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Solution of 2D Volterra-Fredholm Integral Equations of First Kind by using Discretization Method

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Abstract.: This paper is devoted to present a simple but efficient numerical method for solving the 2D Volterra-Fredholm integral equation (VFIE). Both mixed and separate types of VFIEs are considered. The under consideration technique rely on approximating the unknown function with 2D Bernstein polynomials. The proposed technique has advantage in reducing the computational burden. To show the accuracy and convergence of this method, some results are also established. Finally the effectiveness of the technique is demonstrated by applying the technique on some numerical tests.

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1. Introduction

Volterra and Fredholm integral equations are well-known integral equations that arise in various fields of science such as population dynamics, kinetic, theory of gases, radiations and communication theory. The major investigator of the theory related to these integral equations is Vita Volterra (1860-1940) and Ivar Fredholm (1866-1927). The VFIE of the first kind is of the form

$$\begin{aligned} f(v,w) &= \lambda_1 \int_a^w \int_c^v k_1(y,v,z,w)u(y,z)dydz \\ &\quad +\lambda_2 \int_a^b \int_c^d k_2(y,v,z,w)u(y,z)dydz. \qquad (1.\ 1) \end{aligned}$$

The mixed VFIE of the first kind is of the form

$$f(v,w) = \lambda \int_a^w \int_b^c k(y,v,z,w)u(y,z)dydz. \tag{1. 2}$$

Where a, b, c, and d are constants and λ, λ_1 and λ_2 are constant parameters. The function $f(v,w)$, the kernels $k(y,v,z,w)$, $k_1(y,v,z,w)$ and $k_2(y,v,z,w)$ are known functions and $u(y,z)$ is the unknown function to be determined. There is a growing interest in finding the solution of two-dimensional integral equations of first and second kind in recent years. Most of the work has been done on second kind of integral equations. Alipour and Rostamy used Bernstein polynomials for obtaining the solution of Abels integral equation [3]. Maleknejad and Hashminzadeh used Bernstein operational matrix approach for solving the system of higher order linear Volterra-Fredholm integro-differential equations [1]. Khan et al. presented the discretization method for solving 2-dimensional Volterra integral equation, which is based on $2D$ Bernstein polynomial [6]. Now a day, the researchers are trying to find the solution of $2D$ integral equations by using some approximation techniques. In 2017, Khan et al. found numerical solution of mixed Volterra-Fredholm integral equations based on Bernstein basis functions [5] for one dimensional. Hetmaniok et al. used Homotopy technique for solving $2D$ integral equations of the second kind [2]. In 2013, Jafaraian et al. used collocation Bernstein polynomial expansion method to solve linear second kind system of Fredholm and Volterra integral equations [4]. C. Heitzinger et al. obtained an algorithm for the smoothing three-dimensional monte carlo ion implantation simulation results [7]. In this paper the Volterra-Fredholm integral equation of first kind is solved numerically by discretization method based on Bernstein basis function. Here, both separate and mixed forms of VFIE are considered. The unknown function is replaced by Bernstein basis function to obtain the approximate solution. The presented technique can give more accurate results if the degree of the Bernstein polynomial is increased. Even at the smallest values of 2D Bernstein's degree the proposed technique give appropriate accuracy which results in the low computational cost. The five sections of this paper are managed as follows. In section 2 the basic concepts are discussed. Section 3, deals with numerical technique based on 2D Bernstein basis functions. In Section 4, we have provided some results about convergence analysis. In the last section some numerical problems are carried out. All the computations are performed using MATLAB.

2. Basic Concepts

Definition 2.1. *(Multivariate Bernstein Polynomials) Let $p_1, p_2, \ldots, p_r \in N$ and h is a function of r variables. Then the multivariate Bernstein polynomials of h is defined as under*

$$B_{h,p_1,\ldots,p_r}(x_1,\ldots,x_r) := \sum_{\substack{0\leqslant k_j \leqslant r_j \\ j\in\{1,\ldots,r\}}} h\left(\frac{k_1}{p_1},\ldots,\frac{k_r}{p_r}\right) \prod_{j=1}^{r} \left(\binom{p_j}{k_j} x_j^{k_j}(1-x_j)^{p_j-k_j}\right).$$

If a function $h : [a,b]\times[c,d] \to R$ is approximated by the Bernstein polynomial then it is defined as

$$B_{m_1,n_1}(h(y,z)) = \sum_{i=0}^{m_1}\sum_{j=0}^{n_1} B_{(i,m_1),(j,n_1)}(y,z) h\left(a + i\frac{b-a}{m_1}, c + j\frac{d-c}{n_1}\right), \tag{2. 3}$$

where

$$B_{(i,m_1),(j,n_1)}(y,z) = \binom{m_1}{i}\binom{n_1}{j}\frac{(y-a)^i(b-y)^{m_1-i}(z-c)^j(d-z)^{n_1-j}}{(b-a)^{m_1}(d-c)^{n_1}},$$

for $i = 0,\ldots,m_1; j = 0,\ldots,n_1$, which is known as the 2D Bernstein polynomial basis with $y \in [a,b]$ and $z \in [c,d]$. Here m_1, n_1 are arbitrary whole numbers. For the properties of 2D Bernstein polynomial see [5]. The asymptotic formula of the Bernstein polynomial approximation is given in the following Theorem.

