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A variety of multivalently analytic functions with complex coefficients and some argument properties of their applications

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Abstract.: The principal goal of this scientific note is first to compose various rational types functions directly connecting to a variety of multivalently functions with complex coefficients, *which* are regular in certain domains in the complex plane, and then to designate numerous argument properties associating with certain applications of those (multivalent) functions.

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1. Background Information and Motivation

By a simple literature survey, it is possible to arrive at various scientific works associating with certain types of multivalent (*or*, r-valent) functions being of the Taylor-Maclaurin series expansions given by the forms:

$$\begin{aligned}\Upsilon &:= \Upsilon(z)\\ &= z^r + w_{1+r}z^{1+r} + w_{2+r}z^{2+r} + \cdots \ \big(r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; w_{1+r} \in \mathbb{C}\big),\end{aligned} \tag{1}$$

which are *regular* in

$$\mathbb{U} := \Big\{ z : z \in \mathbb{C} \text{ and } |z| < 1 \Big\},$$

where the well-known sets:

$$\mathbb{N} \quad , \quad \mathbb{C} \quad \text{and} \quad \mathbb{U}$$

are called the sets of the *natural* numbers, the *complex* numbers and the *open unit* disc, respectively.

More particularly, we write down here that those complex valued functions, *which* have the forms given as in (1) and also play a big part in the theory of complex functions, are also referred by certain different names in the published literature, *which* are called as the *multivalently (or, r-valently) analytic functions with positive coefficients* when *all coefficients* (w_{1+r}) are *positive real* numbers and the *multivalently (or, r-valently) analytic functions with negative coefficients* when *all coefficients* (w_{1+r}) are *negative real* numbers. For all types of these complex functions called as just above, some of the earlier investiga-tions, given in [1, 3, 5, 10, 11, 13, 20, 21, 25, 27], may be presented to the attention of the concerned researchers as certain examples.

Recently, as it has been informed in details by the investigations in [1, 2, 4, 7, 10, 11, 12, 17, 18, 20, 21, 22, 27, 28, 29], various types of the multivalently analytic functions with complex coefficients defined by making use of certain elementary operators may also play a significant roles for both operator theory and analytic function theory. Consequently, for those and our scopes, there is in need of two essential reminders in relation with some elementary operators *which* were earlier constituted by using the multivalently analytic functions being of the forms as in (1). See, especially, the results in the papers presented by [2, 11, 15].

As a first reminder, for an analytic function Υ having the forms like (1), we want to begin with evoking an extensive definition of a differential operator given by

$$\mathcal{D}_\phi^v\{\Upsilon\}(z) := \phi\,\Upsilon^{(v)}(z) + (1-\phi)z\,\Upsilon^{(v+1)}(z), \tag{2}$$

where

$$\begin{aligned}\Upsilon^{(v)}(z) &\equiv \frac{d^v}{dz^v}\Big(\Upsilon(z)\Big)\\ &\equiv \left(\frac{r!}{(r-v)!} + \sum_{u=s+1}^{\infty}\frac{u!}{(u-v)!}w_u z^{u-r}\right)z^{r-v}\end{aligned} \tag{3}$$

and

$$v<r \;,\; 0\le\phi\le 1 \;,\; s\in\mathbb{N} \;,\; r\in\mathbb{N} \;,\; v\in\mathbb{N}_0 \;\text{ and }\; z\in\mathbb{U}. \tag{4}$$

Under the mentioned conditions given in (4) and after certain simple calculations, the following relation can be easily seen that

$$\frac{d}{dz}\Big(\mathcal{D}_\phi^v\{\Upsilon\}(z)\Big) = \Upsilon^{(v+1)}(z) + (1-\phi)\,z\,\Upsilon^{(v+2)}(z), \tag{5}$$

where the function Υ has the complex-series-expansion form expressed by (1).

In the literature, of course, it also is possible to see numerous scientific researches associating with certain special types constituted by some of earlier elementary differential operators (*or*, their implications / applications) that we have also defined just above. For some of them, there also is a need to call up some comprehensive information (*or*, reminders) (*cf.*, *e.g.*, [1, 2, 3, 11, 13, 14, 15]), as follows.

