Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(8)(2020) pp. 1-15

# A Study of Soft Sets with Soft Members and Soft Elements: A New Approach

Muhammad Saeed
Department of Mathematics,
University of Management and Technology, Lahore, Pakistan,
Email: muhammad.saeed@umt.edu.pk

Manzoor Hussain
Department of Mathematics,
University of Management and Technology, Lahore, Pakistan,
Email: manzoorcioal@gmail.com

Abdul Aleem Mughal
Department of Mathematics,
University of Management and Technology, Lahore, Pakistan,
Email: doctoraleem4math@gmail.com

Received: 13 June, 2019 / Accepted: 17 July, 2020 / Published online: 01 August, 2020

**Abstract.** Soft set theory has gained significant worth since its emergence. Soft algebraic structures have been discussed in numerous researches with the help of sub algebraic structures. Properties of soft algebraic structures, somehow, are hard to study in parallel to the study of the classical approach of algebraic structures. In this study, we perform an extensive inspection of the concept of soft elements and soft members in soft sets. By using soft members and soft elements, soft sets operations and soft sets relations are discussed which enables us to study the concept of soft algebraic structures with this new approach. To elaborate on the new introduced concept, Cayley's table for the soft group has been constructed from where some properties of the soft groups are verified.

**AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09**
**Key Words:** Soft Member, Soft Element, Isolated Soft Element , Sub Soft Member, Soft Relations, Soft Binary Operations, Soft Identity Member, Soft Inverse.

## 1. INTRODUCTION

To meet the challenges of the rapidly changing world of Science and technology Zadeh [29] in 1965, proposed the concept of Fuzzy sets to deal with uncertainty. In 1999, Molodtsov

[13] introduced the soft set theory. This pack of concepts opened new directions for the researchers to deal with the changes to many problems in science and technology. After that many mathematicians discussed this concept of uncertainty in their ways. For instance, Maji et al. [12, 14] enhanced this theory and discussed many fundamental concepts of soft sets. Aktas et al. [2] described soft sets and soft groups. Acar et al. (2007) defined soft rings and presented its basic theory [1]. In 2008, Zou et al. presented their work on data analysis as an application of soft sets [28]. During the same year, Feng et al. have studied soft semiring [9]. Some new operations on the soft sets have been developed in 2009 by Ali et al. [5]. In 2010, Babitha et al. [6] explained soft set relations and functions. It would be worth mentioning that in 2010, the foundation of soft rings has been developed by Acar at al. [1]. In 2012 and 2013, Das and Samanta studied soft metrics and some concepts related to real numbers [7, 8]. Since probability theory has a vital roll in data analysis and hence in the field of soft set Fatimah et al. worked on probabilistic and dual probabilistic soft sets and link them with decision making [10]. Soft set theory is also applicable in medical science in this regard recently Alcantud et al. have used soft sets and fuzzy set theory in surgical decision making [4]. Moreover, many researchers have been attracted by the soft set theory and it brought a revolution in the field of computer, mathematics, economics, decision making, business, etc. for instance in 2019 soft sets have been used by Santos-Buitrago et al. in their work on decision making in computational biology under incomplete information [26]. The concepts of hybrid set and its application in decision making are discussed by various authors and some are cited here [17] ,[18], [20], [21], [22],[24],[19], [23]. It would not be a justice to move onward without mentioning the work of Akkram et al. on Fuzzy N-soft sets [3]. Recently, Riaz and Tehrim presented the property of bipolar Fuzzy soft topology via Q-Neighbourhood [16]. A new development in the soft set theory , Rudiments of N-framed soft set [25] has appeared in the work of Saeed et al. The notion of soft element has been introduced in [7] to initiate the idea of Soft real sets, soft real numbers and their properties. After word, this notion has been used by many authors like [15], [11] etc.

In this paper, soft set theory has been studied with soft elements and soft members. The soft set basic operations including union, intersection, soft relations, and some other properties have been discussed in detail with appropriate examples. Soft set relations have also been presented by using these novel concepts of soft elements and soft members. The authors have applied this new approach to develop Cayley's table for soft groups to verify the properties of the group structure.

The main purpose of this project is to identify a soft set with another form of a soft set by using soft elements and soft members. With the help of this identification of a soft set, many areas of pure and applied mathematics would become open to being explored. For example, the normal series in soft groups, radical theory in soft rings etc. are still to be studied. This is because, by using the existing soft sets literature, the binary operations cannot be used to go in side such fields of pure mathematics. However, this hurdle can be crossed by using soft members of a soft set. For this purpose, in section three soft elements and a soft member of a soft set are defined and the sot set has been identified with the collection of soft members. To make this concept more clear, some examples of soft sets in this regard are constructed. In section four, by using the soft members and soft elements the soft set operations are defined and verified with certain examples. The cartesian product of soft sets and soft set relations are discussed in section five. Moreover, in section six, the soft sets binary operations are developed and applied to discuss soft algebraic structures. In addition to it, a soft group is defined naturally and some group-theoretic concepts are verified with an example of a soft group.

## 2. Preliminaries

Before moving ahead it would be worth-full to write some notations that would be used throughout our discussions. The symbol $\xi$ is used to represent the universe, $P(\xi)$ for the power set of $\xi$, $E$ for the set of parameters and $A, B, C$ etc. for the subsets of $E$. The symbols $\eta, \varrho, \gamma$ will be used for functions from subsets of $E$ to $P(\xi)$, and $a, b, c$ etc. are fixed for the mappings from the subsets of $E$ to $\xi$.

2.1. **Definition:** [29, 12] Let $\xi$ be a universal set and $E$ be a set of attributes. If $\eta : A \to P(\xi)$ a mapping, then the pair $(\eta, A)$ is a soft set over $\xi$, where $A \subset E$.

2.2. **Definition:** [12] Let $(\eta, A), (\varrho, B)$ be soft sets over $\xi$, then $(\eta, A)$ is called a soft subset of $(\varrho, B)$, denoted by $(\eta, A) \subset (\varrho, B)$, if $A \subset B$ and $\eta(e) \subset \varrho(e)$ for all $e \in A$.

2.3. **Definition:** [13] The soft sets $(\eta, A)$ and $(\varrho, B)$ are said to be equal soft sets, if $(\eta, A) \subseteqq (\varrho, B)$ and $(\varrho, B) \subseteqq (\eta, A)$.

2.4. **Definition:** [14] Let $(\eta, A), (\varrho, B)$ be soft sets over $\xi$, then the bi intersection $(\eta, A) \Cap (\varrho, B)$, of $(\eta, A) and (\varrho, B)$ is a soft set $(\gamma, C)$ where $C$ represents $A \cap B$ and for all $e \in C,\ \gamma(e) = \eta(e) \cap \varrho(e)$.

2.5. **Definition:** [14] For $(\eta, A), (\varrho, B)$ soft sets over $\xi$, the extended union $(\eta, A) \cup_\epsilon (\varrho, B)$, of $(\eta, A) and (\varrho, B)$ is a soft set $(\gamma, C)$ where $C$ represents $A \cup B$, and $\forall\ e \in C$

$$\gamma(e) = \begin{cases} \eta(e) & \text{if } e \in A - B \\ \varrho(e) & \text{if } e \in B - A \\ \eta(e) \cup \varrho(e) & \text{if } e \in A \cap B. \end{cases}$$

2.6. **Definition:** [12] For $(\eta, A), (\varrho, B)$ soft sets over $\xi$, the restricted union $(\eta, A) \cup_R (\varrho, B)$ of $(\eta, A)$ and $(\varrho, B)$ is a soft set $(\gamma, C)$ where $C$ represents $A \cap B$ and $\forall\ e \in C$, $H(e) = F(e) \cup G(e)$.

## 3. Soft elements and soft members of a soft set

Some new concepts related to the soft sets are being introduced and these concepts are very useful to discuss the soft sets operations, the soft sets relations and the soft algebraic structures in the soft set theory as well as many other branches of pure and applied mathematics that are related to soft sets in a different way. However, the main purpose of this section is to lay down a foundation in the soft set theory which would enable one to study soft abstract algebra classically.

3.1. **Definition:** The soft set $(\eta, A)$ defined over the universe $\xi$ for the set $E$ of attributes, with a mapping, $\eta : A \to P(\xi)$. Also with a mapping $a : A \to \xi$ such that a(e) $\in \eta(e)$ for every $e \in A$, then the pair $(e, a(e))$ is called a **soft element** for the soft set $(\eta, A)$.
The set of all ordered pairs $(e, a(e))$ for each $a(e) \in \eta(e)$ and each $e \in A$ is denoted by $\tilde{a}$, that is, $\tilde{a} = \{(e, a(e)) \quad | \quad e \in A, a(e) \in \eta(e)\}$ is called **soft member** of $(\eta, A)$.
The number of soft elements in each soft member of the soft set $(\eta, A)$ depends upon the cardinality of $A$ and the definition of $\eta : A \to P(\xi)$
**Note:**The above definition is for the case when the set E is finite, however Zorn's Lemma is required for the case where E is not finite.