Theorem 2.2. *(Asymptotic Formula) Let $h : [e_1, f_1] \times [e_2, f_2] \times \ldots \times [e_r, f_r] \to R$ be a C^2 function and $Y \in [e_1, f_1] \times [e_2, f_2] \times \ldots \times [e_r, f_r]$ then*

$$\begin{aligned}\lim_{m_1\to\infty} m_1((B_{n_1,n_2,\ldots,n_m}h(Y)) - h(Y)) &= \sum_{i=1}^{r}\frac{(x_i-e_i)(f_i-x_i)}{2}\frac{\partial^2 h(Y)}{\partial x_i^2}\\ &\le \frac{1}{8}\sum_{i=1}^{r}((e_i-f_i)^2)\frac{\partial^2 h(Y)}{\partial x_i^2}.\end{aligned}$$

Proof. See [7]. □

Theorem 2.3. *(Uniform Convergence) Let $h : [e_1, f_1] \times [e_2, f_2] \times \ldots \times [e_r, f_r] \to R$ be a continuous function and $X \in [e_1, f_1] \times [e_2, f_2] \times \ldots \times [e_r, f_r]$. Then the multivariate Bernstein polynomials $B_{n_1,n_2,\ldots,n_m}f(X)$ converge uniformly to h for $n_1, n_2, \ldots, n_m$.*

Proof. See [7]. □

In above stated asymptotic formula if we take $h : [a,b] \times [c,d] \to R$ be a C^2 function and $(y,z) \in [a,b] \times [c,d]$ then it becomes:

$$\lim_{m\to\infty} m((B_{m,n}h)(y,z) - h(y,z)) \le \frac{(b-a)^2}{8}\frac{\partial^2 h(y,z)}{\partial y^2} + \frac{(d-c)^2}{8}\frac{\partial^2 h(y,z)}{\partial z^2}. \quad (2.\ 4)$$

3. PROPOSED NUMERICAL METHOD FOR 2-DIMENSIONAL VFIE'S

In this section the proposed technique is applied on both the separate and mixed type of 2D Volterra-Fredholm integral equations.

3.1. Separate kind of Volterra-Fredholm integral equation.

Consider the 2D Volterra-Fredholm integral equation of first kind defined as

$$\begin{aligned}f(v,w) &= \lambda_1\int_a^w\int_c^v k_1(y,v,z,w)u(y,z)dydz\\ &\quad+\lambda_2\int_a^b\int_c^d k_2(y,v,z,w)u(y,z)dydz,\ v\in[a,b] \text{ and } w\in[c,d].\end{aligned} \quad (3.\ 5)$$

Where $f(v,w)$ and $k_2(y,v,z,w)$, $k_1(y,v,z,w)$ are analytic functions and $u(v,w)$ is the unknown function.
In order to obtain the numerical solution of above mentioned integral equation, replace the

unknown function by $u(y,z)$ and $u(v,w)$ by Bernstein plonomials given in (2.3).
So the equation will be

$$
\begin{aligned}
f(v,w) &= \sum_{i=0}^{m_1}\sum_{j=0}^{n_1} u\left(a+i\frac{b-a}{m_1}, c+j\frac{d-c}{n_1}\right)\alpha_{ij}\left[\lambda_1\int_a^w\int_c^v k_1(y,v,z,w)\beta_{ij}(y,z)dydz\right.\\
&\left.-\lambda_2\int_a^b\int_c^d k_2(y,v,z,w)\beta_{ij}(y,z)dydz\right], \qquad (3.\ 6)
\end{aligned}
$$

where $\alpha_{ij} = \frac{\binom{m_1}{i}\binom{n_1}{j}}{(b-a)^{m_1}(d-c)^{n_1}}$ and $\beta_{ij}(v,w) = (v-a)^i(b-v)^{m_1-i}(c-w)^j(d-w)^{n_1-j}$.
To obtain the values of $u(a+i\frac{b-a}{m_1}, a+j\frac{b-a}{n_1}), i = 0,\ldots,m_1, j = 0,\ldots,n_1$, the above equation in the form of a linear system of equations is written by substituting v as $v_p = a+(b-a)\frac{p}{m_1}+\epsilon$, $p = 0,\ldots,m_1-1$, $v_{m_1} = b-\epsilon$, w as $w_q = c+(d-c)\frac{q}{n_1}+\epsilon$, $q = 0,\ldots,n_1-1$ and $w_{n_1} = d-\epsilon$, where ϵ is a very small positive number.
The above system of linear equations can be written in matrix form as $SX = T$, where the matrices S, X and T are given as under:

$$
\begin{aligned}
S &= \alpha_{ij}\left[\lambda_1\int_a^w\int_b^v k_1(y,v,z,w)\beta_{ij}(y,z)dydz\right.\\
&\left.-\lambda_2\int_a^b\int_c^d k_2(y,v,z,w)\beta_{ij}(y,z)dydz\right], \quad i = 0,\ldots,m_1, j = 0,\ldots,n_1,
\end{aligned}
\qquad (3.\ 7)
$$

$$X = \left[u\left(a+i\frac{b-a}{m_1}, c+j\frac{d-c}{n_1}\right)\right]^t,$$

$$T = [f(v_p,w_q)]^t, \quad p = 0,\ldots,m_1, \quad q = 0,\ldots,n_1,$$

where $u\left(a+i\frac{b-a}{m_1}, c+j\frac{d-c}{n_1}\right), i = 0,\ldots,m_1, j = 0,\ldots,n_1$ can be written as $u_{m_1,n_1}\left(a+i\frac{b-a}{m_1}, c+j\frac{d-c}{n_1}\right)$, $i = 0,\ldots,m_1, j = 0,\ldots,n_1$ which are our solutions in nodes $\left(a+i\frac{b-a}{m_1}, c+j\frac{d-c}{n_1}\right), i = 0,\ldots,m_1, j = 0,\ldots,n_1$ and by using it in equation (2. 3), we have $B_{m_1,n_1}(f_{m_1,n_1}; v_p, w_q)$, $p = 0,\ldots,m_1, q = 0,\ldots,n_1$ which is the solution of 2D VFIE given in (3. 5).

