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Generalization of Bosonic Quantum Tunneling with Quantum Effects

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Abstract.: The Hawking radiation via quantum tunneling process for scalar particles in $(3+1)$ and $(2+1)$ dimensional BHs is studied by considering Hamilton-Jacobi technique and WKB method. We obtain the required tunneling rate of emitted particles and recover the general form of Hawking temperatures, $\hat{T}_H$. Moreover, by utilizing modified Klein Gordon equation, we analyze the quantum corrected tunneling rate and corrected temperatures, $\hat{T}_{\hat{e}-\hat{H}}$ for spin-0 particles in $(3+1)$ and $(2+1)$ dimensional BHs.

AMS (MOS) Subject Classification Codes: 81T20; 81T40; 81Q20
Key Words: Hawking temperature; Klein-Gordon equation; Quantum Gravity effects.

1. Introduction

The region of space with so intense gravity that nothing even light cannot escape from it is known as black hole (BH). At the first time, this idea was introduced by an astronomer John Michell in 1974 [40]. The outer surface of a BH is known as "*event horizon*". When the spacetime curvature becomes infinite at the center of a BH then a gravitational singularity exists. Hawking (1974) [13] proposed that BHs emit thermal radiations in little amount so that they are not entirely black, these radiations known as "*Hawking radiation*". The continuous phenomenon of Hawking radiation causes the decrease in mass and energy of a BH, therefore, eventually BH evaporates, which is called BH evaporation.

It is a well known fact that in quantum field theory a lot of strategies have been introduced to study the Hawking radiation by incorporating the semi-classical approximations. The quantum tunneling [23] is the most convenient strategy to investigate the BH radiation. In this phenomenon the particles have finite probability to cross the event horizon. Quantum tunneling has two main techniques: the first one is *null geodesic technique*, that was initially introduced by Kraus & Wilczek [27] and the other is *Hamilton-Jacobi process*, Angheben et al. was first propounded this idea [1]. Both techniques provide the tunneling probability for BHs via horizon, which can be given by the formula

$$\hat{\Gamma} = e^{-2Im\hat{S}_0},\ (\hat{\hbar} = 1)$$

here $\hat{S}_0$ denotes the action of radiated particle and $\hat{\hbar}$ represents the *Planck constant*.

Many researchers have studied the Hawking radiation spectrum through tunneling process. Zhang and Jiang with his colleagues [57, 19] have studied the tunneling method by following the Parikh-Wilczek framework. Sharif and Javed [55] have investigated the tunneling phenomenon via event horizons for various type of BHs by following the Kerner and Mann framework and studied their expected Hawking temperature. Wajiha and her colleagues [14]-[18] have studied the tunneling phenomenon for different types of particles and calculated the Hawking temperature. The Hawking radiation process for vector, scalar and fermions particles have been widely discussed in literature [38]-[25]. Sakalli and Övgün [46]-[53] have studied the Hawking radiation phenomenon from Rindler modified Schwarzschild BH, Dyonic Reissner Nordström BH, non-stationary metrics, traversable Lorentzian Wormholes, Lorentzian wormholes in $3+1$ dimensions and three dimensional rotating hairy BHs. BHs. Sakalli, Övgün and Mirekhtiary have investigated the gravitational lensing effect on the Hawking radiation of Dyonic BHs [54].

Li and Zhao [29, 30] have studied the Hawking radiation process from linear dilaton BHs as well as neutral rotating AdS BHs in conformal gravity. Li et al. [31] have investigated the Hawking temperature for massive spin-1 bosons from dilaton BHs. Different authors [3]-[26] have been discussed the thermodynamical properties of BH physics under generalized uncertainty principle (GUP). Nozari and Mehdipour [34] calculated the modified tunneling rate for Schwarzschild BH under GUP effects. Haouat and Nouicer [12] discussed the creation of pair of spin-0 particles in an electric field by considering minimal length $\bar{M}_f$. Övgün and Jusufi [35] investigated the Hawking temperature through tunneling process from a warped DGP gravity BH and analyzed the GUP effects on Hawking temperature.