In the light of the above definitions, soft set can now be described as:

3.2. **Definition:** If $A \subset E$ and $P(\xi)$ is a power set of the universe $\xi$ such that $\eta : A \to P(\xi)$ and $a : A \to \xi$, then the collection of all soft members of the form $\tilde{a} = \{(e_i, a(e_i)) \quad | \quad e_i \in A, \text{ and } a(e_i) \in \eta(e_i) \neq \phi\}$ can be identify with a soft set $(\eta, A)$ and we write it as $(\eta, A) = \{\tilde{a} \quad | \quad a : A \to \xi \text{ and } a(e) \in \eta(e) \neq \phi \ \forall \ e \in A\}$.

3.3. **Example:** Let $\xi = \{s_1, s_2, s_3, s_4, s_5\}$ and $E = \{e_1, e_2, e_3, e_4\}$ take $A = \{e_1, e_2\}$ where $A \subset E$. We define $\eta : A \to P(\xi)$ such that $\eta(e_1) = \{s_1, s_2, s_5\}$ and $\eta(e_2) = \{s_2, s_3, s_4\}$. Therefore one can have $a_i : A \to \xi$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ where $\tilde{a_1} = \{(e_1, s_1), (e_2, s_2)\}, \tilde{a_2} = \{(e_1, s_1), (e_2, s_3)\}, \tilde{a_3} = \{(e_1, s_1), (e_2, s_4)\}, \tilde{a_4} = \{(e_1, s_2), (e_2, s_2)\}, \tilde{a_5} = \{(e_1, s_2), (e_2, s_3)\}, \tilde{a_6} = \{(e_1, s_2), (e_2, s_4)\}, \tilde{a_7} = \{(e_1, s_5), (e_2, s_2)\}, \tilde{a_8} = \{(e_1, s_5), (e_2, s_3)\}, \tilde{a_9} = \{(e_1, s_5), (e_2, s_4)\}$ .
Hence the soft set $(\eta, A)$ consists of above nine soft-members, that is, $(\eta, A) = \{a_i/i = 1, 2, ..., 9\}$, where ordered pairs $(e_j, s_k)$ with $j = 1, 2$ and $k = 1, 2, 3, 4, 5$ is called a soft element of the soft members.

3.4. **Example:** Now consider an other soft set $(\varrho, B)$ on the same universe $\xi$ and on the same parametric set $E$ for $B \subset E$ and $B = \{e_2, e_4\}$ as $\varrho : B \to P(\xi)$ and $b_i : B \to \xi$ with $i = 1, 2, 3, 4, 5, 6$ such that $\varrho(e_2) = \{s_2, s_3\}$ and $\varrho(e_4) = \{s_3, s_4, s_5\}$.
Therefore soft members of $(\varrho, B)$ are $\tilde{b_1} = \{(e_2, s_2), (e_4, s_3)\}, \tilde{b_2} = \{(e_2, s_2), (e_4, s_4)\}, \tilde{b_3} = \{(e_2, s_2), (e_4, s_5)\}, \tilde{b_4} = \{(e_2, s_3), (e_4, s_3)\}, \tilde{b_5} = \{(e_2, s_3), (e_4, s_4)\}, \tilde{b_6} = \{(e_2, s_3), (e_4, s_5)\}$ . Thus soft set $(\varrho, B)$ = $\{\tilde{b_1}, \tilde{b_2}, \tilde{b_3}, \tilde{b_4}, \tilde{b_5}, \tilde{b_6}\}$.

3.5. **Remarks:** The clarification of the above concepts would remain incomplete without the following remarks.

(1) Observe that the number of soft-elements (ordered pairs) in each soft member of the soft set is equal to the cardinality of the under laying subset of the parametric set $E$, that is, the number of soft elements in each soft member of $(\eta, A)$ is $3 = |A|$ and the soft member of each soft element in each soft member of $(\varrho, B)$ is $2 = |B|$ where $|A|$ and $|B|$ are the cardinality of $A$ and $B$ respectively.
(2) In example 3.3 and 3.4 the number of soft member in a soft set $(\eta, A)$ is the product of the cardinality of $\eta(e_i)$ for all $e_i \in A$, that is, the number of soft member in $(\eta, A)$ is 9. Indeed $|(\eta, A)| = (|\eta(e_1), (\eta(e_2))|)$ where $|.|$ represents the cardinality of the sets.

3.6. **Definition:** For the soft sets $(\eta, A)$ and $(\varrho, B)$, for all $e \in A$ but $e \notin B$, then the soft elements of $(\eta, A)$ for $e \in A$ are known as **Isolated soft elements** with respect to the soft sets $(\varrho, B)$. Similarly the Isolated soft elements of the soft $(\varrho, B)$ with respect to the soft set $(\eta, A)$ can be defined.

3.7. **Example:** In the above two examples, it can be observed that $e_1 \in$ A but $e_1 \notin$ B hence the soft elements $(e_1, s_1), (e_1, s_2)$ and $(e_1, s_5)$ are in the soft members of the soft set $(\eta, A)$ corresponding to the parameter $e_1$ are **Isolated soft elements** with respect to the soft sets $(\varrho, B)$. Similarly, the soft elements $(e_4, s_3), (e_4, s_4)$ and $(e_4, s_5)$ are in the soft members of the soft set $(\varrho, B)$ corresponding to the parameter $e_4$ are **Isolated soft elements** with respect to the soft sets $(\eta, A)$, since $e_4 \in$ B but $e_4 \notin$ A.

We now present the operations on soft sets using soft elements and soft numbers. We observe that it is compatible with the operations on soft sets that already exist.

## 4. Operations on soft sets using soft members

Here, it is now being endeavored to present the soft set operations by using soft members and soft elements in a natural way which are similar to the crisp set theory.

4.1. **Empty soft member:** A soft member $\tilde{a}$ of a soft set $(\eta, A)$ is said to be empty if for the mapping **a : A** $\rightarrow \xi$ , a(t) $\notin \eta(t)$ for all t in A . All the soft members of a null soft set are empty.

4.2. **Relative whole soft set:** A soft set $(\eta, A)$ whose soft members consist of all the elements of $\xi$. Moreover, a relative whole soft set **(**$\eta, E)$ is called an **absolute soft set**.

4.3. **Example:** Let for a univers $\xi$ = $\{1, 2, 3\}$ and a parametric sub set A = $\{t_1, t_2\}$ set valued function F : A $\rightarrow \xi$ define as $F(t_1)$ = $\xi$ and $F(t_2)$= $\xi$ . It is a relative whole soft set , its soft members are $\tilde{a_1} = \{(t_1, 1), (t_2, 1)\}$, $\tilde{a_2} = \{(t_1, 1), (t_2, 2)\}$, $\tilde{a_3} = \{(t_1, 1), (t_2, 3)\}$, $\tilde{a_4} = \{(t_1, 2), (t_2, 1)\}$, $\tilde{a_5} = \{(t_1, 2), (t_2, 2)\}$, $\tilde{a_6} = \{(t_1, 2), (t_2, 3)\}$ , $\tilde{a_7} = \{(t_1, 3), (t_2, 1)\}$ , $\tilde{a_8} = \{(t_1, 3), (t_2, 2)\}$, $\tilde{a_9} = \{(t_1, 3), (t_2, 3)\}$ .
Observe that , the collection of soft elements corresponding to each t belongs to A form $\xi$. Hence we have $(\eta, A)$ = $\{\tilde{a_1}, \tilde{a_2}, \tilde{a_3}, \tilde{a_4}, \tilde{a_5}, \tilde{a_6}, \tilde{a_7}, \tilde{a_8}, \tilde{a_9}\}$.

4.4. **Sub-soft member:** For the two soft sets $(\eta, A)$ and $(\varrho, B)$ over the same universe $\xi$, where the set A is a subset of set B, a soft member $\tilde{a}$ of $(\eta, A)$ is said to be a sub soft member of a soft member $\tilde{b}$ of $(\varrho, B)$ if each soft element of $\tilde{a}$ is a soft element of $\tilde{b}$ for all t in A.

4.5. **Soft sub Set:** If each soft member $\tilde{a}$ of $(\eta, A)$ there is at least one soft member $\tilde{b}$ of $(\varrho, B)$ such that $\tilde{a}$ is a sub-soft member of $\tilde{b}$ for all t in A, where set A is a subset of the set B, then $(\eta, A)$ is called a soft subset of $(\varrho, B)$.

