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Estimates for Certain Integral Inequalities on (p, q)-calculus

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Abstract. The aim of this paper is to derive a new quantum analogue of an integral identity by using (p, q)-calculus. As a consequence of this identity, some new estimates for Ostrowski type inequality for (p, q)-differentiable η-convex and η-quasi-convex functions are obtained. Moreover, some new estimates for Hermite-Hadamard type inequality for (p, q)-differentiable η-convex and η-quas-iconvex functions are given as well.

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1. Introduction

In mathematics, q-calculus, also known as quantum calculus, is the study of calculus with no limits. In quantum calculus, we obtain q-analogues of mathematical objects that can be recaptured as $q \to 1^-$. quantum calculus first time developed by Jackson in the

early twentieth century, but the history of quantum calculus can be traced back to some much earlier work done by Euler and Jacobi et al. [10]. The subject of quantum calculus has numerous applications in various areas of mathematics and physics such as number theory, orthogonal polynomials, combinatorics, basic hypergeometric functions, quantum theory, mechanics and theory of relativity. Quantum calculus has received exclusive interest by many researchers and hence it is considered to be a corporate subject between mathematics and physics. Interested readers are referred to [5]-[3] for some current advances in the theory of quantum calculus and theory of inequalities in quantum calculus.
In recent articles, Tariboon et al. [32, 33] presented the idea of q-derivatives and q-integral over the definite interval $[\phi, \psi]$ of $\mathbb{R}$ and addressed numerous problems on quantum analogues such as the q-Hlder inequality, the q-Ostrowski inequality, the q-Cauchy-Schwarz inequality, the q-Grss-Cebysev integral inequality, the q-Grss inequality. The most recently, Alp et al. [2], proved q-Hermite-Hadamard inequality, some new q-Hermite-Hadamard inequalities, and generalized q-Hermite-Hadamard inequality, also they studied some integral inequalities which provide quantum estimates for the left part of the quantum analogue of Hermite-Hadamard inequality through q-differentiable convex and quasi-convex functions and other integral inequalities by classical convexity. Most recently, Tunç et al. [35]-[36] derived the notion of (p, q)-calculus on $[\phi, \psi]$ of $\mathbb{R}$. Mathematical formulae of (p, q)-derivative and (p, q)-integral have been derived and new fundamental properties are defined. The results that depend on (p, q)-calculus are the Minkowski inequality, Hölder inequality, Grüss and Grüess-Chebysev inequality and many others. Latif et al. [15], gave Hermite-Hadamard type integral inequalities for convexity and quasi-convexity functions on (p, q)-calculus. Recently, Kunt et al. [21] proposed left part of Hermite-Hadamard inequalities by using (p, q)-differentiable convex as well as quasi-convex functions. Now a days, the interest of researchers is increasing in the topic of (p, q)-calculus because of new results established in literature. Please see, [17]-[11] and the references cited therein.
It is well known that the theory of inequality plays a fundamental role in pure and applied mathematics and has extensive applications. Apart from the larger number of research results of inequalities in classical analysis, there are considerable works on the study of inequalities so that the theory of convex functions was widely discussed and applied to various areas of science. In few years, tremendous research has been witnessed on inequalities along with a large number of articles and many productive applications.

Let $g : J \subset \mathbb{R} \to \mathbb{R}$. A function g is called convex function on J, if the following inequality

$$g(\lambda\phi + (1 - \lambda)\psi) \leq \lambda g(\phi) + (1 - \lambda)g(\psi), \qquad (1.\ 1)$$

holds for all $\phi, \psi \in J$ and $\lambda \in [0, 1]$.

Recently, numerous mathematician have worked on the generalization of classical inequalities through different mathematical approaches. One of the most popular and useful inequalities is the Hermite-Hadamard inequality.

In [9], Hadamard established a popular and appropriate inequality in analysis as:

$$g\left(\frac{\phi + \psi}{2}\right) \leq \frac{1}{\psi - \phi}\int_{\phi}^{\psi} g(\lambda)d\lambda \leq \frac{g(\phi) + g(\psi)}{2},$$

the above famous inequality is called Hermite-Hadamard inequality, see [37]- [31].
The paper is organized as follows: In Sections 2 and 3, we will defined some notions of η-convexity and η-quasi-convexity and (p, q)-calculus. As an auxiliary result, we introduce an identity correlated with (p, q)-differentiable functions. In Section 4 and 5, with the help of the auxiliary result, we will establish our main results and last section is conclusion.

2. Formulations and Basic Facts

Let us recall the formulations and basic facts which are firmly concerned to this paper. A new class of convexity was put forward by Gordji et al. [7]. This new class is known as η-convex function. The class of η-convex function is generalization of classical convex function.

Definition 2.1. *A function $g : J \subset \mathbb{R} \to \mathbb{R}$ is called an η-convex function, If following inequality defined as*

$$g\left((1-\lambda)\phi + \lambda\psi\right) \leq g(\phi) + \lambda\eta(g(\psi), g(\phi)), \tag{2. 2}$$

holds for all $\phi, \psi \in J$ and $\lambda \in [0, 1]$.