As a second reminder, we also need to start with presenting an extra-elementary operator defined by the help of the definition in [15]. (See also [14] and [16].) For its definition, in the light of the information given by (2)-(5), for a multivalently analytic function like $\Upsilon(z)$ as in (1), we then establish the following-differential operator given by

$$\mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z) := \frac{z\,\frac{d}{dz}\Big(\mathcal{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)\Big)}{\mathcal{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)}\,, \tag{6}$$

where $\mathcal{D}_{\phi}^{v}\big\{\Upsilon\big\}(z) \neq 0$ for all $z \in \mathbb{U}$.

Finally, by taking into account those elementary operators (2) and (5) together with (3) and (5), of course, under the suitable conditions indicated by (4), there can be possible to find out a large number of relations designated by several combining of the expressions, *which* are

$$\Upsilon^{(v)}(z) \quad , \quad \mathcal{D}_{\phi}^{v}\big\{\Upsilon\big\}(z) \quad \text{and} \quad \mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z) \tag{7}$$

for any functions $\Upsilon(z)$ being of the series forms given as in (1).

Specially, after some simple operations, it is easy to see that the point $z = 0$ is a *removable singular* point for the complex type operator $\mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)$ (and also all of its special forms (like (8) below)) for both all $v \in \mathbb{N}_0$ and any functions $\Upsilon(z)$ like (1). In the present cases, as is well known, these types of the indicated-special relations between the mentioned operators $\mathrm{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)$ and $\Upsilon^{(v)}(z)$ also play an important role for (Analytic and) Geometric Function Theory. (See [6, 9] for its special topics and see also [6, 8, 9, 14, 16, 19, 23, 26], as examples.) If we want to exemplify here, it is enough to take the value of the parameter v as $v := 0$. Since $\Upsilon^{(v)}(z)$ is the vth derivative of any *multivalently analytic function* $\Upsilon(z)$ specified by (1), for all admissible values of those parameters specified as in (4) and also for any multivalently analytic functions with complex coefficients like the form in (1), only one of those special relationships:

$$\begin{aligned} \mathbf{D}_{\phi}^{0}\big\{\Upsilon\big\}(z) &\equiv \frac{z\,\frac{d}{dz}\Big(\mathcal{D}_{\phi}^{0}\big\{\Upsilon\big\}(z)\Big)}{\mathcal{D}_{\phi}^{0}\big\{\Upsilon\big\}(z)} \\ &\equiv \frac{z\,\frac{d}{dz}\Big(\phi\,\Upsilon(z) + (1-\phi)z\,\Upsilon'(z)\Big)}{\phi\,\Upsilon(z) + (1-\phi)\,z\,\Upsilon'(z)} \\ &\equiv \frac{z\,\Upsilon'(z) + (1-\phi)\,z^2\,\Upsilon''(z)}{\phi\,\Upsilon(z) + (1-\phi)\,z\,\Upsilon'(z)} \end{aligned} \tag{8}$$

can be easily achieved, *which* is closely associated with (Analytic and) Geometric Function Theory as elucidated by the works in [6, 9]. Here, we especially recommend focusing on the whole of the recent paper in [15] as both various implications of the assertions in (2), (3), (5), (6) and (8). The others are omitted here. We bring the identification of the others to the attention of the relevant researchers.

In order to establish our comprehensive results, *which* will be included numerous combining of the related-elementary-differential operators dealing with the mentioned forms in (7), it is time to enter to the second chapter of this paper.

2. Some Results and Related Implications

The proposition (just below) is an important tool and very useful auxiliary theorem proven by M. Nunokawa (see [24]). For some of the earlier results proven by that assertion (just below), one may refer to the papers given in [14, 15, 16] as some examples.