3.2. **Mixed kind of Volterra-Fredholm integral equation.** Same technique can be applied on mixed VFIE of first kind, as given below. Consider the mixed VFIE given in (1. 2)

$$f(v,w) = \lambda\int_a^w\int_b^c k(y,v,z,w)u(y,z)dydz, \quad v\in[a,b] \quad and \quad w\in[c,d], \qquad (3.\ 8)$$

where $f(v,w)$ and $k(y,v,z,w)$ are analytic functions and $u(y,z)$ is unknown function to be determined. To obtain the solution, replace the unknown function by Bernstein Polynomials given in (2.3).

$$f(v,w)=\sum_{i=0}^{m_1}\sum_{j=0}^{n_1}u\left(a+i\frac{b-a}{m_1},c+j\frac{d-c}{n_1}\right)\alpha_{ij}\left[\lambda\int_a^w\int_b^c k_1(y,v,z,w)\beta_{ij}(y,z)dydz\right], \tag{3. 9}$$

where $\alpha_{ij}=\frac{\binom{m_1}{i}\binom{n_1}{j}}{(b-a)^{m_1}(d-c)^{n_1}}$ and $\beta_{ij}(v,w)=(v-a)^i(b-v)^{m_1-i}(c-w)^j(d-w)^{n_1-j}$. To obtain the values of $u\left(a+i\frac{b-a}{m_1},c+j\frac{d-c}{n_1}\right), i=0,\ldots,m_1$ and $j=0,\ldots,n_1$, the above equation can be written in the form of linear system of equations by substituting v as $v_p=a+(b-a)\frac{p}{m_1}+\epsilon$, $p=0,\ldots,m_1-1$, $v_{m_1}=b-\epsilon$, w as $w_q=c+(d-c)\frac{q}{n_1}+\epsilon$, $q=0,\ldots,n_1-1$ and $w_{n_1}=d-\epsilon$, where ϵ is a very small positive number.
The above system of linear equations can be written in matrix form as $CX=D$, where the matrices C, X and D are given as:

$$C=\alpha_{ij}\left[\lambda\int_a^w\int_b^c k_1(y,v,z,w)\beta_{ij}(y,z)dydz\right],\ i=0,\ldots,m_1,j=0,\ldots,n_1, \tag{3. 10}$$

$$\begin{aligned} X &= \left[u\left(a+i\frac{b-a}{m_1},c+j\frac{d-c}{n_1}\right)\right]^t, \\ D &= [f(v_p,w_q)]^t, \quad p=0,\ldots,m_1, \quad q=0,\ldots,n_1, \end{aligned}$$

where $u\left(a+i\frac{b-a}{m_1},c+j\frac{d-c}{n_1}\right)$, $i=0,\ldots,m_1,j=0,\ldots,n_1$ can be written as $u_{m_1,n_1}\left(a+i\frac{b-a}{m_1},c+j\frac{d-c}{n_1}\right)$, $i=0,\ldots,m_1,j=0,\ldots,n_1$ that are our outcomes in nodes $\left(a+i\frac{b-a}{m_1},c+j\frac{d-c}{n_1}\right)$, $i=0,\ldots,m_1,j=0,\ldots,n_1$ and by substituting it in equation (2. 3), we obtained $B_{m_1,n_1}(f_{m_1,n_1};v_p,w_q)$, $p=0,\ldots,m_1,q=0,\ldots,n_1$ that is the solution of 2D mixed VFIE (3. 8).

3.3. **Convergence Analysis.**

Theorem 3.4. *Consider 2DVFIE of first kind defined in (3. 5). Suppose the function $u(v,w)$ is continuous on $[a,b]^2$ and the kernel $k_1(y,v,z,w)$ and $k_2(y,v,z,w)$ are analytic on the square $[a,b]^2\times[c,d]^2$. If the matrix S given in (3. 7) has an inverse then*

$$\begin{aligned} &\sup_{v_p\epsilon[a,b],w_q\epsilon[c,d]}|u(v_p,w_q)-B_{m_1,n_1}(u_{m_1,n_1}(v_p,w_q))| \\ &\leq 1+[1+(E)(b-a)(d-c)]\|S^{-1}\|\left[\frac{(b-a)^2}{8m_1}\|u_{vv}\|+\frac{(d-c)^2}{8n_1}\|u_{ww}\|\right], \end{aligned} \tag{3. 11}$$

where $v_p = a + p\frac{b-a}{m_1}$, $w_q = c + q\frac{d-c}{n_1}$, $p = 0, \ldots, m_1$; $q = 0, \ldots, n_1$, $E = L + M$ *and* $u(v, w)$ *is the actual solution. Here*

$$L = \sup_{y,v\epsilon[a,b],z,w\epsilon[c,d]} |\lambda_1 k_1(y, v, z, w)|, \qquad M = \sup_{y,v\epsilon[a,b],z,w\epsilon[c,d]} |\lambda_2 k_2(y, v, z, w)| \quad and$$

$B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q)$ *is proposed method solution.*

Proof. Consider

$$\begin{aligned} & \sup_{v_p\epsilon[a,b],w_q\epsilon[c,d]} |u(v_p, w_q) - B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q))| \\ = & \sup_{v_p\epsilon[a,b],w_q\epsilon[c,d]} |u(v_p, w_q) - B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q)) \\ & + u_{m_1,n_1}(v_p, w_q) - u_{m_1,n_1}(v_p, w_q)| \end{aligned} \tag{3. 12}$$

$$\begin{aligned} \leq & \sup_{v_p\epsilon[a,b],w_q\epsilon[c,d]} |u(v_p, w_q) - u_{m_1,n_1}(v_p, w_q)| \\ + & \sup_{v_p\epsilon[a,b],w_q\epsilon[c,d]} |u_{m_1,n_1}(v_p, w_q) - B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q))|. \end{aligned} \tag{3. 13}$$

