New integral inequalities of Hermite–Hadamard type in a generalized context

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Abstract.: In this paper, we obtained new integral inequalities of the Hermite–Hadamard type for convex and quasi–convex functions in a generalized context.

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1. Introduction

To speak of convexity is to speak of interdisciplinary subjects. Convexity is a basic notion in geometry, but it is also widely used in other areas of mathematics. The use of convexity techniques appears in many branches of mathematics and science, such as the Theory of Optimization, Theory of Inequalities, Functional Analysis, Mathematical Programming, Theory of Games, Number Theory, Variational Calculus and their interrelationship with these branches have shown today deeper and more fruitful impact. In addition, in recent years, various extensions and generalizations of the classical concept of convexity of both sets and functions have been studied and there are regular meetings and conferences of researchers working in this area.

In our work, we will use the class of quasi–convex functions.
Definition 1.1. A function $\psi : I \to \mathbb{R}$, $I := [v_1, v_2]$ is said to be quasi–convex if $\psi(\theta x + (1 - \theta)y) \leq \max\{\psi(x), \psi(y)\}$ holds for all $x, y \in I$ and $\theta \in [0, 1]$.

It is known that a convex function is a quasi–convex function, but the converse is not always true, for example the floor function is quasi-convex but not convex, $\log(x)$ is concave and quasi-convex, and see an example in [9].

One of the most important inequalities, that has attracted many inequalities experts in the last few decades, is the Hermite–Hadamard inequality:

$$\psi\left(\frac{v_1 + v_2}{2}\right) \leq \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} \psi(\theta)d\theta \leq \frac{\psi(v_1) + \psi(v_2)}{2}$$  \hfill (1. 1)

holds for any function $\psi$ convex on the interval $[v_1, v_2]$.

Inequality (1. 1) gives an estimate for the mean value of a function on a closed interval. This inequality was obtained by Charles Hermite in 1883 and Jacques Salomon Hadamard in 1893 independently of each other. The interested reader is referred to [2–4, 6, 8, 11–14, 16, 18, 20, 21, 24] and references therein for more information and other extensions of the Hermite–Hadamard inequality. All through the work we utilize the classical Gamma functions $\Gamma$ (see [23, 30, 31]) and $\Gamma_k$ (see [8]), where $\Gamma$ is the classic Gamma function and $\Gamma_k$ is called $k$–Gamma function:

$$\Gamma(z) = \int_0^\infty \theta^{z-1}e^{-\theta}d\theta, \Re(z) > 0$$

Unmistakably, if $k \to 1$, we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = \frac{k}{k}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z + k) = z\Gamma_k(z)$. To encourage comprehsion of the subject, we present the definition of Riemann-Liouville fractional integrals (with $0 \leq v_1 < \theta < v_2 \leq \infty$). The first is the classic Riemann-Liouville fractional integrals.

Definition 1.2. The Riemann–Liouville fractional integrals $I_{v_1}^\alpha \psi(x)$ and $I_{v_2}^{\alpha-} \psi(x)$ of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined respectively by:

$$I_{v_1}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_{v_1}^{x} (x - \theta)^{\alpha-1}\psi(\theta)d\theta, \quad x > v_1,$$

$$I_{v_2}^{\alpha-} \psi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{v_2} (\theta - x)^{\alpha-1}\psi(\theta)d\theta, \quad x < v_2,$$

where $\psi \in L[v_1, v_2]$.

Remark 1.3. The Riemann-Liouville integral operator fulfills some essential properties of an operator of this nature, thus we have:

- $d_\theta I_{v_1}^{\alpha+} \psi(\theta) = I_{v_1}^{\alpha} \psi(\theta)$, so $I_{v_1}^{\alpha}$ takes the role of an anti-derivative.
- $I_{v_1}^\alpha (I_{v_2}^{\beta} \psi(\theta)) = I_{v_1}^{\alpha+\beta} \psi(\theta)$, i. e., $I_{v_1}^\alpha$ satisfies the semigroup property.
For every \( x \), the Non-comformable fractional integrals are defined by (right and left, respectively):

\[
I_{v_{1}+}^{\alpha} \psi(x) = \frac{1}{k^{\Gamma(k)(\alpha)}} \int_{v_{1}}^{x} (x - \theta)^{k-1} \psi(\theta) \, d\theta, \quad x > v_{1}, k > 0
\]

\[
I_{v_{2}+}^{\alpha} \psi(x) = \frac{1}{k^{\Gamma(k)(\alpha)}} \int_{x}^{v_{2}} (\theta - x)^{k-1} \psi(\theta) \, d\theta, \quad x < v_{2}, k > 0,
\]

where \( \psi \in L[v_{1}, v_{2}] \).

