha, TRS\alpha') \leq \text{c}\, d_5(ATRS\alpha, ATRS\alpha'). \quad (2.20)$$

Taking w=$BATRS\alpha$ and v=$BTRS\alpha'$ in equation (2.5) we get

$$d_5(ATRS\alpha, ATRS\alpha') \leq \text{c}\, d_6(BATRS\alpha, BATRS\alpha'). \quad (2.21)$$

Taking x=$CBATRS\alpha$ and w=$BATRS\alpha'$ in equation (2.6) we get

$$d_6(BATRS\alpha, BATRS\alpha') \leq \text{c}\, d_1(CBATR\alpha, CBATR\alpha').\ = \text{c}d_1(\alpha, \alpha')(2.22)$$

Using equations $(2.17), (2.18), (2.19), (2.20), (2.21)$ and (2.22) we have

$$d_1(\alpha, \alpha') \leq \text{c}\, d_2(S\alpha, S\alpha') \leq c^2\, d_3(RS\alpha, RS\alpha')$$

$$\leq c^3\, d_4(TRS\alpha, TRS\alpha')\ \leq c^4\, d_5(ATRS\alpha, ATRS\alpha')$$

$$\leq c^5\, d_6(BATRS\alpha, BATRS\alpha')$$

$$\leq c^6\, d_1(\alpha, \alpha')$$

which is impossible since $0 \leq \text{c} < 1$.

Thus $d_1(\alpha, \alpha')$ =0 i.e. $\alpha = \alpha'$ i.e. α is unique fixed point of CBATRS.
$\beta, \gamma, \delta, \sigma, \rho$ are unique fixed point of SCBATR, RSCBAT, TRSCBA, ATRSCB and BATRSC respectively.
This complete the proof.

Corollary 2.1: Let$(Z_1, d_1), (Z_2, d_2)$, (Z_3, d_3) and (Z_4, d_4) be complete metric spaces and they are complete. LetA_1 is mapping from Z_1 toZ_2, A_2 is mapping from $Z_2$to Z_3, $A_3$is mapping fromZ_3 to Z_4 and $A_4$is mapping from Z_4 to Z_1 satisfying the inequalities:

$d_1(A_4A_3A_2A_1z_1, A_4A_3A_2A_1z_1')$

$c\,max\{d_1(z_1, z_1')$,$d_1(z_1, A_4A_3A_2A_1z_1)$, $d_1(z_1', A_4A_3A_2A_1z_1')$, $d_2(A_1z_1, A_1z_1')$,

$d_3(A_2A_1z_1, A_2A_1z_1')$,$d_4(A_3A_2A_1z_1, A_3A_2A_1z_1'),\}$

$d_2(A_1A_4A_3A_2z_2, A_1A_4A_3A_2z_2')$
$c\, max\{d_2(z_2, z_2')$,$d_2(z_2, A_1A_4A_3A_2z_2)$, $d_2(z_2', A_1A_4A_3A_2z_2')$, $d_3(A_2z_2, A_2z_2')$,
$d_4(A_3A_2z_2, A_3A_2z_2')$, $d_1(A_4A_3A_2A_1z_2, A_4A_3A_2z_2')\}$

$d_3(A_2A_1A_4A_3z_3, A_2A_1A_4A_3z_3')$
$c\, max\{d_3(z_3, z_3')$,$d_3(z_3, A_2A_1A_4A_3z_3)$, $d_3(z_3', A_2A_1A_4A_3z_3')$, $d_4(A_3z_3, A_3z_3')$,
$d_1(A_4A_3Az_3, A_4A_3z_3')$, $d_2(A_1A_4A_3z_3, A_1A_4A_3z_3')\}$

$d_4(A_3A_2A_1A_4z_4, A_3A_2A_1A_4z_4')$
$c\, max\{d_4(z_4, z_4')$,$d_4(z_4, A_3A_2A_1A_4z_4)$, $d_4(z_4', A_3A_2A_1A_4z_4')$, $d_1(A_4z_4, A_4z_4')$,
$d_2(A_1A_4Az_4, A_1A_4z_4')$, $d3(A_2A_1A_4z_4, A_2A_1A_4z_4')\}$

$\forall\ z_1, z_1' \in Z_1,\ z_2, z_2' \in Z_2, z_3, z_3' \in Z_3$ and $z_4, z_4' \in Z_4$,where $0 < c < 1$,
then$A_4A_3A_2A_1$ has a unique fixed point $\alpha_1 \in Z_1$,
$A_1A_4A_3A_2$ has a unique fixed point $\alpha_2 \in Z_2$,
$A_2A_1A_4A_3$ has a unique fixed point $\alpha_3 \in Z_3$,
$A_3A_2A_1A_4$ has a unique fixed point $\alpha_4 \in Z_4$.
Further $A_1\alpha_1 = \alpha_2, A_2\alpha_2 = \alpha_3$,
$A_3\alpha_3 = \alpha_4\ \ A_4\alpha_4 = \alpha_1$.

3. Conclusion

In this paper we obtain the a related fixed point theorem for six metric spaces using contractive type mapping.

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