Since the inequality

$$\|B_{m_1,n_1}(u(v, w)) - u(v, w)| \leq \frac{(v-a)(b-v)}{2m_1}\|u_{vv}\| + \frac{(w-c)(d-w)}{2n_1}\|u_{ww}\| \tag{3. 14}$$

gives the following bound,

$$\begin{aligned} & \sup_{v_p\epsilon[a,b],w_q\epsilon[c,d]} |u_{m_1,n_1}(v_p, w_q) - B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q))| \\ & \leq \frac{(b-a)^2}{8m_1}\|u_{vv}\| + \frac{(d-c)^2}{8n_1}\|u_{ww}\|. \end{aligned} \tag{3. 15}$$

To complete the proof it is left to find the following bound

$$\sup_{v_p\epsilon[a,b],w_q\epsilon[c,d]} |(u(v_p, w_q)) - (u_{m_1,n_1}(v_p, w_q))|.$$

Since, we have

$$\begin{aligned} f(v, w) &= \lambda_1 \int_a^w \int_c^v k_1(y, v, z, w) B_{m_1,n_1}(u(y, z)) dydz \\ &\quad + \lambda_2 \int_a^b \int_c^d k_2(y, v, z, w) B_{m_1,n_1}(u(y, z)) dydz. \end{aligned} \tag{3. 16}$$

If we replace $u(v, w)$ with $u_{m_1,n_1}(v, w)$ then $f(v, w)$ will become $\hat{f}(v, w)$ as

$$\begin{aligned} \hat{f}(v, w) &= \lambda_1 \int_a^w \int_c^v k_1(x, v, y, w) B_{m_1,n_1}(u_{m_1,n_1}(y, z)) dydz \\ &\quad + \lambda_2 \int_a^b \int_c^d k_2(y, v, z, w) B_{m_1,n_1}(u_{m_1,n_1}(y, z)) dydz. \end{aligned} \tag{3. 17}$$

If we replace v by v_p and w by w_q then

$$f(v_p, w_q) = u(v_p, w_q)S \tag{3. 18}$$

and

$$\hat{f}(v_p, w_q) = u_{m_1,n_1}(v_p, w_q)S, \tag{3. 19}$$

where S is the matrix defined in (3. 7). This implies

$$u(v_p, w_q) - u_{m_1,n_1}(v_p, w_q) = S^{-1}[f(v_p, w_q) - \hat{f}(v_p, w_q)]. \tag{3. 20}$$

From (3. 20) we have

$$\sup_{v_p \in [a,b], w_q \in [c,d]} |(u(v_p, w_q) - u_{m_1,n_1}(v_p, w_q))| \leq \|S^{-1}\| \max |f(v_p, w_q) - \hat{f}(v_p, w_q)|. \tag{3. 21}$$

Now we find a bound for

$$\max |f(v_p, w_q) - \hat{f}(v_p, w_q)|. \tag{3. 22}$$

Let we take

$$\begin{aligned} f(v,w) &= \lambda_1 \int_a^w \int_c^v k_1(y,v,z,w)u(y,z)dydz \\ &\quad +\lambda_2 \int_a^b \int_c^d k_2(y,v,z,w)u(y,z)dydz \end{aligned} \tag{3. 23}$$

and

$$\begin{aligned} \hat{f}(v,w) &= \lambda_1 \int_a^w \int_c^v k_1(y,v,z,w)B_{m_1,n_1}(u_{m_1,n_1}(y,z))dydz \\ &\quad +\lambda_2 \int_a^b \int_c^d k_2(y,v,z,w)B_{m_1,n_1}(u_{m_1,n_1}(y,z))dydz. \end{aligned} \tag{3. 24}$$

From (3. 23) and (3. 24)

$$\begin{aligned} f(v,w) - \hat{f}(v,w) &= \lambda_1 \int_a^w \int_c^v k_1(y,v,z,w)(u(y,z) - B_{m_1,n_1}(u_{m_1,n_1}(y,z)))dydz \\ &+\lambda_2 \int_a^b \int_c^d k_2(y,v,z,w)(u(y,z) - B_{m_1,n_1}(u_{m_1,n_1}(y,z)))dydz. \end{aligned} \tag{3. 25}$$

Which implies

$$\begin{aligned} &\sup |f(v,w) - \hat{f}(v,w)| \leq + \sup |\lambda_1 k_1(y,v,z,w)(u(v,w) - B_{m_1,n_1}(u_{m_1,n_1}(v,w)))| \\ &+ \sup |\lambda_2 k_2(y,v,z,w)(u(v,w) - B_{m_1,n_1}(u(v,w)))|. \end{aligned} \tag{3. 26}$$

Suppose

$$\sup_{v \in [a,b], w \in [c,d]} |\lambda_1(k_1(y,v,z,w))| = L \tag{3. 27}$$

and

$$\sup_{v \in [a,b], w \in [c,d]} |\lambda_2(k_2(y,v,z,w))| = M, \tag{3. 28}$$

Then

$$\sup |f(v,w) - \hat{f}(v,w)| \quad \leq \quad \sup |u(v,w) - B_{m_1,n_1}(u(v,w))(1+E)|, \tag{3. 29}$$

where $E = L + M$. Moreover

$$\sup_{v_p \epsilon [a,b], w_q \epsilon [c,d]} |f(v_p, w_q) - \hat{f}_{m_1,n_1}(v_p, w_q)| \leq \left(\frac{(b-a)^2}{8m_1} \|u_{vv}\| + \frac{(d-c)^2}{8n_1} \|u_{ww}\| \right) (1+E). \tag{3. 30}$$

By putting this in (3. 21), we get

$$\sup_{v_p \epsilon [a,b], w_q \epsilon [c,d]} |u(v_p, w_q) - u_{m_1,n_1}(v_p, w_q)| \leq \left(\frac{(b-a)^2}{8m_1} \|u_{vv}\| + \frac{(d-c)^2}{8n_1} \|u_{ww}\| \right) \|S^{-1}\|(1+E). \tag{3. 31}$$