4.6. **Union of Soft Sets:** For any two soft sets $(\eta, A)$ and $(\varrho, B)$ over a unifiers U their union is denoted by $(\eta, A)\tilde{\cup}(\varrho, B)$ and defined as each soft member of $(\eta, A)\tilde{\cup}(\varrho, B)$ contains the soft elements of $(\eta, A)$ only if t belongs to A or of $(\varrho, B)$ only if t belongs B or of any $(\eta, A)$ or $(\varrho, B)$ for all t belongs to A and B. Observe that both soft sets $(\eta, A)$ and $(\varrho, B)$ are soft subsets of $(\eta, A)\tilde{\cup}(\varrho, B)$.

4.7. **Restricted Union of Soft Sets:** For any two soft sets $(\eta, A)$ and $(\varrho, B)$ over a universe $\xi$ their union is denoted by $(\eta, A)\tilde{\cup}(\varrho, B)$ and defined as each soft member of $(\eta, A)\tilde{\cup}_R(\varrho, B)$ contains the soft elements of $(\eta, A)$ or of $(\varrho, B)$ or of both $(\eta, A)$ and $(\varrho, B)$ for all t belongs to both A and B. Observe that both soft sets $(\eta, A)$ and $(\varrho, B)$ may not be soft subsets of $(\eta, A)\tilde{\cup}_R(\varrho, B)$.

4.8. **Intersection of Soft Sets:** For any two soft sets $(\eta, A)$ and $(\varrho, B)$ over a universe $\xi$ their intersection is denoted by $(\eta, A)\tilde{\cap}(\varrho, B)$ and defined as each soft member of $(\eta, A)\tilde{\cap}(\varrho, B)$ contains the soft elements of both $(\eta, A)$ and $(\varrho, B)$ for all t belongs to both A and B.

4.9. **Extended Intersection of Soft Sets:** For any two soft sets $(\eta, A)$ and $(\varrho, B)$ over a universe $\xi$ their union is denoted by $(\eta, A)\tilde{\cap}_E(\varrho, B)$ and defined as each soft member of $(\eta, A)\tilde{\cap}_E(\varrho, B)$ contains the soft elements of $(\eta, A)$ only if t belongs to A or of $(\varrho, B)$ only if t belongs B or of $(\eta, A)$ and $(\varrho, B)$ for all t belongs to both A and B.

4.10. **Complement of a Soft Set:** The complement of a soft set $(\eta, A)$ is also a soft set $(\acute{\eta}, A)$ such that if $\tilde{u}$ is a soft member in $(\acute{\eta}, A)$ , where $\acute{\eta}$ is the crisp complement of $\eta$ for all t in A , then there exist a soft member $\tilde{u^c}$ in $(\eta, A)$ such that $\tilde{u} \cap \tilde{u^c} = \phi$ . This means is there is a soft element $(t, a(t)) \in \tilde{u}$ implies that $\exists\ (t, \acute{a}(t)) \in \tilde{u^c}$ such that a(t) $\in \eta(t)$ and $\acute{a}(t) \in \acute{\eta}(t)$ of course a(t) and $\acute{a}(t)$ are disjoint for all t $\in$ A and hence $(\eta, A)\ \tilde{\cap}\ (\acute{\eta}, A) = \Phi$ .

4.11. **Proposition:** A soft set $(\acute{\eta}, A)$ is a soft complement of the soft set $(\eta, A)$ if and only if $(\eta, A)\ \tilde{\cap}\ (\acute{\eta}, A) = \Phi$ .
**Proof:** Let $(\acute{\eta}, A)$ be a soft complement of $(\eta, A)$ thus by definition $(\eta, A)\ \tilde{\cap}\ (\acute{\eta}, A) = \Phi$ . Conversely, suppose that $(\eta, A)\ \tilde{\cap}\ (\acute{\eta}, A) = \Phi$, now it is to show that $(\acute{\eta}, A)$ is a soft complement of $(\eta, A)$. For this let $\tilde{u} \in (\eta, A)$ so for each $(t, a(t)) \in \tilde{u}$ and a(t) $\in \eta(t)$ for all t $\in$ A (by definition) and hence a(t) $\notin \acute{\eta}(t)$ for all t $\in$ A , therefore $\tilde{u}$ is not a soft member of $(\acute{\eta}, A)$ since $\tilde{u}$ is arbitrary thus $(\acute{\eta}, A)$ is a soft complement of $(\eta, A)$ .

4.12. **Theorem:** If $(\acute{\eta}, A)$ is a soft complement of $(\eta, A)$ then $(\eta, A)\ \tilde{\cup}\ (\acute{\eta}, A)$ is an absolute soft set over U.

**Proof:** It is obvious and left for readers to verify.

4.13. **Theorem:** A soft set $(\eta, A)$ is a sub-soft set of $(\varrho, B)$ if and only if $\eta(t) \in \varrho(t)$ for all t in A.
**Proof:** The proof can be observed directly from the definition of the soft set and the definition of the soft sub-member.

The restricted intersection $(\eta, A)\tilde{\cap}_R(\varrho, B) = (\gamma, C)$ discussed in above two examples 3.3 and 3.4 where $C$ is equal to $A \cap B = \{e_2\}$ and $\gamma : C \to P(\xi)$ is $\eta(e_2) = \{s_2, s_3\}$ therefore $(\gamma, C)$ has two soft members $\tilde{c_1} = \{(e_2, s_2)\}$ and $\tilde{c_2} = \{(e_2, s_3)\}$ for c :C $\to \xi$ with $i = 1, 2$ . Hence $(\eta, A)\tilde{\cap}_R(\varrho, B) = \{\tilde{c_1}, \tilde{c_2}\}$.
For the union of soft sets $(\eta, A)$ and $(\varrho, B)$ in the view of the definition 4.6 is $(\eta, A)\tilde{\cup}(\varrho, B) = (\gamma, C)$ where $C = A \cup B = \{e_1, e_2, e_4\}$ and $\gamma : C \to P(\xi)$ is defined as $\gamma(e_1) = \{s_1, s_2, s_5\}, \gamma(e_2) = \{s_2, s_3, s_4, s_5\}, \gamma(e_4) = \{s_3, s_4, s_5\}$ . Therefore, the number of soft elements in each soft member in $(\gamma, C)$ is $|c| = 3$ and the number of soft members of the soft set $(\gamma, C)$ is $|(\gamma, C)| = |(\gamma(e_1))(\gamma(e_2))(\gamma(e_3))| = 3 \times 4 \times 3 = 36$.

4.14. **Example:** Let $\xi = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ be a universal set and $E = \{t_1, t_2, t_3, t_4\}$ be a set of parametric family. Suppose that $A = \{t_1, t_2\}$ and $B = \{t_1, t_2, t_3\}$ are subsets of $E$. Consider $(\eta, A), (\varrho, B)$ over the universe $\xi$ where $\eta(t_1) = \{s_1, s_3\}, \eta(t_2) = \{s_3, s_6\}$ and $\varrho(t_1) = \{s_1, s_3\}, \varrho(t_2) = \{s_3, s_6\}, \varrho(t_3) = \{s_4, s_5\}$. Hence, (by definition 2.2) $(\eta, A) \subseteq (\varrho, B)$ however it shall also be shown by using (definition 4.5) of soft subset. The soft members of $(\eta, A)$, are $\tilde{a_1} = \{(t_1, s_1), (t_2, s_3)\}, \tilde{a_2} = \{(t_1, s_1), (t_2, s_6)\}, \tilde{a_3} = \{(t_1, s_3), (t_2, s_3)\}, \tilde{a_4} = \{(t_1, s_3), (t_2, s_6)\}$. So soft set $(\eta, A)$ in the form of its soft members is written as $(\eta, A) = \{\tilde{a_1}, \tilde{a_2}, \tilde{a_3}, \tilde{a_4}\}$. The soft members of $(\varrho, B)$, are $\tilde{b_1} = \{(t_1, s_1), (t_2, s_3), (t_3, s_4)\}, \tilde{b_2} = \{(t_1, s_1), (t_2, s_6), (t_3, s_4)\}, \tilde{b_3} = \{(t_1, s_3), (t_2, s_3), (t_3, s_4)\}$, $\tilde{b_4} = \{(t_1, s_3), (t_2, s_6), (t_3, s_4)\}, \tilde{b_5} = \{(t_1, s_1), (t_2, s_3), (t_3, s_5)\}, \tilde{b_6} = \{(t_1, s_1), (t_2, s_6), (t_3, s_5)\}$, $\tilde{b_7} = \{(t_1, s_3), (t_2, s_3), (t_3, s_5)\}, \tilde{b_8} = \{(t_1, s_3), (t_2, s_6), (t_3, s_5)\}$.
This implies that $(\varrho, B) = \{\tilde{b_1}, \tilde{b_2}, \tilde{b_3}, \tilde{b_4}, \tilde{b_5}, \tilde{b_6}, \tilde{b_7}, \tilde{b_8}\}$.