Non-comformable fractional integral definitions ([20]):

**Definition 1.5.** Let \( \alpha \in \mathbb{R} \) and \( 0 < v_{1} < v_{2} \). For each function \( \psi \in L[v_{1}, v_{2}] \), we define

\[
N_{v_{1}}^{\alpha} \psi(x) = \int_{u}^{x} \theta^{-\alpha} \psi(\theta) \, d\theta, \text{ for every } x, u \in [v_{1}, v_{2}].
\]

**Definition 1.6.** Let \( \alpha \in \mathbb{R} \) and \( v_{1} < v_{2} \). For each function, \( \psi \in L[v_{1}, v_{2}] \). Let us define the Non-comformable fractional integrals

\[
N_{v_{1}}^{\alpha} \psi(x) = \int_{v_{1}}^{x} (x - \theta)^{-\alpha} \psi(\theta) \, d\theta,
\]

\[
N_{v_{2}}^{\alpha} \psi(x) = \int_{x}^{v_{2}} (\theta - x)^{-\alpha} \psi(\theta) \, d\theta
\]

for every \( x \in [v_{1}, v_{2}] \). Here, for \( \alpha = 0 \) \( N_{v_{1}}^{0} \psi(x) = N_{v_{2}}^{0} \psi(x) = \int_{v_{1}}^{v_{2}} \psi(\theta) \, d\theta \).

In different works, integral operators have been used [1, 7, 27], which come from local differential operators of a different nature. Thus we have the Non-comformable fractional integrals ([15, 20]). Next, we present the weighted integral operators, which will provide a basis for our work.

**Definition 1.7.** Let \( \psi \in L[v_{1}, v_{2}] \) and \( w : I = [0, 1] \to \mathbb{R} \), be a continuous and positive function, with first and second order derivatives piecewise continuous on \( I \), taking in 0 and 1 we take the lateral derivatives, right and left respectively, and \( w(0) = w(1) = 0 \). Then, the weighted integral operators are defined by (right and left, respectively):

\[
\Delta^{\alpha} I_{v_{1}+}^{\alpha} \psi(x) = \int_{v_{1}}^{x} w'(\frac{x - \theta}{\Delta}) \psi(\theta) \, d\theta, \quad x > v_{1},
\]

\[
\Delta^{\alpha} I_{v_{2}+}^{\alpha} \psi(x) = \int_{x}^{v_{2}} w'(\frac{\theta - x}{\Delta}) \psi(\theta) \, d\theta, \quad x < v_{2},
\]

where \( \Delta = v_{2} - v_{1} \).

**Remark 1.8.** Given the generality of the kernel \( w'(t) \) considered in the previous Definition, it may be that some of the integral operators obtained as particular cases of our definition do not satisfy some of the properties referred to the classical fractional operators, for example, the semi-group law. It is clear that in the cases that we indicate next in the Remark 1.9, if the referred properties are satisfied.
Remark 1.9. Depending on the form of the function \( w(\theta) \) from (1.2), we get different integral operators:

1. If we take \( w = w(\theta, \alpha) = \frac{\Delta^{\alpha-1} \theta^{\alpha+1}}{\alpha(\alpha + 1) \Gamma(\alpha)} \), we get Riemann-Liouville fractional integrals;

2. If we take \( w = w(\theta, \alpha) = \frac{\Delta^{\alpha-1} \theta^{\alpha+1}}{\alpha(\alpha + 1) \Gamma(\alpha)} k \), we get \( k \)-Riemann-Liouville fractional integrals;

3. If we take \( w = w(\theta, \alpha) = \frac{\Delta^{\alpha-2} \theta^{2-\alpha}}{(1-\alpha)(2-\alpha)} \), we get non-comformable fractional integrals;

4. If we take \( w = w(\theta, 0) \), i.e. \( w'' = 1 \) we obtain the classical Riemann integrals.

In this paper, we obtained new variants of the inequality (1.1) within the framework of the weighted integral operators of the Definition 1.7 for convex and quasi-convex functions.

2. Main Results

Then, we can formulate our first result, which was used throughout the work.

Lemma 2.1. Let \( \psi \) be a real function defined on some interval \( I \subset \mathbb{R} \), twice differentiable on \( I^c, v_1, v_2 \in I^c, v_1 < v_2 \). If \( \psi'' \in L[v_1, v_2] \) and \( w(0) = w(1) = 0 \), then the equality:

\[
(w'(0)\psi(v_2) - w'(1)\psi(v_1)) + \frac{1}{\Delta} \left[ w \int_{v_1}^{v_2} \psi(v) \right]
\]

is valid.