Note that, if we put $\eta(g(\psi), g(\phi)) = g(\psi) - g(\phi)$ in inequality (2. 2), then we get inequality (1. 1).

Definition 2.2. [7] *A function $g : J \subset \mathbb{R} \to \mathbb{R}$ is called an η-convex function, If following inequality defined as*

$$g\left((1-\lambda)\phi + \lambda\psi\right) \leq \max\left\{g(\phi), g(\phi) + \eta(g(\psi), g(\phi))\right\},$$

holds for all $\phi, \psi \in J$ and $\lambda \in [0, 1]$.

Gordji et al. in [7] presented the following inequality
Let $g : [\phi, \psi] \to \mathbb{R}$ be an η-convex function such that η be bounded above on $g([\phi, \psi]) \times g([\phi, \psi])$, then

$$\begin{aligned} & g\left(\frac{\phi+\psi}{2}\right) - \frac{1}{2}\int_{\phi}^{\psi} \eta(g(\phi+\psi-\lambda), g(\lambda))d\lambda \\ \leq \ & \frac{1}{\psi-\phi}\int_{\phi}^{\psi} g(\lambda)d\lambda \leq \frac{g(\phi)+g(\psi)}{2} + \frac{\eta\left(g(\phi), g(\psi)\right) + \eta\left(g(\psi), g(\phi)\right)}{4}. \end{aligned}$$

For more information about η-convex functions, see [7, 8].

Moreover, in the literature Ostrowski inequality is another one of the most significant mathematical inequalities. Many mathematicians have worked on and around it in several different ways with many applications in Analysis and Probability etc. It is stated in [23] as follows:

Let $g : [\phi, \psi] \to \mathbb{R}$ be a continuously differentiable function on $[\phi, \psi]$ and $||g^{'}||_{\infty} = \sup_{z \in (\phi,\psi)} \left|g^{'}(z)\right| < \infty$, then

$$\left|g(x) - \frac{1}{\phi-\psi}\int_{\phi}^{\psi} g(z)dz\right| \leq \left[\frac{(x-\phi)^2 + (\psi-x)^2}{2(\phi-\psi)}\right] \left\|g^{'}\right\|_{\infty}$$

holds for all $x \in [\phi, \psi]$.

Now we defined some basic definitions and properties on q-calculus which is first time introduced by Tariboon et al. [32, 33].

Definition 2.3. [33] *A function $g : [\phi, \psi] \to \mathbb{R}$ is said to be continuous, quantum derivative of g at $\lambda \in [\phi, \psi]$ with $0 < q < 1$ is characterized by the expression*

$$_{\phi}D_q g(\lambda) = \frac{g(\lambda) - g(q\lambda + (1-q)\phi)}{(1-q)(\lambda - \phi)}, \quad \lambda \neq \phi.$$

we have $_{\phi}D_q g(\phi) = \lim_{\phi \to \lambda} D_q g(\lambda)$.

Definition 2.4. [33] *A function $g : [\phi, \psi] \to \mathbb{R}$ is said to be continuous, quantum-integral over $[\phi, \psi]$ with $0 < q < 1$ is characterized by the expression*

$$\int_{\phi}^{\lambda} g(x)_{\phi} d_q x = (1-q)(\lambda - \phi) \sum_{n=0}^{\infty} q^n g\left(q^n \lambda + (1-q^n)\phi\right)$$

for $\lambda \in [\phi, \psi]$. If $\gamma \in (\phi, \lambda)$, then q-integral on $[\gamma, \lambda]$ is stated as

$$\int_{\gamma}^{\lambda} g(x)_{\phi} d_q x = \int_{\phi}^{\lambda} g(x)_{\phi} d_q x - \int_{\phi}^{\gamma} g(x)_{\phi} d_q x.$$

Alp et al. [2], addressed generalized q-Hermite-Hadamard type inequality on quantum calculus:

Theorem 2.5. [2] *Let $g : [\phi, \psi] \to \mathbb{R}$ be a quantum convex function over $[\phi, \psi]$ with $q \in (0, 1)$. Then we have*

$$g\left(\frac{q\phi + \psi}{1+q}\right) \leq \frac{1}{\psi - \phi} \int_{\phi}^{\psi} g(x)_{\phi} d_{p,q} x \leq \frac{qg(\phi) + g(\psi)}{1+q}.$$

Motivated by this ongoing research on quantum analogues for q-differentiable convex functions. Tunç et al. [35] derived notion of (p, q)-calculus on the intervals $[\phi, \psi]$ of $\mathbb{R}$.