So by using (3. 13), (3. 15) and (3. 31)

$$\sup_{v_p \epsilon [a,b], w_q \epsilon [c,d]} |u(v_p, w_q) - B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q))| \leq (1 + (1+E))(b-a)(d-c)) \left[\frac{(b-a)^2}{8m_1} \|u_{vv}\| + \frac{(d-c)^2}{8n_1} \|u_{ww}\| \right] \|S^{-1}\|.$$

So the proof is completed. □

Theorem 3.5. *Consider two-dimensional mixed VFIE of first kind given in (3. 8). Suppose the kernel $k(y, v, z, w)$ is continuous on the square $[a, b]^2 \times [c, d]^2$ and $u(v, w)$ is continuous on $[a, b]^2$. If the C matrix defined in (3. 10) has an inverse then*

$$\sup_{v_p \epsilon [a,b], w_q \epsilon [c,d]} |u(v_p, w_q) - B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q))| \leq 1 + [1 + L(b-a)(d-c)]\|C^{-1}\| \left[\frac{(b-a)^2}{8m_1} \|u_{vv}\| + \frac{(d-c)^2}{8n_1} \|u_{ww}\| \right], \tag{3. 32}$$

where $v_p = a + p\frac{b-a}{m_1}$, $w_q = c + q\frac{d-c}{n_1}$, $p = 0, \ldots, m_1$, $q = 0, \ldots, n_1$ and $u(v, w)$ is exact solution.
Here $L = \sup_{y,v \epsilon [a,b], z,w \epsilon [c,d]} |\lambda k(y, v, z, w)|$ and $B_{m,n}(u_{m,n}(v_p, w_q)$ is solution by given method.

Proof. We have the relations in (3. 13) and (3. 15). So to complete the proof it is just remaining to obtain a bound for

$$\sup_{v_p \epsilon [a,b], w_q \epsilon [c,d]} |u(v_p, w_q) - (u_{m_1,n_1}(v_p, w_q))|. \tag{3. 33}$$

Consider

$$f(v, w) = \lambda \int_a^w \int_a^b k(y, v, z, w) B_{m_1,n_1}(u; y, z) dydz, \ v \in [a, b] \text{ and } w \in [c, d]. \tag{3. 34}$$

If we replace $u(v, w)$ with $u_{m_1,n_1}(v, w)$ then $\hat{f}(v, w)$ is defined as

$$\hat{f}(v, w) = \lambda \int_a^w \int_a^b k(y, v, z, w) B_{m_1,n_1}(u_{m_1,n_1}; y, z) dydz. \tag{3. 35}$$

Now if replace v and w with v_p and w_q respectively, then

$$f(v_p, w_q) = u(v_p, w_q)C \tag{3. 36}$$

and

$$\hat{f}(v_p, w_q) = u_{m_1,n_1}(v_p, w_q)C, \tag{3. 37}$$

where C is the matrix defined in (3. 10). So that

$$\sup_{v_p \epsilon [a,b], w_q \epsilon [c,d]} |u(v_p, w_q) - u_{m_1,n_1}(v_p, w_q)| \leq \|C^{-1}\| \max |f(v_p, w_q) - \hat{f}(v_p, w_q). \tag{3. 38}$$

Next we need to obtain $\max |f(v_p, w_q) - \hat{f}(v_p, w_q)$. Here

$$f(v, w) = \lambda \int_a^w \int_a^b k(y, v, z, w)u(y, z)dydz, \tag{3. 39}$$

and

$$\hat{f}(v, w) = \lambda \int_a^w \int_a^b k(y, v, z, w)B_{m_1,n_1}(u; v, w)dydz. \tag{3. 40}$$

.

Now consider

$$\begin{aligned} &\sup |f(v, w) - \hat{f}(v, w)| \\ &\leq \sup |\lambda \int_a^w \int_a^b k(y, v, z, w)[u(y, z) - B_{m_1,n_1}(u; y, z)]dydz|. \end{aligned} \tag{3. 41}$$

Which implies

$$\sup |f(v, w) - \hat{f}(v, w)| \leq \sup |\lambda k(y, v, z, w)[u(v, w) - B_{m_1,n_1}(u; v, w)]|. \tag{3. 42}$$

It further gives

$$\sup |f(v, w) - \hat{f}(v, w)| \leq \frac{(b - a)^2}{8m_1} + \frac{(d - c)^2}{8n_1}(\|u_{vv}\| + \|u_{ww}\|)(1 + L). \tag{3. 43}$$

Moreover

$$\sup |u(v_p, w_q) - u_{m_1,n_1}(v_p, w_q)| \leq \frac{(b - a)^2}{8m_1} + \frac{(d - c)^2}{8n_1}(\|u_{vv}\| + \|u_{ww}\|)(1 + L)\|C^{-1}\|. \tag{3. 44}$$

Finally

$$\begin{aligned} &\sup_{v_p \epsilon [a,b], w_q \epsilon [c,d]} |u(v_p, w_q) - B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q))| \\ \leq \quad & 1 + [1 + L(b - a)(d - c)]\|C^{-1}\| \left[\frac{(b - a)^2}{8m_1}\|u_{vv}\| + \frac{(d - c)^2}{8n_1}\|u_{ww}\| \right]. \end{aligned}$$

This complete the proof. □

Lemma 3.6. *Let's suppose that $\|S - I\| = \delta < 1$, where I is the identity matrix of order $(m_1 + 1) \times (n_1 + 1)$ and $\|\cdot\|$ is the maximum norm of rows. Then*

$$\|S^{-1}\| \leq \frac{1}{1-\delta}, \quad \mathit{Cond}(S) \leq \frac{1 + \gamma_1(b-a)(d-c) + \gamma_2(b-a)(d-c)}{1-\delta},$$

where

$$\max_{p,q} |\lambda_1 k_1(v_p, y, w_q, z)| = \gamma_1, \max_{p,q} |\lambda_2 k_2(v_p, y, w_q, z)| = \gamma_2,$$
$$\max_{p,q} |\sum_{i=0}^{m_1} \sum_{j=0}^{n_1} \alpha_{ij} \beta_{ij}(v_p, w_q)| = 1.$$