However, in this paper, we have worked on generalization of scalar particles in detail and provide the complete analysis and comparison of our results with literature. In this regard, we analyze the general formula for Hawking temperature and its modified quantum corrected form by using the quantum tunneling method of spin-0 particles. For this purpose, we use Hamilton-Jacobi technique and apply the WKB approximation to the field equations of scalar particles. After this we calculate tunneling probability of charged radiated particles and their corresponding Hawking temperatures. In order to study the quantum gravity effects, we utilize the generalized Klein-Gordan equation incorporating GUP effects and recover the accompanying quantum corrected temperature for $(3+1)$ and $(2+1)$ dimensional BHs.

2. Generalization of Charged Spin-0 Particles Tunneling in $(3+1)$ Dimensions

The metric for 4-dimensional BH can be expressed as

$$d\hat{s}^2 = -\hat{A}(r,\hat{\theta})d\hat{t}^2 + \frac{1}{\hat{B}}(r,\hat{\theta})d\hat{r}^2 + \hat{C}(r,\hat{\theta})d\hat{\theta}^2 + \hat{D}(r,\hat{\theta})d\hat{\varphi}^2 - 2\hat{F}(r,\hat{\theta})d\hat{t}d\hat{\varphi}, \quad (2.\ 1)$$

where $A,\ B,\ C,\ D$ and F are functions of r and $\hat{\theta}$. By considering $\hat{B}(r,\hat{\theta}) = 0$, we can obtain the horizons of BH. The angular velocity at event horizon can be calculated by the formula:

$$\Omega_H = \frac{\hat{F}}{\hat{D}}. \quad (2.\ 2)$$

To calculate the tunneling rate of charged scalar particles the Klein-Gordon equation with charge $\hat{q}$ and scalar field $\hat{\Phi}$ is defined as

$$\hat{g}^{\hat{\mu}\hat{\nu}}\left(\hat{\partial}_{\hat{\nu}}-\iota\hat{q}\hat{A}_{\hat{\nu}}\right)\left(\hat{\partial}_{\hat{\mu}}-\iota\hat{q}\hat{A}_{\hat{\mu}}\right)\hat{\Phi}-\hat{m}^2\hat{\Phi}=0, \tag{2. 3}$$

here $\hat{g}^{\hat{\mu}\hat{\nu}}$, $\hat{m}$ and $\hat{A}_{\hat{\mu}}$ represents the contra-variant metric tensor, mass of radiated particles and vector potential, respectively,

By applying the WKB method to Eq.(2. 3), we consider the following ansatz.

$$\hat{\Phi}(\hat{t},\hat{r},\hat{\theta},\hat{\varphi})=e^{\left(\frac{\iota}{\hbar}\hat{S}_0(\hat{t},\hat{r},\hat{\theta},\hat{\varphi})+\hat{S}_1(\hat{t},\hat{r},\hat{\theta},\hat{\varphi})+O(\hat{\hbar})\right)}. \tag{2. 4}$$

The Eq.(2. 3) becomes

$$\hat{g}^{\hat{\mu}\hat{\nu}}(\hat{\partial}_{\hat{\mu}}\hat{S}_0-\hat{q}\hat{A}_{\hat{\mu}})(\hat{\partial}_{\hat{\nu}}\hat{S}_0-\hat{q}\hat{A}_{\hat{\nu}})+\hat{m}^2=0. \tag{2. 5}$$

After putting the values of $\hat{g}^{\hat{\mu}\hat{\nu}}$ and $\hat{A}_{\hat{\mu}}$ into the above equation, we get

$$\begin{aligned} &- \frac{(\hat{\partial}_{\hat{t}}\hat{S}_0-\hat{A}_{\hat{t}}\hat{q})^2}{\hat{G}(\hat{r},\hat{\theta})}+\hat{B}(\hat{\partial}_{\hat{r}}\hat{S}_0)^2-\frac{2\hat{F}}{\hat{A}\hat{D}}(\hat{\partial}_{\hat{t}}\hat{S}_0-\hat{q}\hat{A}_{\hat{t}})(\hat{\partial}_{\hat{\varphi}}\hat{S}_0-\hat{q}\hat{A}_{\varphi}) \\ &+ \frac{\hat{A}}{\hat{D}\hat{G}}(\hat{\partial}_{\hat{\varphi}}\hat{S}_0-\hat{q}\hat{A}_{\hat{\varphi}})^2+\hat{C}^{-1}(\hat{\partial}_{\hat{\theta}}\hat{S}_0)^2+\hat{m}^2=0. \end{aligned} \tag{2. 6}$$