Lemma 1. (see [24]) *Let $\Omega(z)$ be a function with complex variable that is regular in the disc $\mathbb{U}$ satisfies the condition $\Omega(0) = 1$, and also let exist a point $z_0 \in \mathbb{U}$ such that*

$$\Re e\Big(\Omega(z)\Big) > 0 \quad \text{when} \quad |z| < |z_0| < 1, \tag{9}$$

$$\Omega(z_0) \neq 0 \tag{10}$$

and

$$\Re e\Big(\Omega(z_0)\Big) = 0. \tag{11}$$

Then

$$z\Omega'(z)\Big|_{z:=z_0} = i\Delta\Omega(z_0), \tag{12}$$

where Δ is a real number with $|\Delta| \geq 1$.

By both considering Lemma 1 and using various extensive combining determined by certain (elementary) operators in (7) (see [15] and also [14, 16]), we now can both create and then prove an extensive result consisting of various arguments of certain rational types of the regular functions like (1), *which* is being the following theorem (below).

Theorem 1. *Under the conditions of the definitions presented in* (1), (2), (3), (5) *and* (6) *and also for the appropriate values of the confined parameters stylized by* (4), *the following proposition then holds.*

$$\text{Arg}\left(\frac{z\frac{d}{dz}\Big(\mathbf{D}_\phi^v\{\Upsilon\}(z)\Big)}{\mathbf{D}_\phi^v\{\Upsilon\}(z)}\right) \notin \Big[-\frac{\pi}{2}, 0\Big) \tag{13}$$

$$\Longrightarrow \Re e\Big(\mathbf{D}_\phi^v\{\Upsilon\}(z)\Big) > (r-v)\mathcal{M}, \tag{14}$$

where

$$v < r \quad , \quad 0 \leq \mathcal{M} < \frac{1}{r-v} \quad \text{and} \quad z \in \mathbb{U}. \tag{15}$$

Proof. First of all, we have to use Lemma 1 to prove the theorem above. For this, it is essential that the function specified in Lemma 1 is firstly created. Let us now try to create the necessary designs. Under the conditions presented in (4) and in the light of the expressions set out in (2), (5) and (6) together with (3), we can easily get that

$$
\begin{aligned}
\mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z) &= \frac{z\frac{d}{dz}\Big(\mathcal{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)\Big)}{\mathcal{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)} \\
&= \frac{z\frac{d}{dz}\Big[\Upsilon^{(v)}(z)+(1-\phi)\,z\,\Upsilon^{(v+1)}(z)\Big]}{\phi\,\Upsilon^{(v)}(z)+(1-\phi)\,z\,\Upsilon^{(v+1)}(z)} \\
&= \frac{z\,\Upsilon^{(v+1)}(z)+(1-\phi)\,z^{2}\,\Upsilon^{(v+2)}(z)}{\phi\,\Upsilon^{(v)}(z)+(1-\phi)\,z\,\Upsilon^{(v+1)}(z)},
\end{aligned}
\tag{16}
$$

where $z \in \mathbb{U}$ and the related function $\Upsilon := \Upsilon(z)$ is constituted by the form like (1). By take cognizance of vth derivative operator in (3) for the result in (16) and, thereby, after some extensive complex calculations, in a similar ways *which* were earlier considered in [16], and also under the conditions presented by (15), for the proof of the related theorem, let us then consider the following function $\Omega(z)$ given by the implicit form in:

$$
\mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z) = (r-v)\Big[\mathcal{M}+\big(1-\mathcal{M}\big)\Omega(z)\Big]. \tag{17}
$$

It is obvious that the mentioned function $\Omega(z)$ is both regular in $\mathbb{U}$ and also ensures the condition $\Omega(0) = 1$ as pointed out in the related lemma. Therefore, it follows from (17) that

$$
\frac{z\frac{d}{dz}\Big(\mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)\Big)}{\mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)} = \frac{\big(1-\mathcal{M}\big)z\,\Omega'(z)}{\mathcal{M}+\big(1-\mathcal{M}\big)\Omega(z)} \quad \big(\forall z \in \mathbb{U}\big). \tag{18}
$$