Proof. Integrating by parts, we obtain

\[
\int_{0}^{1} w(\theta)\psi''(\theta v_1 + (1 - \theta)v_2) d\theta = \left\{ \frac{1}{\Delta^2} [w'(0)\psi(v_2) - w'(1)\psi(v_1)] + \frac{1}{\Delta} \int_{0}^{1} w''(\theta)\psi(\theta v_1 + (1 - \theta)v_2) d\theta \right\}.
\]

By putting \( z = \theta v_1 + (1 - \theta)v_2 \), so \( dz = (v_1 - v_2) d\theta \); with this change of variables and rearranging the terms, we obtain equality (2.3).

Similarly, we can prove the validity of the next lemma:

Lemma 2.2. Let \( \psi \) be a real function defined on some interval \( I \subset \mathbb{R} \), twice differentiable on \( I^c, v_1, v_2 \in I^c, v_1 < v_2 \). If \( \psi'' \in L[v_1, v_2] \) and \( w(0) = w(1) = 0 \), then the equality:

\[
(w'(0)\psi(v_1) - w'(1)\psi(v_2)) + \frac{1}{\Delta} \left[ w \int_{v_1}^{v_2} \psi(v) \right]
\]

is valid.
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is valid.

**Remark 2.3.** If we take \( w(\theta) = 1 - (1 - \theta)^{a+1} - \theta^{a+1} \), then from (2. 3 ), we get the result in [29] (see Lemma 4.1) and in [5] (see Lemma 2.2).

**Remark 2.4.** The above result contains as a particular case of Lemma 1 of [2] and Lemma 4 of [8] with \( w(\theta) = \theta (1 - \theta) \).

The first fundamental result of our work is the following. For simplicity let us denote

\[
\begin{align*}
L^+ (HH) &= \int_0^1 w(\theta) \left| \psi'(\theta) \right| d\theta + \int_0^1 \frac{1}{\Delta} \left[ w \Delta I_{v_{1}} \psi'(v_2) \right] d\theta \\
L^- (HH) &= \int_0^1 w(\theta) \left| \psi'(\theta) \right| d\theta + \int_0^1 \frac{1}{\Delta} \left[ w \Delta I_{v_{2}} \psi'(v_1) \right] d\theta
\end{align*}
\]

(2. 5)

**Theorem 2.5.** If, in addition to the conditions of Lemma 2.1, \( |\psi''| \) is quasi-convex on \([v_1, v_2]\), then the inequality

\[
\left| L^+ (HH) \right| \leq \Delta^2 \cdot B \cdot \max \{|\psi''(v_1)|, |\psi''(v_2)|\}
\]

(2. 6)

is valid, with \( B = \int_0^1 w(\theta) d\theta, \Delta = v_2 - v_1 \).

**Proof.** From the quasi-convexity of \( |\psi''| \) and Lemma 2.1, we get

\[
\left| L^+ (HH) \right| \leq \Delta^2 \int_0^1 w(\theta) |\psi''(\theta v_1 + (1 - \theta)v_2)| d\theta \leq \Delta^2 \max \{|\psi''(v_1)|, |\psi''(v_2)|\} \int_0^1 w(\theta) d\theta = \Delta^2 \cdot B \cdot \max \{|\psi''(v_1)|, |\psi''(v_2)|\}.
\]

Which is what was required to prove.

**Remark 2.6.** If we consider that \( w(\theta) = \theta (1 - \theta) \), then this result becomes Theorem 3 of [2].

From now on, we will use the well-known Hölder Inequality and its consequence, the so-called Power Mean Inequality, for more details we recommend the reader consult [10, 17]. In particular, we will use the following result.

**Theorem 2.7.** (Power–mean integral inequality). Let \( q \geq 1 \) and \( 1/p + 1/q = 1 \). If \( \psi \) and \( \phi \) are real functions defined on \([v_1, v_2]\) and if \( |\psi|, |\psi| \) are integrable functions on \([v_1, v_2]\) then

\[
\int_{v_1}^{v_2} |\psi(\theta) \phi(\theta)| d\theta \leq \left( \int_{v_1}^{v_2} |\psi(\theta)| d\theta \right)^{1 - \frac{1}{q}} \left( \int_{v_1}^{v_2} |\phi(\theta)|^q d\theta \right)^{\frac{1}{q}}.
\]

We can improve the previous result if we impose additional conditions to the quasi-convexity of \( |\psi''| \).
Theorem 2.8. Under assumptions of Lemma 2.1, if \( |\psi''|^q \) is quasi-convex on \([v_1, v_2]\), for \( q > 1 \), then the inequality

\[
|L^+(HH)| \leq B_p \cdot \Delta^2 \cdot (\max \{ |\psi''(v_1)|^q, |\psi''(v_2)|^q \})^{\frac{1}{q}}
\]

(2. 7)
is valid, with \( \frac{1}{p} + \frac{1}{q} = 1 \), \( B_p = \left( \int_0^1 w^p(\theta) d\theta \right)^\frac{1}{p} \) and \( \Delta = v_2 - v_1 \).