Definition 2.6. [35] *A function $g : [\phi, \psi] \to \mathbb{R}$ is said to be continuous, then (p, q)-differentiable function of g at $\lambda \in [\phi, \psi]$ with $0 < q < p \leq 1$ is defined as:*

$$_{\phi}D_{p,q} g(\lambda) = \frac{g(p\lambda + (1-p)\phi) - g(q\lambda + (1-q)\phi)}{(p-q)(\lambda - \phi)}, \quad \lambda \neq \phi.$$

we have $_{\phi}D_{p,q} g(\phi) = \lim_{\phi \to \lambda} D_{p,q} g(\lambda)$.

Definition 2.7. [35] *A function $g : [\phi, \psi] \to \mathbb{R}$ is said to be continuous, the definite (p, q)-integral over $[\phi, \psi]$ with $0 < q < p \leq 1$ is defined as:*

$$\int_{\phi}^{\lambda} g(x)_{\phi} d_{p,q} x = (p-q)(\lambda - \phi) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} g\left(\frac{q^n}{p^{n+1}} \lambda + \left(1 - \frac{q^n}{p^{n+1}}\right)\phi\right)$$

for $\lambda \in [\phi, \psi]$. If $c \in (\phi, \lambda)$, then it is defined as

$$\int_{c}^{\lambda} g(x)_{\phi} d_{p,q} x = \int_{\phi}^{\lambda} g(x)_{\phi} d_{p,q} x - \int_{\phi}^{c} g(x)_{\phi} d_{p,q} x.$$

Latif et al. [22] introduced Hermit Hadamard type inequality for (p,q)-calculus as follows:

Theorem 2.8. [22] *Let $g:[\phi,\psi]\to\mathbb{R}$ be a convex differentiable function on $[\phi,\psi]$ and $0<q<p\leq 1$. Then we have*

$$g\left(\frac{q\phi+p\psi}{p+q}\right)\leq\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi}g(x)_{\phi}d_{p,q}x\leq\frac{qg(\phi)+pg(\psi)}{p+q}.$$

Latif et al. [22] also presented a Lemma which is engaged with (p,q)-trapezoidal type inequality as follows:

Lemma 2.9. *Let $g:[\phi,\psi]\to\mathbb{R}$ be a (p,q)- differentiable function on (ϕ,ψ). If ${}_{\phi}D_{p,q}g$ is continuous and integrable on $[\phi,\psi]$ where $0<q<p\leq 1$, then we get*

$$\begin{aligned}H_g(\phi,\psi;p;q)&=\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi}g(x)_{\phi}d_{p,q}x-\frac{qg(\phi)+pg(\psi)}{p+q}\\&=\frac{q(\psi-\phi)}{p+q}\int_0^1(1-(p+q))_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)_0d_{p,q}\lambda.\end{aligned}$$

Ostrowski type inequalities for q-differentiable convex functions presented by Noor et al. in [27].

Lemma 2.10. *Let $g:[\phi,\psi]\to\mathbb{R}$ be a q-differentiable function on (ϕ,ψ). If ${}_{\phi}D_{q}g$ is continuous and integrable on $[\phi,\psi]$ where $0<q<1$, then*

$$\begin{aligned}M_g(\phi,\psi;q)&=g(x)-\frac{1}{\psi-\phi}\int_{\phi}^{\psi}g(u)_{\phi}d_qu\\&=\frac{q(x-\phi)^2}{\psi-\phi}\int_0^1\lambda_{\phi}D_qg(\lambda x+(1-\lambda)\phi)_0d_q\lambda\\&+\frac{q(\psi-x)^2}{\psi-\phi}\int_0^1\lambda_{\phi}D_qg(\lambda x+(1-\lambda)\psi)_0d_q\lambda,\end{aligned}$$

holds for all $x\in[\phi,\psi]$.

In this context, the actual motivation of this paper is introducing a certain (p,q)-integral inequalities by using η-convex and η-quasi-convex functions. These are obtained as special cases when $p=1$ and $q\to 1$.

3. A KEY LEMMA

Lemma 3.1. *Let $g:[\phi,\psi]\to\mathbb{R}$ be a (p,q)- differentiable function on (ϕ,ψ). If ${}_{\phi}D_{p,q}g$ is continuous and integrable on $[\phi,\psi]$ where $0<q<p\leq 1$, then*

$$\begin{aligned}K_g(\phi,\psi;p;q)&=g(x)-\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi}g(u)_{\phi}d_{p,q}u\\&=\frac{q(x-\phi)^2}{(\psi-\phi)}\int_0^1\lambda_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\phi)_0d_{p,q}\lambda\\&+\frac{q(\psi-x)^2}{(\psi-\phi)}\int_0^1\lambda_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\psi)_0d_{p,q}\lambda,\end{aligned}$$

holds for all $x \in [\phi, \psi]$.