By assuming separation of variables method the action of radiated particle is given as

$$\hat{S}_0=-(\hat{E}-\hat{j}\hat{\Omega}_H)\hat{t}+\hat{R}(\hat{r},\hat{\theta})+\hat{k}\hat{\phi}, \tag{2. 7}$$

After applying the above action in Eq.(2. 6), we obtain

$$\frac{(\hat{E}-\hat{\Omega}_H\hat{j}-\hat{A}_{\hat{t}}\hat{q})^2}{(\hat{r}-\hat{r}_+)\hat{G}_{\hat{r}}}+(\hat{r}-\hat{r}_+)\hat{B}_{\hat{r}}\hat{R}_{\hat{r}}^2+\frac{(\hat{k}-\hat{q}\hat{A}_{\hat{\varphi}})^2}{\hat{D}}+\hat{C}^{-1}\hat{R}_{\hat{\theta}}^2+\hat{m}^2=0. \tag{2. 8}$$

In order to calculate the radial part $\hat{R}(\hat{r})$ for fix $\hat{\theta}=\hat{\theta}_{\frac{\pi}{2}}$, we solve above equation and obtain

$$\hat{R}_{\pm}(\hat{r})=\pm\int\sqrt{\frac{(\hat{E}-\hat{\Omega}_H\hat{j}-\hat{q}\hat{A}_{\hat{t}})^2+\hat{A}\hat{m}^2}{\hat{G}\hat{B}}}d\hat{r}, \tag{2. 9}$$

For the imaginary part, we calculate the above expression in the following form

$$\text{Im}\hat{R}_+(\hat{r})=\pm\pi\frac{(\hat{E}-\hat{e}\hat{A}_0-\hat{\Omega}_H\hat{j})}{\sqrt{\hat{G}_{\hat{r}}\hat{B}_{\hat{r}}}}\ . \tag{2. 10}$$

The tunneling rate for radiated particles from above Eq.(2. 10) can be obtained as

$$\begin{aligned} \hat{\Gamma} &= \frac{\hat{\Gamma}_{emission}}{\hat{\Gamma}_{absorption}}=\frac{\exp\left[-\frac{2}{\hbar}(Im\hat{R}_++Im\hat{\Theta})\right]}{\exp\left[-\frac{2}{\hbar}(Im\hat{R}_-+Im\hat{\Theta})\right]}=\exp\left[-\frac{4}{\hat{\hbar}}Im\hat{R}_+\right], \\ &= \exp\left[\frac{-4\pi(\hat{E}-\hat{e}\hat{A}_0-\hat{\Omega}_H\hat{j})}{\sqrt{\hat{G}_{\hat{r}}\hat{B}_{\hat{r}}}}\right], \end{aligned}$$

by considering the Boltzmann formula $\hat{\Gamma}_{\hat{B}} = \exp\left[(\hat{E} - \hat{e}\hat{A}_0 - \hat{\Omega}_H \hat{j})/\hat{T}_{\hat{H}}\right]$, the expected Hawking temperature $\hat{T}_{\hat{H}}$ at the horizon $\hat{r}_+$ is obtained as

$$\hat{T}_{\hat{H}} = \frac{\hat{\kappa}}{2\pi} = \left[\frac{\sqrt{\hat{G}_{\hat{r}}\hat{B}_{\hat{r}}}}{4\pi}\right]. \tag{2. 11}$$

3. Tunneling of Charged Scalar Particles in $(2+1)$ Dimensions

Now, we will discuss the generalization of tunneling of charged scalar particles in $(2+1)$ dimensions. The metric for $(2+1)$-dimensional BH can be defined as

$$d\hat{s}^2 = -\hat{A}d\hat{t}^2 + \frac{1}{\hat{B}}d\hat{r}^2 + \hat{C}d\hat{x}^2, \tag{3. 12}$$

The equation of motion for scalar particles can be given as

$$\hat{g}^{\hat{\mu}\hat{\nu}}\left(\hat{\partial}_{\hat{\nu}} - \frac{\iota\hat{q}}{\hbar}\hat{A}_{\hat{\nu}}\right)\left(\hat{\partial}_{\hat{\mu}} - \frac{\iota\hat{q}}{\hbar}\hat{A}_{\hat{\mu}}\right)\hat{\Phi} - \frac{1}{\hat{\hbar}^2}\hat{m}^2\hat{\Phi} = 0. \tag{3. 13}$$