Proof. From Hölder’s inequality, in its integral version, and Lemma 2.1, we have

\[
|L^+(HH)| \leq \Delta^2 \int_0^1 w(\theta) |\psi''(\theta v_1 + (1 - \theta)v_2)| d\theta
\]

\[
\leq \Delta^2 \left( \int_0^1 w(\theta)^p d\theta \right)^\frac{1}{p} \left( \int_0^1 |\psi''(\theta v_1 + (1 - \theta)v_2)|^q d\theta \right)^\frac{1}{q}
\]

\[
\leq \Delta^2 \cdot B_p \cdot (\max \{ |\psi''(v_1)|^q, |\psi''(v_2)|^q \})^{\frac{1}{q}}.
\]

Which is what was required to prove. \( \square \)

Remark 2.9. Theorem 4 of [2] is easily obtained from the previous result if we put \( w(\theta) = \theta(1 - \theta) \).

A more general variant of the previous theorem, is given in the following result.

Theorem 2.10. Under assumptions of Lemma 2.1, if \( |\psi''|^q \) is quasi-convex on \([v_1, v_2]\) for \( q \geq 1 \), then the inequality

\[
|L^+(HH)| \leq B \cdot \Delta^2 \cdot (\max \{ |\psi''(v_1)|^q, |\psi''(v_2)|^q \})^{\frac{1}{q}}
\]

(2. 8)
is valid, with \( B \) and \( \Delta \) is like in Theorem 2.5.

Proof. Taking into account the of Lemma 2.1 and the power mean inequality (other form of Hölder’s inequality) and \( \frac{1}{p} + \frac{1}{q} = 1 \) for \( q \geq 1 \), we have

\[
|L^+(HH)| \leq \Delta^2 \int_0^1 w(\theta) |\psi''(\theta v_1 + (1 - \theta)v_2)| d\theta
\]

\[
= \Delta^2 \int_0^1 \left[ w(\theta) \right]^{\frac{1}{p} + \frac{1}{q}} |\psi''(\theta v_1 + (1 - \theta)v_2)| d\theta
\]

\[
\leq \Delta^2 \left( \int_0^1 \left[ w(\theta) \right]^{\frac{1}{p}} d\theta \right)^{\frac{1}{p} - \frac{1}{q}} \left( \int_0^1 \left[ w(\theta) \right]^{\frac{1}{q}} d\theta \right)^{\frac{1}{q}}
\]

\[
= \Delta^2 \cdot B \cdot (\max \{ |\psi''(v_1)|^q, |\psi''(v_2)|^q \})^{\frac{1}{q}}.
\]

Which is what was required to prove. \( \square \)

Remark 2.11. We can verify, without much difficulty, that Theorem 5 of [2] is a particular case of the previous result if we make \( w(\theta) = \theta(1 - \theta) \).

The following theorem is obvious.
Theorem 2.12. If, in addition to the conditions of Lemma 2.1, $|\psi''|$ is convex on $[v_1, v_2]$, then the inequality

$$|L^+(HH) + L^-(HH)| \leq \Delta^2 \cdot B \cdot (|\psi''(v_1)| + |\psi''(v_2)|)$$

is valid, with $B = \int_0^1 w(\theta) d\theta$ and $\Delta = v_2 - v_1$.

Proof. By taking into account equalities (2.3), (2.4) and the properties of the modulus from the condition of convexity of the function $|\psi''|$, we get:

$$|L^+(HH) + L^-(HH)| = \Delta^2 \left| \int_0^1 w(\theta) \psi''(\theta v_1 + (1 - \theta) v_2) d\theta \right|$$

$$\leq \Delta^2 \int_0^1 w(\theta) |\psi''(\theta v_1 + (1 - \theta) v_2) + \psi''((1 - \theta) v_1 + \theta v_2)| d\theta$$

$$\leq \Delta^2 \int_0^1 w(\theta) |\theta \psi''(v_1) + (1 - \theta) \psi''(v_2) + (1 - \theta) \psi''(v_1) + \theta \psi''(v_2)| d\theta$$

$$= \Delta^2 \int_0^1 w(\theta) |\psi''(v_1) + \psi''(v_2)| d\theta$$

$$\leq \Delta^2 (|\psi''(v_1)| + |\psi''(v_2)|) \int_0^1 w(\theta)d\theta = \Delta^2 \cdot B \cdot (|\psi''(v_1)| + |\psi''(v_2)|).$$

The proof is completed. □

Remark 2.13. We can verify, without much difficulty, that Theorem 3.1 (for $s = 1$ and $m = 1$) of [3] is a special case of the Theorem 2.12 if we make $w(\theta) = \theta^\alpha (1 - \theta)$.