Proof. Firstly we established a bound for $\|S\|$. Where

$$\begin{aligned} \|S\| &= \max_{p,q} \left| \sum_{i=0}^{m_1} \sum_{j=0}^{n_1} \alpha_{ij} \left[\int_a^{w_q} \int_c^{v_p} k_1(v_p, y, w_q, z) \beta_{ij}(y, z) dy dz \right.\right. \\ & \left.\left. + \int_a^b \int_c^d k_2(v_p, y, w_q, z) dy dz \right] \right|. \end{aligned} \tag{3. 45}$$

Since the sum of Bernstein basis function is equal to one so that we obtain

$$\begin{aligned} & \|S\| = 1 + \max_{p,q} |\lambda_1 \int_a^{w_q} \int_c^{v_p} k_1(v_p, y, w_q, z) dy dz| \\ & + \max_{p,q} |\lambda_2 \int_a^b \int_c^d k_2(v_p, y, w_q, z) dy dz|. \end{aligned} \tag{3. 46}$$

This implies that

$$\|S\| \leq [1 + \gamma_1(b-a)(d-c) + \gamma_2(b-a)(d-c)].$$

Now to obtain a bound for $\|S^{-1}\|$, consider $H = S - I$, so that $\|H\| = \|S - I\| = \delta < 1$.
As $S = I + H$ and $S^{-1} = (I + H)^{-1}$ implies $\|S^{-1}\| = \|(I + H)^{-1}\|$.
Further by using the geometric series formula of infinite sum, we get

$$\|S^{-1}\| \leq \frac{1}{1 - \|H\|} \leq \frac{1}{1-\delta}.$$

So, the condition number is given as

$$\text{Cond}(S) = \|S\|.\|S^{-1}\| \leq \frac{1 + \gamma_1(b-a)(d-c) + \gamma_2(b-a)(d-c)}{1-\delta}.$$

□

Lemma 3.7. *Let's suppose that $\|C - I\| = \xi < 1$, where I is the identity matrix of order $(m_1 + 1) \times (n_1 + 1)$ and $\|\cdot\|$ is the maximum norm of rows. Then*

$$\|C^{-1}\| \leq \frac{1}{1-\xi}, \quad \mathit{Cond}(C) \leq \frac{1 + \gamma(b-a)(d-c)}{1-\xi},$$

where

$$\max_{p,q} |\lambda k(v_p, y, w_q, z)| = \gamma, \quad and \quad \max_{p,q} \left| \sum_{i=0}^{m_1} \sum_{j=0}^{n_1} \alpha_{ij} \beta_{ij}(v_p, w_q) \right| = 1.$$

Proof. Firstly we established a bound for $\|C\|$. Where

$$\|C\| = \max_{p,q} \left| \sum_{i=0}^{m_1} \sum_{j=0}^{n_1} \alpha_{ij} \left[\int_a^{w_q} \int_b^{c} k(v_p, y, w_q, z) \beta_{ij}(y, z) dydz \right] \right|. \tag{3. 47}$$

Since the sum of Bernstein basis function is equal to 1 so that we obtain

$$\|C\| = 1 + \max_{p,q} |\lambda \int_a^{w_q} \int_b^{c} k(v_p, y, w_q, z) dydz|. \tag{3. 48}$$

This implies that

$$\|C\| \leq [1 + \gamma(b-a)(d-c)].$$

Next we need to obtain a bound for $\|C^{-1}\|$. Consider $A = C - I$, so that $\|A\| = \|C - I\| = \xi < 1$. As $C = I + A$ and $C^{-1} = (I + A)^{-1}$ implies $\|C^{-1}\| = \|(I + A)^{-1}\|$.
Further by using the geometric series formula of infinite sum, we get

$$\|C^{-1}\| \leq \frac{1}{1 - \|A\|} \leq \frac{1}{1 - \xi}.$$

So, the condition number is given as

$$\text{Cond}(C) = \|C\|.\|C^{-1}\| \leq \frac{1 + \gamma(b-a)(d-c)}{1 - \xi}.$$

□

4. ERROR EVALUATION

Here we present the following result for error evaluation of the proposed technique. This result will examine the accuracy and efficiency of the technique for the solution of both mixed and separate types of two dimensional VFIEs.

Proposition 1:

Suppose that $u(v, w)$ is the exact solution of integral equation and $B_{m_1,n_1}(u_{m_1,n_1}(v, w))$ is the numerical solution based on Bernstein polynomial of degree m_1 and n_1. The absolute error between exact and approximate solution is given as

$$|e_{m_1,n_1}(v_p, w_q)| = |u(v_p, w_q) - B_{m_1,n_1}(u_{m_1,n_1}(v_p, w_q))|, \tag{4. 49}$$

where $v_p = a + p\frac{b-a}{m_1}$, $p = 0, \ldots, m_1$ and $w_q = c + q\frac{d-c}{n_1}$, $q = 0, \ldots, n_1$.