Now, for the waiting proof, we presume that there exists a complex point z_0 in $\mathbb{U}$ supplying the condition in (11), namely,

$$
\Re e\Big(\Omega\big(z_0\big)\Big) = 0 \quad \big(\Omega(z_0) \neq 0\big), \tag{19}
$$

which also requires

$$
\Omega\big(z_0\big) = i\lambda \quad \big(\lambda \neq 0\big).
$$

By applying the assertions of Lemma 1 to the statement given by (18) and also taking into account the expression just above, for all $\mathcal{M}$ $\big(0 \leq \mathcal{M} < 1\big)$, we then get that

$$
\begin{aligned}
&\text{Arg}\left(\left.\frac{z\frac{d}{dz}\Big(\mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)\Big)}{\mathbf{D}_{\phi}^{v}\big\{\Upsilon\big\}(z)}\right|_{z:=z_0}\right) \\
&\qquad = \text{Arg}\left(\left.\frac{\big(1-\mathcal{M}\big)z\,\Omega'(z)}{\mathcal{M}+\big(1-\mathcal{M}\big)\,\Omega(z)}\right|_{z:=z_0}\right) \\
&\qquad = \text{Arg}\left(\frac{\big(1-\mathcal{M}\big)\,i\,\Delta\,\Omega(z_0)}{\mathcal{M}+\big(1-\mathcal{M}\big)\,\Omega(z_0)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \operatorname{Arg}\left(-\frac{\big(1-\mathcal{M}\big)\,\Delta\,\lambda}{\mathcal{M}+i\big(1-\mathcal{M}\big)\,\lambda}\right)\\
&= \operatorname{Arg}\left(-\frac{\lambda\Delta\mathcal{M}\big(1-\mathcal{M}\big)}{\big|\mathcal{M}+i\big(1-\mathcal{M}\big)\,\lambda\big|^2}+i\frac{\lambda\Delta\big(1-\mathcal{M}\big)^2}{\big|\mathcal{M}+i\big(1-\mathcal{M}\big)\,\lambda\big|^2}\right)\\
&= \operatorname{Arctan}\left(\frac{\mathcal{M}-1}{\mathcal{M}}\right)\\
&\begin{cases} = -\frac{\pi}{2} & \text{when } \mathcal{M} \to 0+ \\ \in \Big(-\frac{\pi}{2}, 0\Big) & \text{when } \quad 0<\mathcal{M}<1/(r-v) \end{cases} . \qquad (20)
\end{aligned}
$$

But, the result composed by (20) is contradiction with the inequality of the related theorem, *which* is the result given by (13). To say it simply that there is *no* a point z_0 in $\mathbb{U}$ satisfying the condition in (19). Thus, it also shows that the inequality:

$$\Re e\Big(\Omega(z)\Big) > 0$$

must be satisfied for all $z \in \mathbb{U}$. Thusly, the implicit form defined in (17) immediately follows that the inequality:

$$
\begin{aligned}
\Re e\Big(\mathbf{D}^{v}_{\phi}\{\Upsilon\}(z)\Big) &= \Re e\Big((r-v)\big[\mathcal{M}+\big(1-\mathcal{M}\big)\Omega(z)\big]\Big)\\
&= (r-v)\Big[\mathcal{M}+\big(1-\mathcal{M}\big)\Re e\Big(\Omega(z)\Big)\Big]\\
&> (r-v)\mathcal{M},
\end{aligned}
$$

which also is the inequality given by (14). Thus, this finishes the desired proof.

As we particularly highlighted in the first part, the main purpose of this research has been to reach a very comprehensive fundamental conclusion by using various derivative operators. Exactly so it has been done, and theorem 1 has been presented and then proven. Naturally, of course, there will be possible to present various comprehensive conclusions (*or* recommendations) regarding many specific consequences of the main result generated here. Suggestions to be presented will be possible to get them by considering the appropriate selection of all parameters mentioned in the first section. Although determining all those possible situations is a long task, we want to realize a number of our suggestions as certain implications.

As the first implication of this investigation, under the conditions given by (4) and also in the light of the definition (6) constituted by certain combining of the elementary operators given by (7), for any multivalently analytic functions $\Upsilon := \Upsilon(z)$ like the form in (1), the main result, namely, Theorem 1 can then be rearranged by the following implication, *which* is Proposition 1 (just below).