By assuming WKB method, we have the ansatz

$$\hat{\Phi}(\hat{t}, \hat{r}, \hat{x}) = \exp\left[\frac{\iota}{\hat{\hbar}}\hat{S}_0(\hat{t}, \hat{r}, \hat{x})\right], \tag{3. 14}$$

After substituting the values of $g^{\hat{\mu}\hat{\nu}}$ and $\hat{A}_\mu$, Eq. (3. 12) takes the following form

$$-\quad \frac{(\hat{\partial}_{\hat{t}}\hat{S}_0 - \hat{A}_{\hat{t}}\hat{q})^2}{\hat{A}} + \hat{B}(\hat{\partial}_{\hat{r}}\hat{S}_0)^2 + \frac{(\hat{\partial}_{\hat{t}}\hat{S}_0 - \hat{A}_{\hat{t}}\hat{q})^2}{\hat{C}} + \hat{m}^2 = 0. \tag{3. 15}$$

We consider the particle's action in the form

$$\hat{S}_0(\hat{t}, \hat{r}, \hat{x}) = -(\hat{E} - \hat{j}\hat{\Omega}_H)\hat{t} + \hat{N}\hat{x} + \hat{R}(\hat{r}). \tag{3. 16}$$

By substituting (3. 16) in Eq.(3. 15), we obtain

$$\frac{(\hat{E} - \hat{\Omega}_H j - \hat{q}\hat{A}_t)^2}{\hat{A}} + \hat{B}\hat{R}_r^2 + \frac{\hat{N}^2}{\hat{C}} + \bar{m}^2 = 0. \tag{3. 17}$$

After solving the above equation, we have

$$\hat{R}_\pm(\hat{r}) = \pm\int\sqrt{\frac{(\hat{E} - \hat{\Omega}_H\hat{j} - \hat{q}\hat{A}_{\hat{t}})^2 + \hat{A}\hat{m}^2}{\hat{A}\hat{B}}}d\hat{r}, \tag{3. 18}$$

From above equation we calculate the imaginary part in the form

$$\mathrm{Im}\hat{R}_+(\hat{r}) = \pm\pi\frac{(\hat{E} - \hat{e}\hat{A}_0 - \hat{\Omega}_H\hat{j})}{\sqrt{\hat{A}_{\hat{r}}\hat{B}_{\hat{r}}}}. \tag{3. 19}$$

The probability rate for radiated particles is obtained in the form

$$\begin{aligned}\hat{\Gamma} &= \frac{\hat{\Gamma}_{emission}}{\hat{\Gamma}_{absorption}} = \frac{\exp\left[-\frac{2}{\hbar}(Im\hat{R}_+ + Im\hat{C})\right]}{\exp\left[-\frac{2}{\hbar}(Im\hat{R}_- + Im\hat{C})\right]} = \exp\left[-\frac{4}{\hat{\hbar}}Im\hat{R}_+\right], \\ &= \exp\left[\frac{-4\pi(\hat{E} - \hat{e}\hat{A}_0 - \hat{\Omega}_H\hat{j})}{\sqrt{\hat{A}_{\hat{r}}\hat{B}_{\hat{r}}}}\right],\end{aligned}$$

The corresponding Hawking temperature can be derived as

$$\hat{T}_{\hat{H}} = \frac{\hat{\kappa}}{2\pi} = \left[\frac{\sqrt{\hat{A}_{\hat{r}}\hat{B}_{\hat{r}}}}{4\pi}\right]. \tag{3. 20}$$

It is the general formula to derive the standard Hawking temperature.

4. Quantum Corrections of Scalar Particles in $(3+1)$ dimensions

In order to discuss the quantum corrected temperature at the event horizon the generalized Klein-Gordan equation only for first order of $\hat{\beta}$, is given as

$$-(\hat{\hbar}\iota)^2\hat{\partial}^{\hat{t}}\hat{\partial}_{\hat{t}}\hat{\Phi} = [\hat{m}^2 + (\hat{\hbar}\iota)^2\hat{\partial}^i\hat{\partial}_i][\hat{m}^2 + 1 - 2\hat{\beta}(\hat{\hbar}\iota)^2\hat{\partial}^i\hat{\partial}_i]\hat{\Phi}. \tag{4. 21}$$