Proof. Applying Definition 2.6 and Definition 2.7 , we have

$$\begin{aligned} &\frac{(x-\phi)^2}{(\psi-\phi)}\int_0^1 \lambda_\phi D_{p,q} g(\lambda x+(1-\lambda)\phi)_0 d_{p,q}\lambda \\ &+\frac{(\psi-x)^2}{(\psi-\phi)}\int_0^1 \lambda_\phi D_{p,q} g(\lambda x+(1-\lambda)\psi)_0 d_{p,q}\lambda \\ &= I_1 + I_2. \end{aligned} \tag{3. 3}$$

Now

$$\begin{aligned} I_1 &= \frac{(x-\phi)^2}{\psi-\phi}\int_0^1 \frac{g(p\lambda x+(1-p\lambda)\phi)-g(q\lambda x+(1-q\lambda)\phi)}{(p-q)(x-\phi)}{}_0 d_{p,q}\lambda \\ &= \frac{x-\phi}{\psi-\phi}\left[\begin{array}{c} \sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} g\left(\frac{q^n}{p^n}x+\left(1-\frac{q^n}{p^n}\right)\phi\right) \\ -\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} g\left(\frac{q^{n+1}}{p^{n+1}}x+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\phi\right) \end{array}\right] \\ &= \frac{x-\phi}{\psi-\phi}\left[\begin{array}{c} \frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n} g\left(\frac{q^n}{p^n}x+\left(1-\frac{q^n}{p^n}\right)\phi\right) \\ -\frac{1}{q}\sum_{n=1}^{\infty}\frac{q^n}{p^n} g\left(\frac{q^n}{p^n}x+\left(1-\frac{q^n}{p^n}\right)\phi\right) \end{array}\right] \end{aligned} \tag{3. 4}$$

$$\begin{aligned} &= \frac{x-\phi}{\psi-\phi}\left[\begin{array}{c} \frac{1}{q}g(x)-\left(\frac{1}{q}-\frac{1}{p}\right) \\ \times\sum_{n=0}^{\infty}\frac{q^n}{p^n} g\left(\frac{q^n}{p^n}x+\left(1-\frac{q^n}{p^n}\right)\phi\right) \end{array}\right] \\ &= \frac{(x-\phi)}{q\,(\psi-\phi)}g(x)-\frac{1}{pq(\psi-\phi)}\int_\phi^{px+(1-p)\phi} g(u)_\phi d_{p,q}u. \end{aligned} \tag{3. 5}$$

Similarly

$$I_2 = \frac{(\psi-x)}{q\,(\psi-\phi)}g(x)-\frac{1}{pq(\psi-\phi)}\int_x^{p\psi+(1-p)x} g(u)_\phi d_{p,q}u. \tag{3. 6}$$

Substituting (3.5) and (3.6) into (3.3), and multiplying the resulting identity by q gives the desired identity. □

4. Main Results

Theorem 4.1. *Let* $g : J \subset \mathbb{R} \to \mathbb{R}$ *be a* (p, q)*-differentiable function on* J^o *with* $0 < q < p \le 1$*. If* ${}_\phi D_{p,q} g$ *is continuous and integrable on* $[\phi, \psi]$ *such that* $|{}_\phi D_{p,q} g|$ *is an* η*-convex* *function on $[\phi,\psi]$, where $\phi,\psi\in J^{\circ}$ with $\phi<\psi$, then*

$$
\begin{aligned}
&\left|K_g\left(\phi,\psi;p;q\right)\right| \\
&\leq \frac{q\left(x-\phi\right)^2}{\left(\psi-\phi\right)}\left[\frac{1}{p+q}\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|+\frac{1}{p^2+pq+q^2}\eta\left(\left|{}_{\phi}D_{p,q}g\left(x\right)\right|,\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|\right)\right] \\
&+\frac{q\left(\psi-x\right)^2}{\left(\psi-\phi\right)}\left[\frac{1}{p+q}\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|+\frac{1}{p^2+pq+q^2}\eta\left(\left|{}_{\phi}D_{p,q}g\left(x\right)\right|,\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|\right)\right]
\end{aligned}
$$

holds for all $x\in[\phi,\psi]$.

Proof. Utilizing Lemma 3.1 and the fact that $|{}_{\phi}D_{p,q}g|$ is η-convex function on $[\phi,\psi]$, we get