3. SOME METHODOLOGICAL OBSERVATIONS

Throughout the work, we have pointed out the generality of our results since they contain as particular cases, several of which are known from the literature. However, we can reformulate our Definition 1.7 and cover other known results. For example, it suffices to use a version of our integral operator:

Definition 3.1. Let $\psi \in L[v_1, v_2]$ and let $w$ be a continuous and positive function, $w : I = [0, 1] \rightarrow \mathbb{R}$, with the first and second order derivatives piecewise continuous on $I$ and $w(0) = 0$. Then, the weighted integral operators are defined by (right and left side, respectively):

$$w \int_{v_1}^x I^+_{v_1} \psi(x) = \int_{v_1}^x w'' \left( \frac{x - \theta}{2} \right) \psi(\theta) d\theta, \quad x > v_1,$$

$$w \int_{v_2}^x I^-_{v_2} \psi(x) = \int_x^{v_2} w'' \left( \frac{\theta - x}{2} \right) \psi(\theta) d\theta, \quad x < v_2,$$

where $\Delta = v_2 - v_1$. So, we have
Lemma 3.2. Let $\psi$ be a function of real variables defined on the interval $I$, $I \subset \mathbb{R}$, twice differentiable on $I$, $v_1, v_2 \in I$, $v_1 < v_2$ and $w(0) = 0$. If $\psi'' \in L[v_1, v_2]$, then we have the following equality:

\[ w'(0) \psi(v_1) + \psi(v_2) - w'(1) \psi \left( \frac{v_1 + v_2}{2} \right) \]

\[ + \frac{1}{\Delta} \left[ \frac{w}{2} I_{v_1+\Delta} + \psi(v_2) + \frac{w}{2} I_{v_1-\Delta} - \psi(v_1) \right] \]

\[ = \Delta^2 \int_0^1 w(\theta) \left[ \psi'' \left( \frac{\theta}{2} v_1 + \frac{2 - \theta}{2} v_2 \right) + \psi'' \left( \frac{2 - \theta}{2} v_1 + \frac{\theta}{2} v_2 \right) \right] d\theta, \]

where $\Delta = v_2 - v_1$.

Proof. Analogous to the proof of Lemma 2.1 in [18]. By integrating by parts and changing variables under the resulting integrals, we obtain:

\[ \int_0^1 w(\theta) \psi'' \left( \frac{\theta}{2} v_1 + \frac{2 - \theta}{2} v_2 \right) d\theta \]

\[ = -\frac{2w(1)}{\Delta} \psi' \left( \frac{v_1 + v_2}{2} \right) - \frac{4w'(1)}{\Delta^2} \psi \left( \frac{v_1 + v_2}{2} \right) + \frac{4w'(0)}{\Delta^2} \psi(v_1) + \psi(v_2) \]

\[ + \frac{8}{\Delta^3} \int_{v_1+\Delta}^{v_2} \psi'' \left( \frac{v_2 - z}{2} \right) \psi(z) dz \]

\[ = -\frac{2w(1)}{\Delta} \psi' \left( \frac{v_1 + v_2}{2} \right) - \frac{4w'(1)}{\Delta^2} \psi \left( \frac{v_1 + v_2}{2} \right) + \frac{4w'(0)}{\Delta^2} \psi(v_1) + \psi(v_2) \]

\[ + \frac{8}{\Delta^3} \left[ \frac{w}{2} I_{v_1+\Delta} + \psi(v_2) \right]. \]

Similarly, for the second integral, we will have

\[ \int_0^1 w(\theta) \psi'' \left( \frac{2 - \theta}{2} v_1 + \frac{\theta}{2} v_2 \right) d\theta = -\frac{2w(1)}{\Delta} \psi' \left( \frac{v_1 + v_2}{2} \right) \]

\[ - \frac{4w'(1)}{\Delta^2} \psi \left( \frac{v_1 + v_2}{2} \right) + \frac{4w'(0)}{\Delta^2} \psi(v_1) + \psi(v_2) \]

\[ + \frac{8}{\Delta^3} \left[ \frac{w}{2} I_{v_1-\Delta} - \psi(v_1) \right]. \]

By adding (3.10) and (3.11) and multiplying by $(v_2 - v_1)^2$ we get (3.9). The proof is completed.

Remark 3.3. In a particular case, if we take $w(\theta) = \theta^{\alpha+1}$, then from (3.9), we get the result in [28] (see Lemma 1).

Remark 3.4. In a particular case, if we take $w(\theta) = \theta^2$, then from (3.9), we get the result in [26] (see Lemma 2).