The line element is expressed as

$$d\hat{s}^2 = -\hat{A}d\hat{t}^2 + \frac{1}{\hat{B}}d\hat{r}^2 + \hat{C}d\hat{\theta}^2 + \hat{D}d\hat{\varphi}^2. \tag{4. 22}$$

The wave function for radiated particles is assumed by

$$\hat{\Phi}(\hat{t},\hat{r},\hat{\theta},\hat{\varphi}) = \left[\frac{\iota}{\hat{\hbar}}\hat{S}_0(\hat{t},\hat{r},\hat{\theta},\hat{\varphi})\right], \tag{4. 23}$$

After solving and assuming $\hat{\hbar}$ only for leading order in above Eq.(4. 21), we get

$$\frac{1}{\hat{A}}(\hat{\partial}_{\hat{t}}\hat{S}_0)^2 = \left[\hat{B}(\hat{\partial}_{\hat{r}}\hat{S}_0)^2 + \frac{1}{\hat{C}}(\hat{\partial}_{\hat{\theta}}\hat{S}_0)^2 + \frac{1}{\hat{D}}(\hat{\partial}_{\hat{\varphi}}\hat{S}_0)^2 + \hat{m}^2\right] \times \left[1 - 2\hat{\beta}\left(\hat{B}(\hat{\partial}_{\hat{r}}\hat{S}_0)^2 + \frac{1}{\hat{C}}(\hat{\partial}_{\hat{\theta}}\hat{S}_0)^2 + \frac{1}{\hat{D}}(\hat{\partial}_{\hat{\varphi}}\hat{S}_0)^2 + \hat{m}^2\right)\right]. \tag{4. 24}$$

The action of radiated particle is defined as

$$\hat{S}_0 = -(\hat{E} - \hat{\Omega}_H\hat{j})\hat{t} + \hat{R}(\hat{r},\hat{\theta}) + \hat{k}\hat{\varphi}. \tag{4. 25}$$

It is also important to mention that, we cannot separate $\hat{R}(\hat{r},\hat{\theta})$ as $\hat{R}(\hat{r})\hat{\Theta}(\hat{\theta})$. So, after fixing $\hat{\theta} = \hat{\theta}_0$, the Eq.(5. 31) implies

$$\hat{P}_0(\hat{\partial}_{\hat{r}}\hat{R})^4 + \hat{P}_1(\hat{\partial}_{\hat{r}}\hat{R})^2 + \hat{P}_2 = 0, \tag{4. 26}$$

where

$$\begin{aligned}
\hat{P}_0 &= -2\hat{\beta}\hat{B}^2,\ \hat{P}_1 = \hat{B}\left(1 - 4\hat{\beta}\frac{\hat{k}^2}{\hat{D}} - 4\hat{\beta}\hat{m}^2\right),\\
\hat{P}_2 &= \hat{m}^2 + \frac{\hat{k}^2}{\hat{D}} - 2\hat{\beta}\frac{\hat{k}^4}{\hat{D}^2} - 4\hat{\beta}\hat{m}^2\frac{\hat{k}^2}{\hat{D}} - 2\hat{\beta}\hat{m}^4 - \frac{(\hat{E} - \hat{\Omega}_H\hat{j})^2}{\hat{A}}.
\end{aligned}$$

After solving Eq.(4. 26) the radial part of particle's action is given follows

$$\hat{R}_{\pm}(\hat{r}) = \pm\int \frac{d\hat{r}}{\sqrt{\hat{B}\hat{A}}}\left[1+\hat{\beta}\left(\hat{m}^2+\frac{(\hat{E}-\hat{\Omega}_H\hat{j})^2}{\hat{A}}+\frac{\hat{k}^2}{\hat{D}}\right)\right]\times \sqrt{(\hat{E}-\hat{\Omega}_H\hat{j})^2-\hat{m}^2\hat{A}-\frac{\hat{k}^2\hat{A}}{\hat{D}}+2\hat{\beta}\left(\frac{\hat{k}^4\hat{A}}{\hat{D}^2}+\frac{2m^2\hat{k}^2\hat{A}}{\hat{D}}+\hat{m}^4\hat{A}\right)}. \quad (4.\ 27)$$