$$
\begin{aligned}
\left|K_g\left(\phi,\psi;p;q\right)\right| &= \left|\frac{q\left(x-\phi\right)^2}{\psi-\phi}\int_0^1\lambda_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\phi)_0d_{p,q}\lambda\right. \\
&\left.+\frac{q\left(\psi-x\right)^2}{\psi-\phi}\int_0^1\lambda_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\psi)_0d_{p,q}\lambda\right| \\
&\leq \frac{q\left(x-\phi\right)^2}{\psi-\phi}\int_0^1\lambda\left|{}_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\phi)\right|{}_0d_{p,q}\lambda \\
&+\frac{q\left(\psi-x\right)^2}{\psi-\phi}\int_0^1\lambda\left|{}_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\psi)\right|{}_0d_{p,q}\lambda \\
&\leq \frac{q\left(x-\phi\right)^2}{\psi-\phi}\int_0^1\lambda\left[\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|+\lambda\eta\left(\left|{}_{\phi}D_{p,q}g\left(x\right)\right|,\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|\right)\right]{}_0d_{p,q}\lambda \\
&+\frac{q\left(\psi-x\right)^2}{\psi-\phi}\int_0^1\lambda\left[\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|+\lambda\eta\left(\left|{}_{\phi}D_{p,q}g\left(x\right)\right|,\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|\right)\right]{}_0d_{p,q}\lambda \\
&\leq \frac{q\left(x-\phi\right)^2}{\psi-\phi}\left[\frac{1}{p+q}\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|+\frac{1}{p^2+pq+q^2}\eta\left(\left|{}_{\phi}D_{p,q}g\left(x\right)\right|,\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|\right)\right] \\
&+\frac{q\left(\psi-x\right)^2}{\psi-\phi}\left[\frac{1}{p+q}\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|+\frac{1}{p^2+pq+q^2}\eta\left(\left|{}_{\phi}D_{p,q}g\left(x\right)\right|,\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|\right)\right].
\end{aligned}
$$

Hence, we get the outcome that we need. □

Theorem 4.2. *Let $g:J\subset\mathbb{R}\to\mathbb{R}$ be a (p,q)-differentiable function on J^o with $0<q<p\leq 1$. If ${}_{\phi}D_{p,q}g$ is continuous and integrable on $[\phi,\psi]$ such that $|{}_{\phi}D_{p,q}g|^r$ is an η-convex function on $[\phi,\psi]$ where $\phi,\psi\in J^{\circ}$ with $\phi<\psi$, then*

$$
\begin{aligned}
&\left|K_g\left(\phi,\psi;p;q\right)\right| \leq \frac{q\left(x-\phi\right)^2}{\left(\psi-\phi\right)}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\left(\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^r+\frac{1}{p+q}\right. \\
&\left.\times\eta\left(\left|{}_{\phi}D_{p,q}g\left(x\right)\right|^r,\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^r\right)\right)^{\frac{1}{r}}+\frac{q\left(\psi-x\right)^2}{\left(\psi-\phi\right)}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}} \\
&\times\left(\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|^r+\frac{1}{p+q}\eta\left(\left|{}_{\phi}D_{p,q}g\left(x\right)\right|^r,\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|^r\right)\right)^{\frac{1}{r}}
\end{aligned}
$$

holds for all $x \in [\phi, \psi]$ *and for* $s, r > 1$, $\frac{1}{s} + \frac{1}{r} = 1$.

Proof. Utilizing Lemma 3.1 , application of Hölder inequality and η-convexity of $|_{\phi}D_{p,q}g|^r$ on $[\phi, \psi]$, we get

$$
\begin{aligned}
&|K_g(\phi,\psi;p;q)| = \left|\frac{q(x-\phi)^2}{\psi-\phi}\int_0^1 \lambda_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\phi)_0 d_{p,q}\lambda\right.\\
&\left.+\frac{q(\psi-x)^2}{\psi-\phi}\int_0^1 \lambda_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\psi)_0 d_{p,q}\lambda\right|\\
&\le \frac{q(x-\phi)^2}{\psi-\phi}\left(\int_0^1 \lambda^s\right)^{\frac{1}{s}}\left(\int_0^1 |_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\phi)|^r {}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\\
&+\frac{q(\psi-x)^2}{\psi-\phi}\left(\int_0^1 \lambda^s\right)^{\frac{1}{s}}\left(\int_0^1 |_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\psi)|^r {}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\\
&\le \frac{q(x-\phi)^2}{\psi-\phi}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\\
&\times\left(\int_0^1 \left[|_{\phi}D_{p,q}g(\phi)|^r+\lambda\eta\left(|_{\phi}D_{p,q}g(x)|^r, |_{\phi}D_{p,q}g(\phi)|^r\right)\right] {}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\\
&+\frac{q(\psi-x)^2}{\psi-\phi}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\\
&\times\left(\int_0^1 \left[|_{\phi}D_{p,q}g(\psi)|^r+\lambda\eta\left(|_{\phi}D_{p,q}g(x)|^r, |_{\phi}D_{p,q}g(\psi)|^r\right)\right] {}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\\
&\le \frac{q(x-\phi)^2}{\psi-\phi}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\\
&\times\left(|_{\phi}D_{p,q}g(\phi)|^r+\frac{1}{p+q}\eta\left(|_{\phi}D_{p,q}g(x)|^r, |_{\phi}D_{p,q}g(\phi)|^r\right)\right)^{\frac{1}{r}}\\
&+\frac{q(\psi-x)^2}{\psi-\phi}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\\
&\times\left(|_{\phi}D_{p,q}g(\psi)|^r+\frac{1}{p+q}\eta\left(|_{\phi}D_{p,q}g(x)|^r, |_{\phi}D_{p,q}g(\psi)|^r\right)\right)^{\frac{1}{r}}.
\end{aligned}
$$