The soft elements corresponding to the parameter $t_3$ in each member of the soft set $(\varrho, B)$ are isolated with respect to the soft members of the soft set $(\eta, A)$. Therefore,by definition 4.4, the soft members of the soft set $(\eta, A)$ are sub soft members of the soft members of the soft set $(\varrho, B)$. Hence for each $\tilde{a} \in (\eta, A)$, there is at least one $\tilde{b} \in (\varrho, B)$ such that $\tilde{a} \subseteq \tilde{b}$ observe that $\tilde{a_1} \subseteq \tilde{b_1}$ and also $\tilde{a_1} \subseteq \tilde{b_5}$ similarly, it can also be observed for the other members of $(\eta, A)$ . Hence, by definition 4.5, it implies that $(\eta, A) \subseteq (\varrho, B)$.

4.15. **Proposition:** A soft set $(\eta, A)$ is a soft subset of $(\varrho, B)$ if and only iff for each $\tilde{a} \in (\eta, \varrho)$, there exist at least one $\tilde{b} \in (\varrho, B)$, such that $\tilde{a}$ is a sub-soft member of $\tilde{b}$.
**Proof:** Suppose that the soft $(\eta, A)$ is a soft subset of $(\varrho, B)$ then by the definition of soft subset if $\tilde{a} \in (\eta, \varrho)$ then there is a soft member $\tilde{b} \in (\varrho, B)$ such that $\tilde{a}$ is a sub member of $\tilde{b}$ . Conversly; suppose that for each $\tilde{a} \in (\eta, \varrho)$, there exist at least one $\tilde{b} \in (\varrho, B)$, such that $\tilde{a}$ is a sub-soft member of $\tilde{b}$. Then by definition of soft subset $(\eta, A)$ is a soft subset of $(\varrho, B)$.

As for as equal soft sets are concerned, it can be classified in two ways. Firstly, the two soft sets $(\eta, A)$ and $(\varrho, B)$ are said to be absolute equal if $A = B$ and $\gamma(t) = \eta(t)$ are identical for all $t \in A = B$, secondly the soft set $(\eta, A)$ and $(\varrho, B)$ are casually equal if $A = B$ but $\gamma(t)$ and $\eta(t)$ are identical for some $t \in A = B$,as some of the soft elements are Isolated for other $t \in A = B$, then we write it as $(\eta, A) =_c (\varrho, B)$ .

4.16. **Definition:** [5] If $\xi$ is a universe, $E$ a set of attributes and $A$ is a subset of $E$, then

(1) $(\eta, A)$ is known as a relative null soft set with respect to $A$ if $\eta(t) = \phi\ \forall\ t \in A$ and it is denoted by $\tilde{N}_A$.
(2) A soft set $(\eta, A)$ is called a relative whole soft set or a universal set with respect to $A$ if $\eta(t) = \xi$ for all $t \in A$ and it is denoted by $\tilde{\xi}_A$.
(3) An absolute soft set over $\xi$ and it is denoted by $\tilde{\xi}_E$ is a relative whole soft set with respect to $E$.

4.17. **Example:** Let $\xi = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and $E = \{t_1, t_2, t_3\}$ such that $A = \{t_1, t_3\}$. The soft set $(\eta, A)$ over $\xi$ is such that $\eta : A \to P(\xi)$ is defined as $\eta(t)$ is an empty sub set of $\xi$ for all $\in$ A and $a : A \to \xi$ is an empty mapping, that is, $a(t)$ for all

$t \in A$. This implies each soft member of $(\eta, A)$ is an empty collection of the soft elements , therefore $(\eta, A)$ is an empty soft set with respect to $A$.
However, if $(\eta, A)$ is defined as $\eta(t_1) = \xi, \eta(t_3) = \xi$ then each soft members of $(\eta, A)$ has two soft elements, however $(\eta, A)$ consists of 36 soft members and it is a whole set with respect to $A$.
On the other hand, the absolute whole soft set $\tilde{\xi}_E$ consists of 216 members and each member has three soft elements.
The following proposition is analogous of the proposition in [27].

4.18. **proposition:** If $\xi$ is a universe, $E$ a set of attribute, and $A, B, C$ are subsets of $E$. Moreover, $(\eta, A), (\varrho, B)$ and $(\gamma, C)$ are soft sets over $\xi$, then

(1) $(\eta, A) \subseteq \tilde{\xi}_A \subseteq \tilde{\xi}_E$
(2) $\tilde{N}_A \subseteq (\eta, A)$
(3) $(\eta, A) \subseteq (\varrho, B)$ and $(\varrho, B) \subseteq (\gamma, C) \Rightarrow (\eta, A) \subseteq (\gamma, C)$
(4) $(\eta, A) \cong (\varrho, B)$ and $(\varrho, B) \cong (\gamma, C) \Rightarrow (\eta, A) \cong (\gamma, C)$
(5) $(\eta, A) =_c (\varrho, B)$ and $(\varrho, B) =_c (\gamma, C) \Rightarrow (\eta, A) =_c (\gamma, C)$

*Proof.* These can be proved by using the soft elements and the soft members. □

4.19. **Definition:** [5] For a soft set $(\eta, A)$, the relative complement of $(\eta, A)$ is also another soft set $(\eta^{'}, A)^r$ where $\eta^{'} : A \to P(\xi)$ is given by $\eta^{'}(e) = \xi - \eta(e)$ for all $e \in A$.
The following example gives the clear picture of the relative complement of a soft set $(\eta, A)$ defined on a subset $A$ of parameters over the universe.

4.20. **Example:** Consider a soft set $(\eta, A)$ for the set of parameter $E = \{t_1, t_2, t_3\}$ and the universe $\xi = \{s_1, s_2, s_3, s_4\}$. Let $A \subseteq E$ and $A = \{t_1, t_3\}$ and define $\eta : A \to P(\xi)$ as $\eta(t_1) = \{s_2, s_3\}, \eta(t_2) = \{s_1, s_3\}$ and also $a_i : A \to \xi$ for $i = 1, 2, 3, 4$ where $a_1(t_1) = s_2$ , $a_2(t_1) = s_3$ , $a_3(t_2) = s_1$ and $a_4(t_2) = s_3$. Hence the soft member of $(\eta, A)$ are $\tilde{a_1} = \{(t_1, s_2), (t_2, s_1)\}, \tilde{a_2} = \{(t_1, s_2), (t_2, s_3)\}, \tilde{a_3} = \{(t_1, s_3), (t_2, s_1)\}, \tilde{a_4} = \{(t_1, s_3), (t_2, s_3)\}$.
For the relative complement $\eta^{'} : A \to P(\xi)$, $\eta^{'}(e) = \xi - \eta(e)$ for each $e \in A$. This implies $\eta^{'}(t_1) = \{s_1, s_4\}$ and $\eta^{'}(t_2) = \{s_2, s_4\}$. Therefore, the complement maps $a_i^{'} : A \to \xi$ for $i = 1, 2, 3, 4$ are $a_1^{'}(t_1) = s_1$ , $a_2^{'}(t_1) = s_4$ , $a_3^{'}(t_2) = s_2$ , $a_4^{'}(t_4) = s_4$ . Hence the soft members of $(\eta, A)^{'}$ are $\tilde{a_1}^{'} = \{(t_1, s_1), (t_2, s_2)\}, \tilde{a_2}^{'} = \{(t_1, s_1), (t_2, s_4)\}, \tilde{a_3}^{'} = \{(t_1, s_4), (t_2, s_2)\}, \tilde{a_4}^{'} = \{(t_1, s_4), (t_2, s_4)\}. This implies (\eta, A)^{'} = \{\tilde{a_1}^{'}, \tilde{a_2}^{'}, \tilde{a_3}^{'}, \tilde{a_4}^{'}\}$.
We will observe here $(\eta^{'}, A)^r$ is giving clearly different picture from the classical concept in the set theory, however we can observe that $(\eta, A)^r \tilde{\cap} (\eta, A) = \tilde{N}_A$.