After solving the above integral, and taking $\hat{\beta}$ just for leading order, we get the following result at event horizon $\hat{r}_+$,

$$\text{Im}\hat{R}(\hat{r}_+) = \pm\pi\left(\frac{(\hat{E}-\hat{\Omega}_H\hat{j})^2}{\hat{A}'(\hat{r}_+)}\right)(1+\hat{\beta}\hat{\Im}), \quad (4.\ 28)$$

here

$$\hat{\Im} = \hat{m}^2+\frac{(\hat{E}-\hat{\Omega}_H\hat{j})^2}{\hat{A}'}+\frac{\hat{k}^2}{\hat{D}'}.$$

The corrected tunneling rate for scalar particles can be defined as

$$\hat{\Gamma} = = \frac{\hat{\Gamma}_{emission}}{\hat{\Gamma}_{absorption}} = \frac{\exp\left[-\frac{2}{\hbar}(Im\hat{R}_+ + Im\hat{\theta})\right]}{\exp\left[-\frac{2}{\hbar}(Im\hat{R}_- + Im\hat{\theta})\right]} = \exp\left[-\frac{4}{\hat{\hbar}}Im\hat{R}_+\right],$$
$$= \exp\left[-\frac{4\pi}{\hat{A}'(\hat{r}_+)}(\hat{E}-\hat{\Omega}_H\hat{j})\times(1+\hat{\beta}\hat{\Im})\right]. \quad (4.\ 29)$$

By applying Boltzmann formula $\hat{\Gamma}_{\hat{B}} = \exp\left[(\hat{E}-\hat{\Omega}_H\hat{j})/\hat{T}_{\hat{e}-\hat{H}}\right]$, the effective Hawking temperature is calculated as

$$\hat{T}_{\hat{e}-\hat{H}} = \frac{\hat{A}'(\hat{r}_+)}{4\pi(1+\hat{\beta}\hat{\Im})} = \hat{T}_0(1-\hat{\beta}\hat{\Im}), \quad (4.\ 30)$$

$\hat{T}_0$ represents the standard temperature of the BH.

5. Quantum Corrections of Scalar Particles in $(2+1)$ dimensions

Now, we will discuss the quantum gravity effects for scalar particles by BHs in $(2+1)$ dimensions. The modified Klein-Gordon Eq. (4. 21) in the background of $(2+1)$ dimensional BH metric (3. 12), gets the form

$$\frac{\hat{\hbar}^2}{\hat{A}(\hat{r})}\frac{\partial^2\Phi}{\partial\hat{t}^2}-\frac{\hat{\hbar}^2}{\hat{C}(\hat{r})}\frac{\partial^2\Phi}{\partial\hat{x}^2}-2\hat{\hbar}^4\hat{\beta}\hat{B}(\hat{r})\frac{\partial^2\hat{\Phi}}{\partial\hat{r}^2}\left[-\hat{B}(\hat{r})\frac{\partial^2\Phi}{\partial r^2}\right]-2\frac{\hat{\hbar}^4\hat{\beta}}{\hat{C}(\hat{r})}\frac{\partial^2\hat{\Phi}}{\partial\hat{x}^2}\left[-\frac{1}{\hat{C}(\hat{r})}\frac{\partial^2\hat{\Phi}}{\partial\hat{x}^2}\right]$$
$$-\hat{\hbar}^2\hat{B}(\hat{r})\frac{\partial^2\hat{\Phi}}{\partial\hat{r}^2}+\hat{m}^2(1-2\hat{\beta}\hat{m}^2)\Phi = 0. \quad (5.\ 31)$$

The wave function for radiated particles can be assumed as

$$\Phi(\hat{t},\hat{r},\hat{x}) = \exp\left[\frac{\iota}{\hat{\hbar}}\hat{S}_0(\hat{t},\hat{r},\hat{x})\right], \quad (5.\ 32)$$

After putting Eq. (5. 32) into Eq. (5. 31) for leading order in $\hat{\hbar}$, we get

$$\frac{1}{\hat{A}(\hat{r})}\left(\frac{\partial\hat{S}_0}{\partial\hat{t}}\right)^2=\hat{B}(\hat{r})\left(\frac{\partial\hat{S}_0}{\partial\hat{r}}\right)^2+\frac{1}{\hat{C}(\hat{r})}\left(\frac{\partial\hat{S}_0}{\partial\hat{x}}\right)^2+\hat{m}^2+\frac{\hat{\beta}}{\hat{C}(\hat{r})^2}\left(\frac{\partial\hat{S}_0}{\partial\hat{x}}\right)^4$$
$$-\hat{\beta}\left[\hat{m}^4-2\hat{B}(\hat{r})^2\left(\frac{\partial\hat{S}_0}{\partial\hat{r}}\right)^4\right]. \quad (5.\ 33)$$