We get our required result. □

Theorem 4.3. *Let* $g : J \subset \mathbb{R} \to \mathbb{R}$ *be a* (p, q)*-differentiable function on* J^o *with* $0 < q < p \le 1$. *If* ${}_{\phi}D_{p,q}g$ *is continuous and integrable on* $[\phi, \psi]$ *such that* $|_{\phi}D_{p,q}g|^r$ *is an*

η-quasi-convex function on $[\phi,\psi]$ where $\phi,\psi\in J$ with $\phi<\psi$, then for $s,r>1$, $\frac{1}{s}+\frac{1}{r}=1$,

$$\begin{aligned}
|K_g(\phi,\psi;p;q)| &\leq \frac{q(x-\phi)^2}{(\psi-\phi)}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\\
&\times\left(\max\left\{\left|{}_{\phi}D_{p,q}g(\phi)\right|^r,\left|{}_{\phi}D_{p,q}g(\phi)+\eta\left({}_{\phi}D_{p,q}g(\phi),{}_{\phi}D_{p,q}g(x)\right)\right|^r\right\}\right)^{\frac{1}{r}}\\
&+\frac{q(\psi-x)^2}{(\psi-\phi)}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\\
&\times\left(\max\left\{\left|{}_{\phi}D_{p,q}g(\psi)\right|^r,\left|{}_{\phi}D_{p,q}g(\phi)+\eta\left({}_{\phi}D_{p,q}g(\phi),{}_{\phi}D_{p,q}g(x)\right)\right|^r\right\}\right)^{\frac{1}{r}},
\end{aligned}$$

holds for all $x\in[\phi,\psi]$.

Proof. Utilizing Lemma 3.1 , application of Hölder inequality and η-quasi-convexity of $\left|{}_{\phi}D_{p,q}g\right|^r$ on $[\phi,\psi]$, we get

$$\begin{aligned}
|K_g(\phi,\psi;p;q)| &= \left|\frac{q(x-\phi)^2}{\psi-\phi}\int_0^1\lambda{}_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\phi){}_0d_{p,q}\lambda\right.\\
&\left.+\frac{q(\psi-x)^2}{\psi-\phi}\int_0^1\lambda{}_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\psi){}_0d_{p,q}\lambda\right|\\
&\leq \frac{q(x-\phi)^2}{\psi-\phi}\left(\int_0^1\lambda^s\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\phi)\right|^r{}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\\
&+\frac{q(\psi-x)^2}{\psi-\phi}\left(\int_0^1\lambda^s\right)^{\frac{1}{s}}\left(\int_0^1\left|{}_{\phi}D_{p,q}g(\lambda x+(1-\lambda)\psi)\right|^r{}_0d_{p,q}\lambda\right)^{\frac{1}{r}}\\
&\leq \frac{q(x-\phi)^2}{(\psi-\phi)}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\\
&\times\left(\max\left\{\left|{}_{\phi}D_{p,q}g(\phi)\right|^r,\left|{}_{\phi}D_{p,q}g(\phi)+\eta\left({}_{\phi}D_{p,q}g(\phi),{}_{\phi}D_{p,q}g(x)\right)\right|^r\right\}\right)^{\frac{1}{r}}\\
&+\frac{q(\psi-x)^2}{(\psi-\phi)}\left(\frac{p-q}{p^{1+s}-q^{1+s}}\right)^{\frac{1}{s}}\\
&\times\left(\max\left\{\left|{}_{\phi}D_{p,q}g(\psi)\right|^r,\left|{}_{\phi}D_{p,q}g(\psi)+\eta\left({}_{\phi}D_{p,q}g(\psi),{}_{\phi}D_{p,q}g(x)\right)\right|^r\right\}\right)^{\frac{1}{r}}.
\end{aligned}$$

The proof is completed. □

Now we calculate new quantum estimates for Hermite-Hadamard type inequalities by using η-convexity, η-quasi-convexity and (p,q)-calculus.

Theorem 4.4. *Let g and h be two non-negative η-convex functions defined on a non-empty interval $[\phi,\psi]$ of real line $\mathbb{R}$, then*

$$\begin{aligned}
&\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi}g(x)h(x){}_{\phi}d_{p,q}x\\
&\leq g(\phi)h(\phi)+\frac{1}{p+q}H_1(\phi,\psi,\eta)+\frac{1}{p^2+pq+q^2}H_2(\phi,\psi,\eta),
\end{aligned}$$

where

$$H_1(\phi,\psi,\eta)=g(\phi)\,\eta(h(\psi),h(\phi))+h(\phi)\,\eta(g(\psi),g(\phi))$$

and

$$H_2(\phi,\psi,\eta)=\eta(h(\psi),h(\phi))\,\eta(g(\psi),g(\phi)).$$

Proof. Since g and h be two non-negative η-convex functions on $[\phi,\psi]$, then

$$g((1-\lambda)\phi+\lambda\psi)\le g(\phi)+\lambda\eta(g(\psi),g(\phi)) \tag{4. 7}$$

and

$$h((1-\lambda)\phi+\lambda\psi)\le h(\phi)+\lambda\eta(h(\psi),h(\phi)). \tag{4. 8}$$