TABLE 1. **Numerical results of problem no. 1**

(v,w)	Exact Solution	Approximate Solution ($m_1 = n_1 = 2$)	Approximate Solution ($m_1 = n_1 = 6$)	Absolute Error ($m_1 = n_1 = 2$)	Absolute Error ($m_1 = n_1 = 6$)
(0.0,0.0)	2	2.0000095439	2.0000000005	0.0000095439	$5.15173E^{-10}$
(0.1,0.1)	2.1052000000	2.0979714866	2.1051708346	0.0071994294	$8.34611E^{-08}$
(0.2,0.2)	2.2214000000	2.2109503334	2.2214027706	$1.04524E^{-02}$	$1.24705E^{-08}$
(0.3,0.3)	2.3499000000	2.3389460785	2.3498588078	$1.09127E^{-02}$	$2.35106E^{-10}$
(0.4,0.4)	2.4918000000	2.4819587238	2.4918246620	$9.86597E^{-03}$	$3.56068E^{-08}$
(0.5,0.5)	0.6487000000	2.6399882694	2.6487212380	$8.73300E^{-03}$	$3.2686E^{-08}$
(0.6,0.6)	2.8221000000	2.8130347151	2.8221187693	$9.08408E^{-03}$	$3.09941E^{-08}$
(0.7,0.7)	3.0138000000	3.0010980611	3.0137526268	$1.26546E^{-02}$	$8.06588E^{-08}$
(0.8,0.8)	3.225500000	3.2041783074	3.2255407938	$2.13626E^{-02}$	$1.34653E^{-07}$
(0.9,0.9)	3.4596000000	3.4222754538	3.4596030109	$3.73276E^{-02}$	$1.00167E^{-07}$
(1.0,1.0)	3.7183000000	3.6553895006	3.7182815876	$6.28923E^{-02}$	$2.40824E^{-07}$

5. NUMERICAL APPLICATIONS

In this section, some numerical problems are considered to solve by using the presented numerical technique based on the 2D Bernstein's basis functions. This will show the accuray of the proposed technique. Numerical outcomes of these examples at the various values of v and w are shown in Tables $1-3$ for $m_1 = n_1 = 2$ and $m_1 = n_1 = 6$. It is also easy to see from Tables 1, 2, and 3 that the presented method is efficient and accurate. Further, the graphs are also presented to show the comparison between the actual and numerical solutions.

Problem 1. Consider the 2D VFIE of first kind

$$4w^2 - 4w + 4we^w = \int_0^w \int_0^1 4wu(y,z)dydz, \tag{5. 50}$$

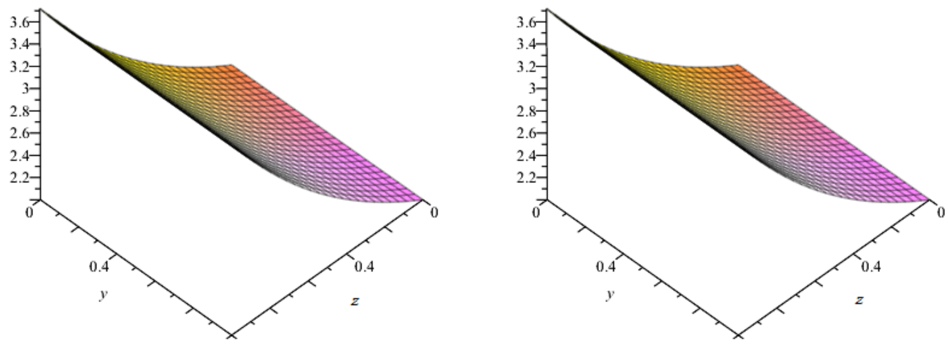
where $u(y,z) = 1+e^z$ is the exact solution of above 2-dimensional VFIE. The approximate solution is obtained by using the technique given in section 3.2. Table 1 shows the accuracy between the exact and the numerical solutions at different points of the domain. Whereas the uniformity and efficiency of the actual and numerical solutions are visualized by Figure 1.

Problem 2. Consider the separate two-dimensional VFIE of first kind

$$\begin{aligned} &-\frac{1}{2}v^3w^3\cos w + \left(2vw - \frac{1}{2}v^3w^3\right)\frac{\sin^2 w}{1+\cos w} + \frac{4vw\cos w - 2vw\cos w\sin^2 w}{1+\cos^2 w} \\ &= \int_0^w \int_0^v vwu(y,z)dydz + \int_0^1 \int_0^1 (y-z)u(y,z)dydz, \end{aligned} \tag{5. 51}$$

where $u(y,z) = 2yz$ is the exact solution. The absolute errors show the efficiency of the proposed technique at distinct knot points which is demonstrated in Table 2. The visualization of the proposed technique for the solution of (5. 51) is shown in Figure 2.

Problem 3. Consider the separate two-dimensional VFIE of first kind



Exact Solution Approximate Solution

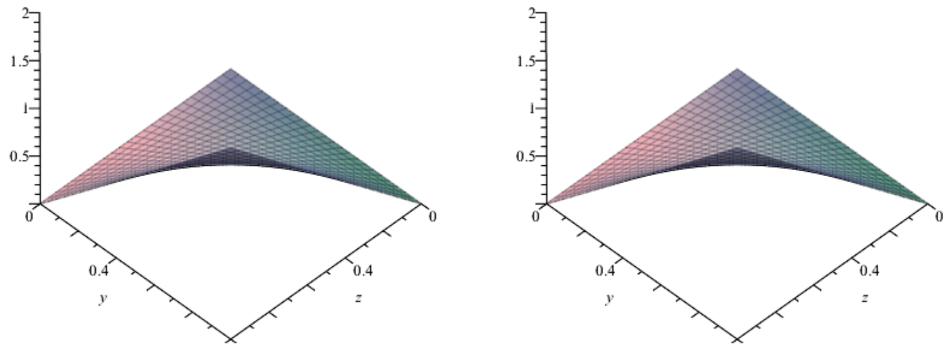
FIGURE 1. *Comparison between the actual and numerical solutions of Problem no. 1 at* $m_1 = n_1 = 2$.