By using Lemma 3.2, one can easily prove the following theorem:
Theorem 3.5. Under assumptions of Lemma 3.2, if $|\psi''|$ is convex on $[v_1, v_2]$, we have the following inequality

$$\left| A + \frac{1}{\Delta} \left[ \frac{\psi}{\Delta} I_{\frac{\alpha + 2}{\alpha + 2}} + \psi(v_2) + \frac{\psi}{\Delta} I_{\frac{\alpha + 2}{\alpha + 2}} - \psi(v_1) \right] \right| \leq \frac{\Delta^2}{8} \cdot B \cdot (|\psi''(v_1)| + |\psi''(v_2)|),$$

with $B = \int_0^1 w(\theta) d\theta$ and $A = \left[ w''(0) \left( \frac{\psi(v_1) + \psi(v_2)}{2} \right) - w'(1) \psi \left( \frac{v_1 + v_2}{2} \right) \right]$. 

Proof. Analogous to the proof of Theorem 2.1 in [18]. From Lemma 3.2 and the properties of the module, we will have:

$$\left| w''(0) \frac{\psi(v_1) + \psi(v_2)}{2} - w'(1) \psi \left( \frac{v_1 + v_2}{2} \right) \right| + \frac{1}{\Delta} \left[ \frac{\psi}{\Delta} I_{\frac{\alpha + 2}{\alpha + 2}} + \frac{\psi}{\Delta} I_{\frac{\alpha + 2}{\alpha + 2}} - \psi(v_1) \right] \leq \frac{\Delta^2}{8} \int_0^1 w(\theta) \left| \psi'' \left( \frac{\theta}{2} v_1 + \frac{2 - \theta}{2} v_2 \right) \right| d\theta.$$ 

From the condition of convexity of the function $|\psi''| :$

$$\leq \left| \psi'' \left( \frac{\theta}{2} v_1 + \frac{2 - \theta}{2} v_2 \right) \right| + \left| \psi'' \left( \frac{2 - \theta}{2} v_1 + \frac{\theta}{2} v_2 \right) \right|$$

$$\leq \frac{\theta}{2} |\psi''(v_1)| + \left( 1 - \frac{\theta}{2} \right) |\psi''(v_2)| + \left( 1 - \frac{\theta}{2} \right) |\psi''(v_1)| + \frac{\theta}{2} |\psi''(v_2)|$$

$$= |\psi''(v_1)| + |\psi''(v_2)|.$$ 

From (3.13), by taking into account (3.14) and the accepted designations, we obtain (3.12). The proof is completed. 

Remark 3.6. If we take $w(\theta) = \theta^{\alpha + 1}$, then we obtain a particular case of Theorem 3.5:

$$\left| \frac{2^{\alpha + 1} \Gamma(\alpha + 2)}{\Delta^\alpha} \left[ I_{\frac{\alpha + 2}{\alpha + 2}} + \psi(v_2) + I_{\frac{\alpha + 2}{\alpha + 2}} - \psi(v_1) \right] - (\alpha + 1) \psi \left( \frac{v_1 + v_2}{2} \right) \right| \leq \frac{\Delta^2}{8 (\alpha + 2)} (|\psi''(v_1)| + |\psi''(v_2)|).$$

This inequality was obtained in [28] (see Theorem 3).

Remark 3.7. If we put $w(\theta) = \theta^{\frac{\alpha}{2} + 1}$, then from Theorem 3.5, we obtain the inequality via $k$–fractional integral:

$$\left| \frac{2^{\frac{\alpha}{2} - 1} \Gamma(\alpha)}{\Delta^\frac{\alpha}{2} \left( \frac{\alpha}{2} + 1 \right)} \left[ k P_{\frac{\alpha + 2}{2}} + \psi(v_2) + k P_{\frac{\alpha + 2}{2}} - \psi(v_1) \right] - \psi \left( \frac{v_1 + v_2}{2} \right) \right|$$

$$\leq \frac{\Delta^2}{8 \left( \frac{\alpha}{2} + 1 \right) \left( \frac{\alpha}{2} + 2 \right)} (|\psi''(v_1)| + |\psi''(v_2)|).$$

This inequality was obtained in [18] (see Corollary 2.1).
**Remark 3.8.** If we put \( w(\theta) = \theta^{2-\alpha} \), then from Theorem 3.5, we obtain the inequality via non-conformable fractional integral:

\[
\left\| (1-\alpha)\Delta^{\alpha-1} \left[ N_3 I_{\alpha+2}^\alpha + \psi(v_2) + N_3 I_{\frac{3}{2}+\alpha}^\alpha - \psi(v_1) \right] - \psi \left( \frac{v_1 + v_2}{2} \right) \right\| \leq \frac{\Delta^2}{8} \left( \frac{1}{1-\alpha} \right) \left( \frac{2}{2-\alpha} \right) \left( |\psi''(v_1)| + |\psi''(v_2)| \right).
\]

**Remark 3.9.** If we put \( w(\theta) = \theta^2 \), then we obtain a particular case of Theorem 3.5:

\[
\left\| \frac{1}{v_2 - v_1} \int_0^1 \psi(\theta) d\theta - \psi \left( \frac{v_2 + v_1}{2} \right) \right\| \leq \frac{(v_2 - v_1)^2}{48} \left( |\psi''(v_1)| + |\psi''(v_2)| \right).
\]

This estimate was obtained in [25] and confirmed in a number of works (for example [3], [4] and [22]).