Multiplying inequality (4.7) to inequality (4.8), we get

$$\begin{aligned} &g((1-\lambda)\phi+\lambda\psi)\,h((1-\lambda)\phi+\lambda\psi)\\ =&\,g(\phi)\,h(\phi)+\lambda g(\phi)\,\eta(h(\psi),h(\phi))+\lambda h(\phi)\,\eta(g(\psi),g(\phi))\\ &+\lambda^2\eta(h(\psi),h(\phi))\,\eta(g(\psi),g(\phi)). \end{aligned} \tag{4. 9}$$

Applying (p,q)-integration on inequality (4.9) with respect to λ on $[0,1]$, we have

$$\begin{aligned} &\frac{1}{p(\psi-\phi)}\int_{\phi}^{p\psi+(1-p)\phi} g(x)h(x)\,{}_{\phi}d_{p,q}x\\ =&\,g(\phi)\,h(\phi)+\frac{1}{p+q}\left[g(\phi)\,\eta(h(\psi),h(\phi))+h(\phi)\,\eta(g(\psi),g(\phi))\right]\\ &+\frac{1}{p^2+pq+q^2}\eta(h(\psi),h(\phi))\,\eta(g(\psi),g(\phi)). \end{aligned}$$

We get our desire result. □

Theorem 4.5. *Let $g:J\subset\mathbb{R}\to\mathbb{R}$ be a (p,q)-differentiable function on J° with $0<q<p\le 1$. If ${}_{\phi}D_{p,q}g$ is continuous and integrable on J such that $|{}_{\phi}D_{p,q}g|$ is an η-convex function on $[\phi,\psi]$, where $\phi,\psi\in J^\circ$ and $\phi<\psi$, then*

$$\begin{aligned} |H_g(\phi,\psi;p;q)|\le&\,\frac{q(\psi-\phi)}{p+q}\left\{\nu_1(p,q)\,|{}_{\phi}D_{p,q}g(\phi)|\right.\\ &\left.+\nu_2(p,q)\,\eta(|{}_{\phi}D_{p,q}g(\psi)|,|{}_{\phi}D_{p,q}g(\phi)|)\right\}, \end{aligned}$$

where

$$\begin{aligned} \nu_1(p,q)&=\frac{2(p+q-1)}{(p+q)^2},\\ \nu_2(p,q)&=\frac{q\left[\left(p^3-2+2p\right)+\left(2p^2+2\right)q+pq^2\right]+2p^2-2p}{(p+q)^3(p^2+pq+q^2)}. \end{aligned}$$

Proof. Utilizing Lemma 2.9 and η-convexity of $|{}_{\phi}D_{p,q}g|$ on $[\phi,\psi]$, we get

$$\begin{aligned}|H_g(\phi,\psi;p;q)| &= \left|\frac{q(\psi-\phi)}{p+q}\int_0^1 (1-(p+q))\,{}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)\,{}_0d_{p,q}\lambda\right| \\ &\leq \frac{q(\psi-\phi)}{p+q}\int_0^1 |(1-(p+q))|\,|{}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)|\,{}_0d_{p,q}\lambda\end{aligned}$$

$$\begin{aligned}&\leq \frac{q(\psi-\phi)}{p+q}\int_0^1 |(1-(p+q))|\left[|{}_{\phi}D_{p,q}g(\phi)|+\lambda\eta\left(|{}_{\phi}D_{p,q}g(\psi)|,|{}_{\phi}D_{p,q}g(\phi)|\right)\right]{}_0d_{p,q}\lambda \\ &\leq \frac{q(\psi-\phi)}{p+q}\left\{|{}_{\phi}D_{p,q}g(\phi)|\int_0^1 |(1-(p+q))|\,{}_0d_{p,q}\lambda\right. \\ &\left.+\eta\left(|{}_{\phi}D_{p,q}g(\psi)|,|{}_{\phi}D_{p,q}g(\phi)|\right)\int_0^1 \lambda\,|(1-(p+q))|\,{}_0d_{p,q}\lambda\right\} \\ &\leq \frac{q(\psi-\phi)}{p+q}\left[\begin{array}{c}\frac{2(p+q-1)}{(p+q)^2}|{}_{\phi}D_{p,q}g(\phi)| \\ +\frac{q\left[\left(p^3-2+2p\right)+\left(2p^2+2\right)q+pq^2\right]+2p^2-2p}{(p+q)^3(p^2+pq+q^2)} \\ \times\eta\left(|{}_{\phi}D_{p,q}g(\psi)|,|{}_{\phi}D_{p,q}g(\phi)|\right)\end{array}\right].\end{aligned}$$