4.21. **proposition:** For the soft set $(\eta, A)$ where $A$ is a subset of $E$ over the universe $\xi$, then

- $(\eta, A)^r \tilde{\cap} (\eta, A) = \tilde{N}_A$
- $((\eta, A)^r)^r \cong (\eta, A)$
- $(\tilde{N}_A)^r \cong (\tilde{\xi}_A)^r$

*Proof.* These can be proved directly by using the definitions. □

4.22. **Definition:** [14]

(1) For $(\eta, A), (\varrho, B)$ soft sets over the same universe $\xi$ then the AND operation, $\wedge$, on $(\eta, A)$ and $(\varrho, B)$ is denoted by $(\eta, A) \wedge (\varrho, B)$ is also soft set $(\gamma, C)$, where $C = A \times B$ and $\gamma(u, v) = \eta(u) \cap \varrho(v) \quad \forall (u, v) \in A \times B$

(2) The OR-operation on $(\eta, A)$ and $(\varrho, B)$ is denoted by $(\eta, A) \vee (\varrho, B) = (\gamma, C)$ , where $C = A \times B$ and $\gamma(u, v) = \eta(u) \cup \varrho(v) \ \forall \ (u, v) \in A \times B$

4.23. **Example:** Let $\xi = \{s_1, s_2, s_3, s_4\}$ and $E = \{u, v, w\}$ and $A, B \subseteq E$ where $A = \{u, v\}$ and $B = \{u, w\}$. The soft sets $(\eta, A)$ and $(\varrho, B)$ are defined as $\eta : A \to P(\xi)$ where $\eta(u) = \{s_1\}$, $\eta(v) = \{s_1, s_3\}$ and $a_i : A \to \xi$ where $a_1(u) = s_1, a_2(v) = s_2, a_3(v) = s_3$ Soft members of $(\eta, A)$ are $\tilde{a_1} = \{(u, s_1), (v, s_1)\}, \tilde{a_2} = \{(u, s_1), (v, s_3)\}$. This means $(\eta, A) = \{\tilde{a_1}, \tilde{a_2}\}$. Now $\varrho : B \to P(\xi)$ is $\varrho(u) = \{s_3, s_4\}, \varrho(w) = \{s_3\}$ and $b_i : B \to \xi$ with $\tilde{b_1} = \{(u, s_3), (w, s_3)\}, \tilde{b_2} = \{(u, s_4), (w, s_3)\}$ . Hence $(\eta, A) \wedge (\varrho, B) = (\gamma, C)$ where $C = A \times B = \{(u, u), (u, w), (v, u), (v, w)\}$. Now $\gamma : C \to P(\xi)$ with $\gamma(u, u) = \phi, \gamma(v, w) = \{s_3\}, \gamma(u, w) = \phi, \gamma(v, u) = \phi$. Therefore, there is only one soft member in $(\eta, A) \wedge (\varrho, B)$. Hence we have $(\eta, A) \wedge (\varrho, B) = \{(u, w), s_3\}$ where $\{(u, w), s_3\}$ is a soft element of this soft set.

## 5. Cartesian Soft Product and Soft Set Relations

The following section contains the cartesian soft product and the soft set relations using the soft members of soft sets.

5.1. **Definition:** For $(\eta, A), (\varrho, B)$ soft sets over a common universe $\xi$ such that $\eta : A \to P(\xi)\,; a : A \to \xi$ and $\varrho : B \to P(\xi)\,; b : B \to \xi$. Therefore $(\eta, A) = \{\tilde{a} \mid a : A \to \xi, \eta : A \to P(\xi), a(e) \in \eta(e) \forall e \in A\}$ and $(\varrho, B) = \{\tilde{b} \mid b : B \to \xi, \varrho : B \to P(\xi), b(e) \in \eta(e) \forall e \in B\}$. Then the cartesian product of $(\eta, A) \times (\varrho, B)$ is a soft set $(\delta, C)$ where C = $A \times B$ such that $\delta : C \to P(\xi \times \xi)$ and $\delta(e_i, e_j) = \eta(e_i) \times \varrho(e_j) \ \forall \ (e_i, e_j) \in C$. Moreover, $c : C \to \xi \times \xi$ such that $c(e_i, e_j) \in \eta(e_i) \times \varrho(e_j) \ \forall \ (e_i, e_j) \in C$. Therefore $\delta(e_i, e_j) = \{(a(e_i), b(e_i)) \mid a(e_i) \in \eta(e_i), b(e_j) \in \varrho(e_j)\}$.So eventually $\tilde{c} = (\tilde{a}, \tilde{b}) = \{((e_i, e_j), (a(e_i), b(e_j)) \mid \forall (e_i, e_j) \in C$ and $(a(e_i), b(e_j)) \in \eta(e_i) \times \varrho(e_j)\}$ is a soft member of $(\delta, C)$.Finally the Cartesian soft product is a soft set $(\delta, C) = \{(\tilde{a}, \tilde{b}) \mid (a, b) : C \to \xi \times \xi, \ \delta : C \to P(\xi \times \xi)\}$.

5.2. **Example:** The cartesian soft product of the soft sets $(\eta, A)$, and $(\varrho, B)$ (In example 4.14) is a soft set $(\delta, C)$ where C = $A \times B$ such that $\delta : C \to P(\xi \times \xi)$ and $\delta(t_i, t_j) = \eta(t_i) \times \varrho(t_j) \ \forall \ (t_i, t_j) \in C$. Moreover, $c : C \to \xi \times \xi$ such that $c(t_i, t_j) \in \eta(t_i) \times \varrho(t_j) \ \forall \ (t_i, t_j) \in C$. Therefore $\delta(t_i, t_j) = \{(a(t_i), b(t_i)) \mid a(t_i) \in \eta(t_i), b(t_j) \in \varrho(t_j)\}$.So eventually $\tilde{c} = (\tilde{a}, \tilde{b}) = \{((t_i, t_j), (a(t_i), b(t_j)) \mid \forall (t_i, t_j) \in C$ and $(a(t_i), b(t_j)) \in \eta(t_i) \times \varrho(t_j)\}$ is a soft member of $(\delta, C)$.Hence the Cartesian soft product is $(\delta, C) = \{(\tilde{a_i}, \tilde{b_j}) \mid \tilde{a_i} \in (\eta, A)\, \tilde{b_j} \in (\varrho, B)\}$. Where $i = 1, 2, 3, 4\, and\, j = 1, 2, 3, ........, 8$. A typical soft member of $(\delta, C)$ has the form $(\tilde{a_2}, \tilde{b_5})$ = { $\{(t_1, s_1), (t_2, s_6)\}$, $\{(t_1, s_1), (t_2, s_3), (t_3, s_5)\}$ } .

5.3. **Definition:** For $(\eta, A), (\varrho, B)$ are soft sets over the universe $\xi$. Then a soft subset of the cartesian product $(\eta, A) \times (\varrho, B)$ is a soft set $(\psi, C)$ which is a soft set relation from $(\eta, A)$ to $(\varrho, B)$, where $C = A_1 \times B_1, \ A_1 \subset A, B_1 \subset B$ such that $\psi : A_1 \times B_1 \to P(\xi \times \xi)$ and $h : C \to \xi_1 \times \xi_2$ where $\psi(e_i, e_j) = \eta(e_i) \times \varrho(e_j) \ \forall \ (e_i, e_j) \in C$ with $h(e_i, e_j) \in$

$\psi(e_i,e_j)\,\forall(e_i,e_j)\in C$ and $\psi(e_i,e_j)=\{(a(e_i),b(e_j)) \mid a(e_i)\in\eta(e_i),b(e_j)\in\varrho(e_j)\}$ $\forall e_i\in A_1$ and $\forall e_j\in B_1$. Therefore $\tilde{h}=(\tilde{a},\tilde{b})=\{((e_i,e_j),(a(e_i),b(e_j)) \mid (e_i,e_j)\in C$ and $(a(e_i),b(e_j))\in\eta(e_i)\times\varrho(e_j)\}$ is a soft member of $(\psi,C)$ and hence the soft set relation from $(\eta,A)$ to $(\varrho,B)$ is $(\psi,C)=\{(\tilde{a},\tilde{b}) \mid (a,b):A_1\times B_1\to\xi\times\xi,\,\psi:A_1\times B_1\to P(\xi\times\xi)\}$.

**Note:**

(1) Whenever there is no confusion we can write simply $(a(e_i),b(e_j))$ for the soft elements of the soft members of the soft relation $(\psi,C)$. Moreover, $R$ can also be used for the soft set $(\psi,C)$.