**Theorem 3.10.** Under assumptions of Lemma 3.2, if \( |\psi''|^q \) is convex on \([v_1, v_2]\), we have the following inequality:

\[
\left\| A + \frac{1}{\Delta} \left[ \frac{w}{2} I_{\alpha+2}^\alpha + \psi(v_2) + \frac{w}{2} I_{\alpha+2}^\alpha - \psi(v_1) \right] \right\| \leq \frac{\Delta^2}{8} \cdot B_p \cdot \left[ \left( \frac{|\psi''(v_1)|^q}{4} + \frac{3|\psi''(v_2)|^q}{4} \right)^{\frac{1}{q}} \right. + \left. \left( \frac{3|\psi''(v_1)|^q}{4} + \frac{|\psi''(v_2)|^q}{4} \right)^{\frac{1}{q}} \right],
\]

where \( A = \left[ w'(0) \left( \frac{\psi(v_1) + \psi(v_2)}{2} \right) - w'(1) \psi \left( \frac{v_1 + v_2}{2} \right) \right], \frac{1}{p} + \frac{1}{q} = 1, \forall q, p > 1 \) and \( B_p = \left( \int_0^1 w(\theta)^q d\theta \right)^{\frac{1}{q}}. \)

**Proof.** Analogous to the proof of Theorem 2.2 in [18]. If we use the triangle inequality to the right-hand side of (3.9), we obtain:

\[
\left\| w'(0) \frac{\psi(v_1) + \psi(v_2)}{2} - w'(1) \psi \left( \frac{v_1 + v_2}{2} \right) \right\| \leq \frac{\Delta^2}{8} \int_0^1 w(\theta) \left[ \left| \psi'' \left( \frac{\theta}{2} v_1 + \frac{2-\theta}{2} v_2 \right) \right| + \left| \psi'' \left( \frac{2-\theta}{2} v_1 + \frac{\theta}{2} v_2 \right) \right| \right] d\theta.
\]
Consider the means as arbitrary real numbers like to make a final Remark, on the application of the results obtained for special means. With respect to the firstly indicated direction, we would two directions: first, by imposing additional conditions on the function \( w \) Hermite-Hadamard Inequality.

entiable convex and quasi-convex functions, which were the generalizations of the classic Hermite-Hadamard inequalities for the functions, whose second derivatives are convex and L media and from \([28]\) (Theorem 4) is a special case of Theorem 3.10 if we put \( = \Delta \theta \).

By using the well-known H"{o}lder integral inequality and since \( |\psi''(\theta)|^q \) is a convex function for the right-side (3.15), we get:

\[
\begin{align*}
\frac{\Delta^2}{8} & \int_0^1 w(\theta) \left| \psi'' \left( \frac{\theta}{2} v_1 + \frac{2-\theta}{2} v_2 \right) \right| d\theta + \left| \psi'' \left( \frac{2-\theta}{2} v_1 + \frac{\theta}{2} v_2 \right) \right| d\theta \\
\leq \frac{\Delta^2}{8} \left( \int_0^1 w^p(\theta) d\theta \right) \frac{1}{p} \left[ \left( \int_0^1 \left| \psi''(v_1) \right|^q \right)^{\frac{1}{q}} + \left( \int_0^1 \left| \psi''(v_2) \right|^q \right)^{\frac{1}{q}} \right] \\
& \quad + \left( \int_0^1 \left| \psi''(v_1) \right|^q \right)^{\frac{1}{q}} \left( \int_0^1 \frac{2-\theta}{2} d\theta + \int_0^1 \frac{\theta}{2} d\theta \right) \frac{1}{q} \\
& = \frac{\Delta^2 B_P}{8} \left[ \left( \frac{\left| \psi''(v_1) \right|^q}{4} + \frac{3 \left| \psi''(v_2) \right|^q}{4} \right)^{\frac{1}{q}} + \left( \frac{3 \left| \psi''(v_1) \right|^q}{4} + \frac{\left| \psi''(v_2) \right|^q}{4} \right)^{\frac{1}{q}} \right].
\end{align*}
\]

By taking into account (3.15), the proof is completed. \( \square \)