Proposition 1. *Under the conditions of the definitions given by* (2), (3) *and* (5), *and also for the suitable values of the relevant parameters in* (4), *for any functions* $\Upsilon := \Upsilon(z)$ *having the form given in* (1), *if*

$$\operatorname{Arg}\left\{\frac{\mathcal{D}_{\phi}^{v}\{\Upsilon\}(z)}{\frac{d}{dz}\Big(\mathcal{D}_{\phi}^{v}\{\Upsilon\}(z)\Big)}\frac{d}{dz}\left(\frac{z\frac{d}{dz}\Big(\mathcal{D}_{\phi}^{v}\{\Upsilon\}(z)\Big)}{\mathcal{D}_{\phi}^{v}\{\Upsilon\}(z)}\right)\right\} \notin \Big[-\frac{\pi}{2},0\Big)$$

is provided,

$$\Re e\left\{\frac{z\frac{d}{dz}\Big(\mathcal{D}_{\phi}^{n}\{\Upsilon\}(z)\Big)}{\mathcal{D}_{\phi}^{v}\{\Upsilon\}(z)}\right\} > (r-v)\mathcal{M}$$

is also provided under the conditions composed of (15).

As the second-extensive implication of this investigation, under the conditions given by

$$v<r \quad , \quad r\in\mathbb{N} \quad , \quad v\in\mathbb{N}_0 \quad \text{and} \quad z\in\mathbb{U}, \tag{21}$$

and also by taking the value of the parameter ϕ as $\phi := 1$, for an analytic function $\Upsilon := \Upsilon(z)$ like the form given by (1), the mentioned connection between the elementary operators given by

$$\Upsilon^{(v)}(z) \quad \text{and} \quad \mathcal{D}_{\phi}^{v}\{\Upsilon\}(z) \tag{22}$$

immediately requires to the following relations given by

$$\mathcal{D}_{1}^{v}\{\Upsilon\}(z) \equiv \Upsilon^{(v)}(z)$$

and

$$\mathbf{D}_{1}^{v}\{\Upsilon\}(z) \equiv \frac{z\frac{d}{dz}\Big(\Upsilon^{(v)}(z)\Big)}{\Upsilon^{(v)}(z)} \equiv \frac{z\,\Upsilon^{(v+1)}(z)}{\Upsilon^{(v)}(z)},$$

where, of course, $\Upsilon^{(v)}(z) \neq 0$ for all $z \in \mathbb{U}$.

In consideration of these relationships just above, naturally, Theorem 1 (*or*, Proposition 1) the presents us the following-extensive implication, *which* is Proposition 2 (just below).

Proposition 2. *Under the conditions of the definitions given by* (1), (2), (3) *and* (5), *and also for the agreeable values of the mentioned parameters in* (4), *for any multivalent functions* $\Upsilon := \Upsilon(z)$ *given by* (1), *if*

$$\operatorname{Arg}\left\{\frac{\Upsilon^{(v)}(z)}{\Upsilon^{(v+1)}(z)}\frac{d}{dz}\left(\frac{z\,\Upsilon^{(v+1)}(z)}{\Upsilon^{(v)}(z)}\right)\right\} \notin \Big[-\frac{\pi}{2},0\Big)$$

is satisfied,

$$\Re e\left(\frac{z\,\Upsilon^{(v+1)}(z)}{\Upsilon^{(v)}(z)}\right) > (r-v)\mathcal{M}$$

is also satisfied under the conditions comprised of (15).

As the third-extensive implication of this investigation, under the conditions given by (21) and also by taking considering the value of the parameter ϕ as $\phi := 0,$ for an analytic function $\Upsilon := \Upsilon(z)$ like the form given by (1), the connection between the operators given by (22) immediately yields to the following relations given by

$$\mathcal{D}_0^v\{\Upsilon\}(z) \equiv z\,\Upsilon^{(v+1)}(z)$$

and

$$\begin{aligned}\mathbf{D}_0^v\{\Upsilon\}(z) &\equiv 1 + \frac{z\frac{d}{dz}\Big(\Upsilon^{(v+1)}(z)\Big)}{\Upsilon^{(v+1)}(z)} \\ &\equiv 1 + \frac{z\,\Upsilon^{(v+2)}(z)}{\Upsilon^{(v+1)}(z)}\,,\end{aligned}$$

where, of course, *where* $\Upsilon^{(v+1)}(z) \neq 0$ for all $z \in \mathbb{U}$.