The particle's action can be considered in the form

$$\hat{S}_0(\hat{t},\hat{r},\hat{x})=-\hat{E}\hat{t}+\hat{N}\hat{x}+\hat{R}(\hat{r}). \quad (5.\ 34)$$

Here $\hat{R}(\hat{r})=\hat{R}_\circ(\hat{r})+\hat{\beta}\hat{R}_1(\hat{r})$, thus the radial integral $\hat{R}(\hat{r})$ becomes

$$\hat{R}_\pm(\hat{r})=\sqrt{\frac{\hat{E}^2-\hat{A}(\hat{r})\left(\hat{m}^2+\frac{\hat{N}^2}{\hat{C}(\hat{r})}\right)}{\hat{B}(\hat{r})}}(1+\hat{\beta}\hat{\Im}). \quad (5.\ 35)$$

The above equation implies

$$\hat{R}_\pm(\hat{r})=\pm\iota\pi\frac{\hat{E}}{\hat{A}'(\hat{r})}(1+\hat{\beta}\hat{\Im}), \quad (5.\ 36)$$

where $\hat{\Im}>0$ can be given as

$$\hat{\Im}=\left(\frac{\hat{A}(\hat{r})\left(\hat{m}^2+\frac{\hat{N}^4}{\hat{C}(\hat{r})^2}\right)}{\hat{E}^2-\hat{A}(\hat{r})\left(\hat{m}^2+\frac{\hat{N}^2}{\hat{C}(\hat{r})}\right)}-\frac{\hat{E}^2-\hat{A}(\hat{r})\left(\hat{m}^2+\frac{j^2}{\hat{C}(\hat{r})}\right)}{\hat{B}(\hat{r})}\right), \quad (5.\ 37)$$

Here $\hat{R}_-$ and $\hat{R}_+$ stands for radial functions of incoming/outgoing particles, respectively. The probability rate for emitted particles can be obtained as

$$\hat{\Gamma}\ =\ =e^{-\frac{4}{\hbar}Im\hat{R}_+}=\exp\left[-\frac{4\pi\hat{E}}{\hat{A}'(\hat{r}_+)}\times(1+\hat{\beta}\hat{\Im})\right]. \quad (5.\ 38)$$

By utilizing Boltzmann formula $\hat{\Gamma}_{\hat{B}}=\exp\left[-\hat{E}/\hat{T}_{\hat{e}-\hat{H}}\right]$, the corrected temperature can be deduced in the form

$$\hat{T}_{\hat{e}-\hat{H}}=\frac{\hat{A}'(\hat{r}_+)}{4\pi(1+\hat{\beta}\hat{\Im})}=\hat{T}_0(1-\hat{\beta}\hat{\Im}), \quad (5.\ 39)$$

Equation (5. 39) represents the effective Hawking temperature under quantum gravity effects.

6. Conclusion

In this article, we have investigated the tunneling rate and Hawking temperature for spin-0 particles for $(3+1)$ and $(2+1)$ dimensional BHs. Using the Hamilton-Jacobi technique and WKB method, we have considered the Klein-Gordon equation of motion for massive scalar field. Moreover, we also investigate the effective Hawking temperature for spin-0 particles, which looked preserved over charge and energy. By using modified Klein-Gordon equation the effective Hawking temperature $\hat{T}_{\hat{e}-\hat{H}}=\hat{T}_0(1-\hat{\hat{\beta}}\hat{\Im})$ in Eqs.(4.

30) and (5. 39)has been obtained with quantum gravity effect. It is also worth noting that the effects of quantum gravity decelerates the Hawking temperature. By using these general formulas, we can calculate the Hawking temperature and corrected Hawking temperature for any type of black hole in $(3+1)$ and $(2+1)$ dimensions.

In a conclusion, the quantum gravity has attracted more and more attention of physicists. In this paper, we only calculated the tunneling behavior of scalar particles with and without effect of the quantum gravity. In future, we will focus on the other fields of the quantum gravity.

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