We get our required result. □

Theorem 4.6. *Let $g: J\subset\mathbb{R}\to\mathbb{R}$ be a (p,q)-differentiable function on J° such that ${}_{\phi}D_{p,q}g$ be continuous and integrable on J where $0<q<p\leq 1$. If $|{}_{\phi}D_{p,q}g|^r$ is an η-convex function on $[\phi,\psi]$, then*

$$\begin{aligned}|H_g(\phi,\psi;p;q)| &\leq \frac{\psi-\phi}{p+q}\left(\frac{2(p+q-1)}{(p+q)^2}\right)^{1-\frac{1}{r}}\left(\nu_1(p,q)\,|{}_{\phi}D_{p,q}g(\phi)|^r\right. \\ &\left.+\nu_2(p,q)\,\eta\left(|{}_{\phi}D_{p,q}g(\psi)|^r,|{}_{\phi}D_{p,q}g(\phi)|^r\right)\right)^{\frac{1}{r}},\end{aligned}$$

where $\nu_1(p,q)$ and $\nu_2(p,q)$ are as defined in Theorem 4.5 and $r>1$.

Proof. Utilizing Lemma 2.9, application of Hölder inequality and η-convexity of $\left|{}_{\phi}D_{p,q}g\right|^{r}$, we get

$$\begin{aligned}
\left|H_g\left(\phi,\psi;p;q\right)\right| &\leq \frac{q(\psi-\phi)}{p+q}\left(\int_0^1 \left|(1-(p+q))\right| {}_0d_{p,q}\lambda\right)^{1-\frac{1}{r}} \\
&\times\left(\int_0^1 \left|(1-(p+q))\right| \left|{}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)\right| {}_0d_{p,q}\lambda\right)^{\frac{1}{r}} \\
&\leq \frac{q(\psi-\phi)}{p+q}\left(\frac{2\left(p+q-1\right)}{\left(p+q\right)^2}\right)^{1-\frac{1}{r}} \\
&\times\left(\begin{array}{c}
\frac{2(p+q-1)}{(p+q)^2}\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^{r} \\
+\frac{q\left[\left(p^3-2+2p\right)+\left(2p^2+2\right)q+pq^2\right]+2p^2-2p}{(p+q)^3\left(p^2+pq+q^2\right)} \\
\times\eta\left(\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|^{r},\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^{r}\right)
\end{array}\right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Theorem 4.7. *Let* $g : J \subset \mathbb{R} \to \mathbb{R}$ *be a* (p,q)*-differentiable function on* J^o *with* $0 < q < p \leq 1$. *If* ${}_{\phi}D_{p,q}g$ *is continuous and integrable on* J *such that* $\left|{}_{\phi}D_{p,q}g\right|^{r}$ *is an* η*-quasi-convex function on* $[\phi,\psi]$, *where* $\phi,\psi \in J^{\circ}$ *with* $\phi < \psi$, *then*

$$\begin{aligned}
\left|H_g\left(\phi,\psi;p;q\right)\right| \quad &\leq \frac{q(\psi-\phi)}{p+q}\left(\frac{2\left(p+q-1\right)}{\left(p+q\right)^2}\right)\left(\max\left\{\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^{r},\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^{r}\right.\right. \\
&\left.\left.+\eta\left(\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|^{r},\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^{r}\right)\right\}\right)^{\frac{1}{r}},
\end{aligned}$$

holds for $r \geq 1$

Proof. Utilizing Lemma 2.9, application of Hölder inequality and η-quasi-convexity of $\left|{}_{\phi}D_{p,q}g\right|^{r}$ on $[\phi,\psi]$, we get

$$\begin{aligned}
\left|H_g\left(\phi,\psi;p;q\right)\right| \quad &\leq \frac{q(\psi-\phi)}{p+q}\left(\int_0^1 \left|(1-(p+q))\right| {}_0d_{p,q}\lambda\right)^{1-\frac{1}{r}} \\
&\times\left(\int_0^1 \left|(1-(p+q))\right| \left|{}_{\phi}D_{p,q}g(\lambda\psi+(1-\lambda)\phi)\right| {}_0d_{p,q}\lambda\right)^{\frac{1}{r}} \\
&\leq \frac{q(\psi-\phi)}{p+q}\left(\frac{2\left(p+q-1\right)}{\left(p+q\right)^2}\right)\left(\max\left\{\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^{r},\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^{r}\right.\right. \\
&\left.\left.+\eta\left(\left|{}_{\phi}D_{p,q}g\left(\psi\right)\right|^{r},\left|{}_{\phi}D_{p,q}g\left(\phi\right)\right|^{r}\right)\right\}\right)^{\frac{1}{r}}.
\end{aligned}$$

We get our result. □

5. CONCLUSIONS

In this paper, we have obtained some new results for the (p, q)-calculus of Ostrowski and HermiteHadamard type inequalities for (p, q)-integral. Our work has improved the results of [27] and can be reduced to the classical inequality formulas in special cases when $p = 1$ and $q \to 1^-$. It is expected that this paper may stimulate further research in this field.

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