(2) Since $R$ consists of ordered pairs of the soft members of the soft set $(\psi,C)$ that has the form $\tilde{h}=\{(e_i,e_j),(a(e_i),b(e_j)) \mid (e_i,e_j)\in A_1\times B_1\}$ . Hence $\tilde{a}R\tilde{b}$ iff $(\tilde{a},\tilde{b})\in R$ where $\tilde{a}\in(\eta,A),\tilde{b}\in(\varrho,B)$. If $\tilde{a}$ is not related to $\tilde{b}$ then $(\tilde{a},\tilde{b})\notin R$

5.4. **Example:** A soft set relation from the soft set $(\eta,A)$ to the soft set $(\varrho,B)$,( In the example 4.14 ) is a soft subset $(\psi,C)$ of the cartesian product $(\eta,A)\times(\varrho,B)$, where $C=A_1\times B_1,\ for\, A_1\subset A,\ and\, B_1\subset B$ such that $A1=\{t_1\}$ and $B1=\{t_1,t_2\}$ such that $\psi:A_1\times B_1\to P(\xi\times\xi)$ and $h:C\to\xi\times\xi$ where $\psi(t_i,t_j)=\eta(t_i)\times\varrho(t_j)\,\forall\,(t_i,t_j)\in C$ with $h(t_i,t_j)\in\psi(t_i,t_j)\,\forall(t_i,t_j)\in C$ and $\psi(t_i,t_j)=\{(a(t_i),b(t_j)) \mid a(t_i)\in\eta(t_i),b(t_j)\in\varrho(t_j)\}\,\forall t_i\in A_1$ and $\forall t_j\in B_1$. Hence take $\eta(t_1)=\{s_1,s_3\}$ and $\varrho(t_1)=\{s_1,s_3\},\varrho(t_2)=\{s_3,s_6\}$.
The soft members of $(\eta,A_1)$, are $\tilde{a_1}=\{(t_1,s_1)\},\tilde{a_2}=\{(t_1,s_3)\}$. So soft set $(\eta,A)$ in the form of its soft members is written as $(\eta,A_1)=\{\tilde{a_1},\tilde{a_2}\}$.
The soft members of $(\varrho,B1)$, are $\tilde{b_1}=\{(t_1,s_1),(t_2,s_3)\}$, $\tilde{b_2}=\{(t_1,s_1),(t_2,s_6)\}$, $\tilde{b_3}=\{(t_1,s_3),(t_2,s_3)\}$, $\tilde{b_4}=\{(t_1,s_3),(t_2,s_6)\}$. Eventually, $(\varrho,B_1)=\{\tilde{b_1},\tilde{b_2},\tilde{b_3},\tilde{b_4}\}$.
Hence the soft set relation R from the soft set $(\eta,A)$ to the soft set $(\varrho,B)$ is R = $(\psi,C)=\{(\tilde{a_i},\tilde{b_j}) \mid \tilde{a_i}\in(\eta,A_1)\,,\tilde{b_j}\in(\varrho,B_1)\}$. Where $i=1,2\,and\,j=1,2,3,4$. So for as a soft member of this relation is concerned, it has the form $(\tilde{a_2},\tilde{b_3})$ = $\{\ \{(t_1,s_3)\},\{(t_1,s_3),(t_2,s_3)\}\ \}$.

5.5. **Definition:** The soft relation $R$ on $(\eta,A)$ is a soft subset of $(\eta,A)\times(\eta,A)$, that is, $R(\psi,A_1\times A_1)=\{\tilde{h}/h:A_1\times A_1\to\xi\times\xi$ and $\psi:A_1\times A_1\to P(\xi_1\times\xi_2)$ such that $h(e_i,e_j)\in\psi(\xi)\times\psi(\xi)$ for all $(e_i,e_j)\in A_1\times A_1$ where $A_1\in A\}$. In this case, the soft members of $R$ are of the form $(\tilde{a_i},\tilde{a_j})$ that is $(\tilde{a_i}R\tilde{a_j})$ iff $(\tilde{a_i},\tilde{a_j})\in R$.

5.6. **Definition:** Let $R$ be a soft set relation from $(\eta,A)$ to $(\varrho,B)$ , then the domain of $R$ is a soft subset $(\eta_1,A)$ of $(\eta,A)$ such that $(\eta_1,A_1)=\{(a(e_i),b(e_j)) \mid \forall\ e_i\in A_1\in R$ and for some $e_j\in B_1\}$ . The range of $R$ is a soft subset $(\varrho_1,B_1)$ of $(\varrho,B)$ such that $(\varrho_1,B_1)=\{(a(e_i),b(e_j))\in R$ for some $e_i\in A_1$ and for all $e_j\in B_1\}$ . The inverse relation of $R$ is also a soft subset $(\psi^{-1},B_1\times A_1)$ of the soft set $(\varrho,B)\times(\eta,A)$, that is, $R^{-1}=(\psi^{-1},B_1\times A_1)=\{(\tilde{b},\tilde{a}) \mid a:A_1\to\xi_1\ ;\ \eta:A\to P(\xi),b:B_1\to\xi,\varrho:B\to P(\xi),a(e_i)\in\eta(e_i),b(e_i)\in\varrho(e_j)\ \ \forall\ e_i\in A_1,e_j\in B_1\}$.

5.7. **Example:** In the example 5.4, the domain and range of the soft set relation is $(\eta, A1)\ and (\varrho, B1)$ respectively. The inverse relation relation of R is also a soft subset $(\psi^{-1}, B_1 \times A_1)$ of the soft set $(\varrho, B) \times (\eta, A)$, that is, $R^{-1} = (\psi^{-1}, B_1 \times A_1) = \{(\tilde{b_j}, \tilde{a_i}) \quad | \quad \tilde{a_i} \in (\eta, A1), \tilde{b_j} \in (\varrho, B1)\}$. Where $i = 1, 2\ and\ j = 1, 2, 3, 4$. So for as a soft member of this relation is concerned, it has the form $(\tilde{b_3}, \tilde{a_2})$ = { $\{(t_1, s_3), (t_2, s_3)\}$ , $\{(t_1, s_3)\}$ }.

## 6. Applications in Soft Algebraic Structures

The concept of soft member and soft element helpful to study soft algebraic structures. First define a binary operation on $(\eta, A)$.

6.1. **Definition:** [9] The support of a soft set $(\eta, A)$, denoted by $Supp(\eta, A)$, defined as $Supp(\eta, A) = \{e \in A \quad | \quad \eta(e) \neq \phi\}$. If $Supp(\eta, A) \neq \phi$ then the soft set $(\eta, A)$ is called non-null.

6.2. **Definition:** Let $(\eta, A)$ be a non-null soft set over the universe $U$ with some binary operation $*$ then an operation $\tilde{\circledast}$ is a soft binary operation from $(\eta, A) \times (\eta, A)$ to $(\eta, A)$ , that is , $\tilde{\circledast} : (\eta, A) \times (\eta, A) \rightarrow (\eta, A)$ for $(\tilde{a_i}, \tilde{a_j}) \in (\eta, A) \times (\eta, A)$ and $\tilde{a_i} \tilde{\circledast} \tilde{a_j} = \eta_k(a) * \eta_l(a)$ for all $a \in A$ where $*$ is the binary operation defined on the universal set $U$ . Note that the ordered pair $(a, \eta_k(a) * \eta_l(a))$ is a soft element in some soft member. Hence the pair $((\eta, A), \tilde{\circledast})$ is called soft algebraic structure corresponding the soft binary operation $\tilde{\circledast}$.

6.3. **Definition:** If for all $\tilde{a_i} \in (\eta, A)$, there exist $\tilde{a_e} \in (\eta, A)$ such that $\tilde{a_e} \tilde{\circledast} \tilde{a_i} = \tilde{a_i} \tilde{\circledast} \tilde{a_e} = \tilde{a_i}$ then $\tilde{a_e}$ is called a soft identity element of the soft algebraic structure.

6.4. **Definition:** If for all $\tilde{a_i}, \tilde{a_j} \in (\eta, A)$ have the property that $\tilde{a_i} \tilde{\circledast} \tilde{a_j} = \tilde{a_j} \tilde{\circledast} \tilde{a_i}$ then $\tilde{\circledast}$ is called softly commutative.

Moreover, for all $\tilde{a_i}, \tilde{a_j}, \tilde{a_k} \in (\eta, \varrho)$ satisfy $(\tilde{a_i} \tilde{\circledast} \tilde{a_j}) \tilde{\circledast} \tilde{a_k} = (\tilde{a_i} \tilde{\circledast} \tilde{a_k}) \tilde{\circledast} (\tilde{a_i} \tilde{\circledast} \tilde{a_k})$ then $\tilde{\circledast}$ is softly associative. If for any two soft members $\tilde{a_i}$ and $\tilde{a_j}$ of $(\eta, A)$ then soft operation $\tilde{\circledast}$ satisfy that $(\tilde{a_i} \tilde{\circledast} \tilde{a_j}) = (\tilde{a_j} \tilde{\circledast} \tilde{a_i}) = \tilde{a_e}$ where $\tilde{a_e}$ is soft identity in $(\eta, A)$ then $\tilde{a_i}$ is a soft inverse of $\tilde{a_j}$ and vice versa. The soft inverse of the soft member $\tilde{a_i}$ is denoted by $\tilde{a_i}^{-1}$.