**Remark 3.11.** It is not difficult to verify that the inequality

\[
\left| \frac{2^{\alpha-1} \Gamma(\alpha+2)}{\Delta} \left[ I_1^{\alpha+2} + \psi(v_2) - \psi(v_1) \right] - (\alpha+1) \psi \left( \frac{v_1 + v_2}{2} \right) \right|
\]

\[
\leq \frac{\Delta^2}{8 \left( (\alpha+1) p + 1 \right)^{\frac{1}{p}}} \cdot \left[ \left( \frac{\left| \psi''(v_1) \right|^q}{4} + \frac{3 \left| \psi''(v_2) \right|^q}{4} \right)^{\frac{1}{q}} + \left( \frac{3 \left| \psi''(v_1) \right|^q}{4} + \frac{\left| \psi''(v_2) \right|^q}{4} \right)^{\frac{1}{q}} \right].
\]

from \([28]\) (Theorem 4) is a special case of Theorem 3.10 if we put \( w(\theta) = \theta^{\alpha+1} \).

4. Conclusions

In this work, we have obtained some inequalities by using a certain “weighted” integral, which contains several already published results and leaves open new lines of research as we pointed out in the previous section. Throughout the work, we have obtained the Hermite-Hadamard inequalities for the functions, whose second derivatives are convex and quasi-convex, via generalized integrals. To achieve our objectives, we obtained two lemmas, and on this basis, we obtained different types of integral identities for twice differentiable convex and quasi-convex functions, which were the generalizations of the classic Hermite–Hadamard Inequality.

Apart what we previously presented, other formulations of our results can be obtained in two directions: first, by imposing additional conditions on the function \( w(\theta) \) and, secondly, by the other notions of convexity. With respect to the firstly indicated direction, we would like to make a final Remark, on the application of the results obtained for special means. Consider the means as arbitrary real numbers \( v_1 \) and \( v_2 \), \( v_1 \neq v_2 \). Be \( A \), the arithmetic media and \( L \), the generalized \( \log \)-mean

\[
A(a, b) = \frac{a + b}{2} \quad \text{and} \quad L_n(a, b) = \left[ \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{\frac{1}{n}},
\]
then, using the Theorem 2.5 for $w(\theta) = \theta(1 - \theta)$, we obtain the following result.

**Proposition 4.1.** Let $v_1, v_2 \in \mathbb{R}$, $v_1 < v_2$ and $\psi(\theta) = \theta^n, t \in [v_1, v_2], n \in \mathbb{N}, n \geq 2$. Then, the following inequality holds

$$|A(v_1^n, v_2^n) - L_n^n(v_1, v_2)| \leq \frac{(v_2 - v_1)^2 n(n - 1)}{12} \max \left\{ |v_1|^{n-2}, |v_2|^{n-2} \right\}. \quad (4.16)$$

**Proof.** From (2.6), by taking into account (2.5), we get:

$$|L^+(HH)| \leq (v_2 - v_1)^2 \cdot B \cdot \max \{ |\psi''(v_1)|, |\psi''(v_2)| \} \quad (4.17)$$

$$|L^+(HH)| = \left| w'(0)\psi(v_2) - w'(1)\psi(v_1) + \frac{1}{\Delta} \left[ w \Delta I_{v_1} \psi(v_2) \right] \right|$$

$$= \left| v_2^n + v_1^n + \frac{1}{v_2 - v_1} \int_0^1 (-2) \theta^n d\theta \right| = 2 \left| \frac{v_2^n + v_1^n}{2} - \frac{v_2^{n+1} - v_1^{n+1}}{(v_2 - v_1)(n + 1)} \right|$$

$$= 2 |A(v_1^n, v_2^n) - L_n^n(v_1, v_2)|.$$ 

On the other hand, since $\psi''(\theta) = n(n - 1)\theta^{n-2}$, we have:

$$(v_2 - v_1)^2 \cdot B \cdot \max \{ |\psi''(v_1)|, |\psi''(v_2)| \} \quad (4.19)$$

$$= (v_2 - v_1)^2 \left( \int_0^1 \theta(1 - \theta) d\theta \right) n(n - 1) \max \left\{ |v_1|^{n-2}, |v_2|^{n-2} \right\}$$

$$= \frac{(v_2 - v_1)^2}{6} n(n - 1) \max \left\{ |v_1|^{n-2}, |v_2|^{n-2} \right\}.$$ 

From (4.17), by taking into account (4.18) and (4.19), we obtain (4.16). The proof is completed. □

**Remark 4.2.** The obtained score is in line with the score presented in Proposition 1 of [2]. A similar observation is valid for Propositions 2 and 3 of the mentioned work.

**REFERENCES**


New integral inequalities of Hermite–Hadamard type in a generalized context