Through the instrumentality of those relations given just above, accordingly, Theorem 1 (*or*, Proposition 1) also presents us the following-extensive implication, *which* is Proposition 3 (just below).

Proposition 3. *Under the condition of the definition given in* (3), *and also for the acceptable values of the concerned parameters in* (4), *for any multivalently regular functions* $\Upsilon := \Upsilon(z)$ *given by the form in* (1), *if*

$$\mathrm{Arg}\left\{\frac{z\,\Upsilon^{(v+1)}(z)}{\Upsilon^{(v+1)}(z) + z\,\Upsilon^{(v+2)}(z)}\,\frac{d}{dz}\left(\frac{z\,\Upsilon^{(v+2)}(z)}{\Upsilon^{(v+1)}(z)}\right)\right\} \not\in \Big[-\frac{\pi}{2}, 0\Big)$$

is ensured,

$$1 + \Re e\left(\frac{z\,\Upsilon^{(v+2)}(z)}{\Upsilon^{(v+1)}(z)}\right) > (r-v)\mathcal{M}$$

is then ensured under the conditions constituted by (15).

As the last-extensive implication of this investigation, by the help of the relation stated by (8), *or*, equivalently, by taking into account the value of v as $v := 0$ in Theorem 1 (of course, *or*, in Proposition 1), the extensive implication, *which* specially consists of various geometric properties of (multivalently) analytic functions in the open disc $\mathbb{U}$ (see [6, 9]), can easily been established by the following proposition.

Proposition 4. *For a multivalently regular function* $\Upsilon := \Upsilon(z)$ *being of the form in* (1), *if*

$$\mathrm{Arg}\left\{\frac{\frac{d}{dz}\left(\frac{z\,\Upsilon'(z)+(1-\phi)\,z^2\,\Upsilon''(z)}{\phi\,\Upsilon(z)+(1-\phi)\,z\,\Upsilon'(z)}\right)}{\frac{\Upsilon'(z)+(1-\phi)\,z\,\Upsilon''(z)}{\phi\,\Upsilon(z)+(1-\phi)\,z\,\Upsilon'(z)}}\right\} \not\in \Big[-\frac{\pi}{2}, 0\Big)$$

is true,

$$\Re e\left(\frac{z\,\Upsilon'(z)+(1-\phi)\,z^2\,\Upsilon''(z)}{\phi\,\Upsilon(z)+(1-\phi)\,z\,\Upsilon'(z)}\right)>r\mathcal{M}$$

is also true, where

$$0\le\phi\le 1\quad,\quad r\in\mathbb{N}\quad,\quad 0\le\mathcal{M}<\frac{1}{r}\quad\text{and}\quad z\in\mathbb{U}.$$

As concluding remarks, only four of our main result has been constituted in the forms determined by propositions. It also is clear that all of them are related to multivalently functions. In the same time, some of those are also related to multivalently regular functions (like (1)) and some of the others are also related normalized regular function (like (1) with $p:=1$). Of course, some special results highlighted in the mentioned theorem (*or* propositions), as presented above, may be revealed by appropriate selection of parameters. We want to bring these special results to the attention of researchers. Specially, for the related researchers, we offer firstly to run through various type relationships, determined by certain multivalently regular functions with different type coefficients, between our main results and some of the earlier results given by the papers in the references and then to focus on some of them again.

In addition, of course, by making use of various package programs, it is of course possible to illustrate some (more) special cases of the mentioned-special results as their geometrical (*or* analytical) properties, *which* consist of graphics in the forms 2D and 3D in the related complex planes. We also present the creation of the graphical dimension of these special explanations to the attention of interested researchers.

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