**Remark 6.5.** *In the soft algebraic structure $((\eta, A), \tilde{\circledast})$, the binary operation over $U$ remains preserved. A soft member $\tilde{a_e}$ , and is a soft identity iff in each of the soft element of $\tilde{a_e}$ the second character is the identity of $U$ with respect to $*$ for each $a \in A$. The nature of the soft algebraic structure $((\eta, A), \circledast)$ depends upon the nature of the binary relation in the universe $U$. Further, note that keeping in view the last point of the above remark, one can study all the mathematical algebraic structures in which such a soft algebraic operation $\tilde{\circledast}$ is defined.*

6.6. **proposition.** If $((\eta, A), \tilde{\circledast})$ is a soft algebraic structure over a universe $(U, *)$, then

(1) The soft identity member $\tilde{a_e}$ of $(\eta, A)$ with respect to $\tilde{\circledast}$ if it exists then it is unique.
(2) The soft inverse member $\tilde{a_i}^{-1}$ of the soft member $\tilde{a_i}$ of $(\eta, A)$ if exists then it is unique.

*Proof.* It is same as the using algebraic operations. □

6.7. **Definition:** [2] For $(\eta, A)$ a non-null soft set over $U = G$. Then $(\eta, A)$ is said to be a soft set group over $U = G$ if and only if $\eta(a)$ is a subgroup of $U = G$ for all $a \in A$.

6.8. **Definition:** Let $(\eta, A)$ be soft set over a universe $U = G$ where $(G, *)$ is a group with respect to the binary operation $*$ and let $\tilde{\circledast}$ be a soft binary operation defined on $(\eta, A)$, then $((\eta, A), \tilde{\circledast})$ is a soft group if satisfies the following:

(1) $\tilde{a_i}\tilde{\circledast}\tilde{a_j} \in (\eta, A)$ for all $\tilde{a_i}, \tilde{a_j} \in (\eta, A)$ (soft closure law)
(2) $\tilde{a_i}\tilde{\circledast}(\tilde{a_j}\tilde{\circledast}\tilde{a_k}) = (\tilde{a_i}\tilde{\circledast}\tilde{a_j})\tilde{\circledast}\tilde{a_k}$ for all $\tilde{a_i}, \tilde{a_j}, \tilde{a_k} \in (\eta, A)$ (soft associative law)
(3) There exist $\tilde{a_e} \in (\eta, A)$ such that for all $\tilde{a_i} \in (\eta, A)$, we have $\tilde{a_i}\tilde{\circledast}\tilde{a_e} = \tilde{a_e} * \tilde{a_i} = \tilde{a_i}$ (soft identity member)
(4) For each $\tilde{a_i} \in (\eta, A)$ there exist $\tilde{a_i}^{-1} \in (\eta, A)$ such that $\tilde{a_i}\tilde{\circledast}\tilde{a_i}^{-1} = \tilde{a_i}^{-1}\tilde{\circledast}\tilde{a_i} = \tilde{a_e}$. (soft inverse element)

Furthermore, if $\tilde{\circledast}$ is satisfying the commutative property as well then $((\eta, A), \tilde{\circledast})$ will be an abelian soft set group over $U = G$.

The following example will explain the structure of a soft group which is just defined above.

6.9. **Example:** Let $U = G = \{< w >, w^6 = 1\}$, the 6 6th roots of unity and $A \subset E$ where $E$ is a set of attributes. Let $A = \{e_1, e_2\}$ and define a mapping $\eta : A \to P(U)$ such that $\eta(e_1) = \{< w^2 >, w^6 = 1\} = \{1, w^2, w^4\}$ , $\eta(e_2) = \{< w^3 >, w^6 = 1\} = \{1, w^3\}$. Since each $\eta(e_1)$ and $\eta(e_2)$ are subgroups of $U$. So, by definition 6.7, $(\eta, A)$ is a soft group. The soft member of the soft group $(\eta, G)$ are $\tilde{g_e} = \{(e_1, 1), (e_2, 1)\}$, $\tilde{g_1} = \{(e_1, 1), (e_2, w^3)\}$, $\tilde{g_2} = \{(e_1, w^2), (e_2, 1)\}$, $\tilde{g_3} = \{(e_1, w^2), (e_2, w^3)\}$, $\tilde{g_4} = \{(e_1, w^4), (e_2, 1)\}$, $\tilde{g_5} = \{(e_1, w^4), (e_2, w^3)\}$. Hence $(\eta, A) = \{\tilde{g_e}, \tilde{g_1}, \tilde{g_2}, \tilde{g_3}, \tilde{g_4}, \tilde{g_5}\}$. Therefore, we now show that $((\eta, G), \tilde{\odot})$ is a soft group with respect to the soft multiplication binary operation $\tilde{\odot}$ defined on the soft set $(\eta, A)$ using the definition 6.8. First it is explained that how to use $\tilde{\odot}$ for the soft members of $(\eta, A)$, e.g for $\tilde{g_1}, \tilde{g_2} \in (\eta, A)$, we have $\tilde{g_1}\tilde{\odot}\tilde{g_2} = \{(e_1, 1), (e_2, w^3)\}\tilde{\odot}\{(e_1, w^2), (e_2, 1)\} = \{(e_1, (1)(w^2)), (e_2, (w^3)(1))\} = \{(e_1, w^2), (e_2, w^3)\} = \tilde{g_3}$. For the other axioms of the soft group, we consider the following Cayley table.

TABLE 1. Cayley Table

| $\tilde{\odot}$ | $\tilde{g_e}$ | $\tilde{g_1}$ | $\tilde{g_2}$ | $\tilde{g_3}$ | $\tilde{g_4}$ | $\tilde{g_5}$ |
|---|---|---|---|---|---|---|
| $\tilde{g_e}$ | $\tilde{g_e}$ | $\tilde{g_1}$ | $\tilde{g_2}$ | $\tilde{g_3}$ | $\tilde{g_4}$ | $\tilde{g_5}$ |
| $\tilde{g_1}$ | $\tilde{g_1}$ | $\tilde{g_e}$ | $\tilde{g_3}$ | $\tilde{g_2}$ | $\tilde{g_5}$ | $\tilde{g_4}$ |
| $\tilde{g_2}$ | $\tilde{g_2}$ | $\tilde{g_3}$ | $\tilde{g_4}$ | $\tilde{g_5}$ | $\tilde{g_e}$ | $\tilde{g_1}$ |
| $\tilde{g_3}$ | $\tilde{g_3}$ | $\tilde{g_2}$ | $\tilde{g_5}$ | $\tilde{g_4}$ | $\tilde{g_1}$ | $\tilde{g_e}$ |
| $\tilde{g_4}$ | $\tilde{g_4}$ | $\tilde{g_5}$ | $\tilde{g_e}$ | $\tilde{g_1}$ | $\tilde{g_2}$ | $\tilde{g_3}$ |
| $\tilde{g_5}$ | $\tilde{g_5}$ | $\tilde{g_4}$ | $\tilde{g_1}$ | $\tilde{g_e}$ | $\tilde{g_3}$ | $\tilde{g_2}$ |

Observe that the soft member $\tilde{g_e}$ is the soft identity of the soft group $((\eta, A), \tilde{\odot})$, each element of $(\eta, A)$ has an inverse in $(\eta, A)$. It can also be seen from the table 1, that the operation $\tilde{\odot}$ is associative and commutative. Therefore $((\eta, A), \tilde{\odot})$ is a commutative soft group with respect to the soft binary operation $\tilde{\odot}$ on $(\eta, A)$.

6.10. **Definition:** Let $((\eta, A), \tilde{\odot})$ be a soft group with respect to the soft multiplication binary operation $\tilde{\odot}$ then for $\tilde{g} \in (\eta, A)$, the power $m \in Z_+$ of $\tilde{g}$ is defined as $(\tilde{g})^m = \tilde{g}\tilde{\odot}\tilde{g}\tilde{\odot}\tilde{g}\tilde{\odot}...\tilde{g}\tilde{\odot}\tilde{g}\quad m$ times.

6.11. **Definition:** Let $((\eta, A), \tilde{\oplus})$ be a soft group with respect to the soft addition binary operation $\tilde{\oplus}$ then for $\tilde{g} \in (\eta, A)$, the $m$ times of $\tilde{g}$ is defined as $m\tilde{g} = \tilde{g}\tilde{\oplus}\tilde{g}\tilde{\oplus}\tilde{g}\tilde{\oplus}...\tilde{g}\tilde{\oplus}\tilde{g}\quad m$ times.