TABLE 2. **Numerical results of problem no. 2**

(v,w)	**Exact Solution**	**Approximate Solution** $(m_1 = n_1 = 2)$	**Approximate Solution** $(m_1 = n_1 = 6)$	**Absolute Error** $(m_1 = n_1 = 2)$	**Absolute Error** $(m_1 = n_1 = 6)$
(0.0,0.0)	0.0000	$1.0642E^{-13}$	$1.2571E^{-15}$	$1.0642E^{-13}$	$1.2571E^{-15}$
(0.1,0.1)	0.0200	0.0188730125	0.0200000000	$1.1269E^{-3}$	$9.2415E^{-15}$
(0.2,0.2)	0.0800	0.0785129051	0.0799999999	$1.4871E^{-3}$	$1.7310E^{-13}$
(0.3,0.3)	0.1800	0.1773093711	0.1799999999	$2.6906E^{-3}$	$4.0833E^{-13}$
(0.4,0.4)	0.3200	0.3162994712	0.3199999999	$3.7005E^{-3}$	$3.5177E^{-13}$
(0.5,0.5)	0.5000	0.4915286120	0.5000000000	$8.4713E^{-3}$	$4.9409E^{-13}$
(0.6,0.6)	0.7200	0.7188608903	0.7200000000	$1.1391E^{-2}$	$2.4912E^{-12}$
(0.7,0.7)	0.9800	0.9721055411	0.9800000000	$7.8944E^{-2}$	$5.3438E^{-12}$
(0.8,0.8)	1.2800	1.2717454015	1.2800000000	$8.2546E^{-2}$	$7.3419E^{-12}$
(0.9,0.9)	1.6200	1.6155037255	1.6200000000	$4.4962E^{-2}$	$4.3716E^{-11}$
(1.0,1.0)	2.0000	1.9988819351	1.9999999999	$1.1180E^{-3}$	$9.2415E^{-15}$

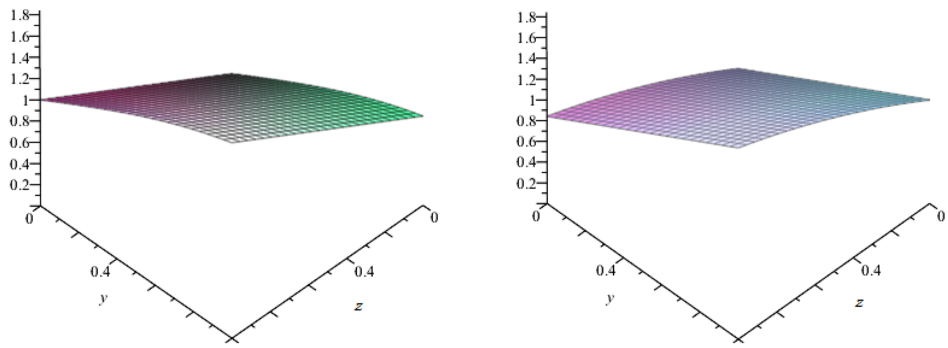
$$\frac{3}{2}vw + \frac{1}{2}v^2w - v\cos w - vw\cos 1 + v = \int_0^w \int_0^v u(y,z)dydz$$
$$- \int_0^1 \int_0^1 vwu(y,z)dydz, \qquad (5.\ 52)$$

where the exact solution of this problem is $u(y,z) = y + \sin z$. Absolute errors for exact and numerical solution can be compared with the help of Tables 3 for various values of (y,z). Figure 3 differentiate between the results of exact and the numerical solutions.



Exact Solution Approximate Solution

FIGURE 2. *Comparison between the actual and numerical solutions of Problem no. 2 at* $m_1 = n_1 = 6$.



Exact Solution Approximate Solution

FIGURE 3. *Comparison between the actual and numerical solutions of Problem no. 3 at* $m_1 = n_1 = 2$.

6. CONCLUSION

In this paper,the unknown function $u(y, z)$ is replaced by two-dimensional Bernstein basis functions to achieve the numerical solution of 2DVFIEs of first kind. It is noted

TABLE 3. **Numerical results of problem no. 3**

(v,w)	**Exact Solution**	**Approximate Solution** ($m_1 = n_1 = 2$)	**Approximate Solution** ($m_1 = n_1 = 6$)	**Absolute Error** ($m_1 = n_1 = 2$)	**Absolute Error** ($m_1 = n_1 = 6$)
(0.0,0.0)	0	-0.0000076238	-0.0000000002	$7.62383E^{-06}$	$2.75798E^{-10}$
(0.1,0.1)	0.1998000000	0.2053693419	0.1998334587	$5.53592E^{-03}$	$4.20962E^{-08}$
(0.2,0.2)	0.3987000000	0.4062153182	0.3986693209	$7.54598E^{-03}$	$9.80520E^{-09}$
(0.3,0.3)	0.5955000000	0.6024825703	0.5955202085	$6.96236E^{-03}$	$4.57540E^{-09}$
(0.4,0.4)	0.7894000000	0.7941251914	0.7894183527	$4.70684E^{-03}$	$1.04348E^{-08}$
(0.5,0.5)	0.9794000000	0.9810991020	0.9794255428	$1.67356E^{-03}$	$4.26885E^{-09}$
(0.6,0.6)	1.1646000000	1.1633620505	1.1646424713	$1.28042E^{-03}$	$2.02104E^{-09}$
(0.7,0.7)	1.3442000000	1.3408736129	1.3442177001	$3.34407E^{-03}$	$1.28867E^{-08}$
(0.8,0.8)	1.5874000000	1.5135951927	1.5173561108	$3.76089E^{-03}$	$1.99750E^{-08}$
(0.9,0.9)	1.6833000000	1.6814900211	1.6833268835	$1.83688E^{-03}$	$2.61226E^{-08}$
(1.0,1.0)	1.8415000000	1.8445231569	1.8414709944	$3.05217E^{-03}$	$9.59324E^{-09}$

that the proposed approximating scheme is accurate and gives excellent numerical outcomes. It is also observed that the presented technique can give more accurate numerical outcomes if the degree of the Bernstein polynomials is increased. Even at the smallest value of m_1 and n_1, the proposed numerical technique gives appropriate accuracy which results in the low computational cost. The presented technique can be extended for the solution of nonlinear and singular integral equation of each kind.

AUTHORS CONTRIBUTION

Both authors approved the manuscript and contributed equally to the writing of this paper.

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