6.12. **Definition.** A soft group $((\eta, A), \tilde{\odot})$ with respect to the soft multiplication binary operation $\tilde{\odot}$ is called a soft cyclic group if it is generated by some of its soft member.

6.13. **Definition:** Order of a soft member $\tilde{g}$ of a soft group $((\eta, A), \tilde{\odot})$ with respect to soft multiplication binary operation $\tilde{\odot}$ means a least positive integer $n$ such that $\tilde{g^n} = \tilde{g}_e$, where $\tilde{g}_e$ is the soft identity member of the soft group $((\eta, A), \tilde{\odot})$.

Now the concept of soft cyclic group is explained in the following example.

6.14. **Example:** Observe that the example 6.9 is a soft cyclic group of order 6. From the table 1 it is clear that the soft members $\tilde{g_3}$ and $\tilde{g_5}$ are the soft generators of the soft cyclic group $((\eta, A), \tilde{\odot})$. The order of each of these two members is 6, that is, $(\tilde{g_3})^6 = \tilde{g}_e$ and $(\tilde{g_5})^6 = \tilde{g}_e$.
Indeed, we have $(\tilde{g_5})^2 = \tilde{g_5}\tilde{\odot}\tilde{g_5} = \tilde{g_2}$, $(\tilde{g_5})^3 = \tilde{g_5}\tilde{\odot}(\tilde{g_5})^2 = \tilde{g_5}\tilde{\odot}\tilde{g_2} = \tilde{g_1}$, $(\tilde{g_5})^4 = \tilde{g_5}\tilde{\odot}(\tilde{g_5})^3 = \tilde{g_5}\tilde{\odot}\tilde{g_1} = \tilde{g_4}$, $(\tilde{g_5})^5 = \tilde{g_5}\tilde{\odot}(\tilde{g_5})^4 = \tilde{g_3}$, $(\tilde{g_5})^6 = \tilde{g_5}\tilde{\odot}(\tilde{g_5})^5 = \tilde{g_5}\tilde{\odot}\tilde{g_3} = \tilde{g}_e$. Similarly, one can also verify that $(\tilde{g_3})^6 = \tilde{g}_e$.
To see the orders of the remaining soft members of the soft group $((\eta, A), \tilde{\odot})$, we have $(\tilde{g_4})^3 = \tilde{g_4}\tilde{\odot}(\tilde{g_4})^2 = (\tilde{g_4}\tilde{\odot}(\tilde{g_4}\tilde{\odot}\tilde{g_4})) = \tilde{g_4}\tilde{\odot}\tilde{g_2} = \tilde{g}_e$. Therefore the order of the soft member $\tilde{g_4}$ is 3. Similarly, one can see that the order of the soft members $\tilde{g_1}$ and $\tilde{g_2}$ are 2 and 3 respectively.

## 7. Conclusion

In this paper, the concepts of soft point and soft member of a soft set are initiated. With the help of soft members and soft elements, the soft set operations and soft set relations are discussed and successful application of soft binary operation has been depicted in the form of Cayley table and soft group structures. All of these ideas provide a basic foundation for the development and further research in the field of soft set theory. As for as, the motivation of this novel concept is concerned, one can extend these concepts to study the structures and properties of different soft algebraic structures. Moreover, soft set theory can be extended in many areas of mathematics like topology, functional analysis, and algebra. It can also open new ways the field of decision-making problems in various practical daily life situations.

## 8. Acknowledgment

Authors are profoundly thankful to the anonymous referees for their outstanding suggestions, nice comments, and amazing directions for the completion of this paper. Without their motivations and appreciation it was hard to achieve this goal.

## REFERENCES

[1] U. Acar, F. Koyuncu and B. Tanay, *Soft sets and soft rings*, Computers and Mathematics with Applications, **59** (2010) 3458-3463.

[2] H. Aktas and N. Cagman, *Soft sets and soft groups*, Information Science, **177** (2007) 2726-2735.

[3] M. Akram and A. Adeel, *Fuzzy N-soft sets: A novel model with applications*, Journal of Intelligent and Fuzzy Systems, **35,** No. 4 (2018) 4757-4771.

[4] J. C. R. Alcantud, G. Varela and B. Santos-Buitrago, *Analysis of survival for lung cancer resections cases with fuzzy and soft set theory in surgical decision making*, Plos One. **14,** No.6 (2019) 218-283.

[5] M.I. Ali, F. Feng , X. Lui, W. K. Min and M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications, **57**(2009) 1574-1553.

[6] K. V. Babitha and J. J. Sunil, *Soft set relations and functions*, Computers and Mathematics with Applications, **60** (2010) 1840-1849.

[7] S. Das and S.K. Samanta, *Soft real sets, soft real numbers and their properties*,The Journal of Fuzzy Mathematics, **20**, No.3 (2012) 551-576.

[8] S. Das and S.K. Samanta, *On Soft metric spaces*, The journal of Fuzzy Mathematics, **6,** No.21(2013) 707-734.

[9] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings*, Computers and Mathematics with Applications, **56** (2008) 2621-2628.

[10] F. Fatimah, D. Rosadi and R. B. F. Hakim, *Probabilistic soft sets and dual probabilistic soft sets in decision-making*, Neural Computing and Applications, **31,** No.2 (2019) 397-407.

[11] S. Goldar and S. Ray, *A Study of Soft Topological Axioms and Soft Compactness by Using Soft Elements*, Journal of New Results in Science, **8,** No.2(2019)53-66.

[12] P. K. Maji and A. R. Roy, *An application of soft sets in a decision making problem*, Computers and Mathematics with Applications, **44** (2002) 1077-1083

[13] D. Molodtsov, *Soft set theory first results*, Computers and Mathematics with Applications, **37** (1999) 19-31.

[14] P. K. Maji, R. Biswas and R. Roy, *Soft set theory*, Comput. Computers and Mathematics with Applications, **45** (2003) 555-562.

[15] S. Ray and S. Goldar, *Soft Set and Soft Group from Classical View Point*, The Journal of the Indian Mathematical Society, 84 (2017) 273-286.

[16] M. Riaz and T.T. Syeda, *Certain Properties of Bipolar Fuzzy Soft Topology Via Q-Neighborhood*, Punjab Uni.j.math.**51,** No.3 (2019) 113-131.

[17] M. Riaz, K. Naeem and M.O. Ahmad, *Novel Concepts of Soft Sets with Applications*, Annals of Fuzzy Mathematics and Informatics, **13,** No.2 (2017) 239-251.

[18] M. Riaz and K. Naeem, *Measurable Soft Mappings*, Punjab Uni. j. math. **48,** No.2 (2016) 19-34.

[19] M. Riaz, M. Saqlain and M. Saeed. (2019). Application of generalized fuzzy TOPSIS in decision making for Neutrosophic Soft set to Predict the Champion of FIFA 2018: A Mathematical Analysis, Punjab Uni.j.math.**51,** No.8 (2019) 141-156

[20] M. Riaz and Z. Fatima, *Certain properties of soft metric spaces*, Journal of Fuzzy Mathematics, **25,** No.3(2017) 543-560.

[21] M. Riaz and M. R. Hashmi, *Fuzzy parameterized fuzzy soft topology with applications*, Annals of Fuzzy Mathematics and Informatics, **13,** No.5 (2017) 593-613.

[22] M. Riaz and M. R. Hashmi, *Certain applications of fuzzy parameterized fuzzy soft sets in decision-making problems*, International Journal of Algebra and Statistics, **5,** No.2(2016) 135-146.

[23] M. Riaz and M. R. Hashmi, *Fuzzy parameterized fuzzy soft compact spaces with decision-making*, Punjab Uni. j. math. **50,** No.2 (2018) 131-145.

[24] M. Saqlain, M. Saeed, M. Raees and F. Smarandache, *Generalization of TOPSIS for neutrosophic hypersoft set using accuracy function and its application*, Neutrosophic Sets and Systems, **27** (2019) 131-137.

[25] M. Saeed, M. R. Ahmad, M. Saqlain and M. Riaz, *Rudiments N-framed soft sets*, Punjab Uni.j.math. **52,** No.5 (2020) 15-30.

[26] B. Santos-Buitrago, A. Riesco and M. Knapp, *Soft set theory for decision making in computational biology under incomplete information*, IEEE Access, **7,** No. 2 (2019) 18183-18193.
[27] A. Sezgin and A.O. Atagun, *On operations of soft sets*, Computers and mathematics with applications, **60**(2011) 1840-1849.
[28] Y. Zou, Z. Xiao and Hashmi, *Data analysis approaches of soft sets under incomplete information.*, Knowledge-Based Systems, **21,** No.8 (2008) 941-945.
[29] L.A. Zadeh *Fuzzy sets*, Information and Control, **8** (1965) 338-353.