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Picture Fuzzy Incidence Graphs with Application

Irfan Nazeer
Department of Mathematics,
University of management and technology, Lahore 54770, Pakistan,
Email: irfannazir779@gamil.com

Tabasam Rashid
Department of Mathematics,
University of management and technology, Lahore 54770, Pakistan,
Email: tabasam.rashid@umt.edu.pk

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Abstract.: In this research article, we initiate the novel idea of picture fuzzy incidence graphs (PFIGs). We explain some innovative notions comprising of picture fuzzy cut-vertices, picture fuzzy bridges, picture fuzzy incidence cutpairs, and picture fuzzy incidence cut-vertices. Some rudimentary theorems and essential results are also examined in the PFIGs. Further, we determine the different concepts together with the order, size, and certain kinds of degrees in PFIG. A new type of PFIGs namely, complete picture fuzzy incidence graphs (CPFIGs) and complement of (PFIGs) are also furnished. A comparative analysis of PFIGs with fuzzy incidence graphs is also presented. Finally, an application of PFIGs in the control of illegal transportation of people from India to America is provided.

AMS (MOS) Subject Classification Codes: 05C40; 05C70; 05C72
Key Words: Fuzzy set, Fuzzy graph, Fuzzy incidence graph, Picture fuzzy graph.

1. Introduction and Preliminaries

Zadeh [50] was the first who gave the idea of fuzzy sets (FSs), which has unlocked the new perspective for the researchers. FSs theory becomes a sturdy area in multiple disciplines including mathematics, computer science, and signal processing. FSs commonly manifest vagueness and enigma in routine life problems. A great number of experts have concentrated on the extensions of FSs and their uses in daily life. But FSs were not without flaws, they only talked about the membership function and missed the non-membership function which is available in intuitionistic fuzzy sets (IFSs) developed by Atanassov [12]. IFSs is one of the paramount extension of FSs and are more convenient and reliable as compared to FSs due to its additional non-membership component. From the last few decades, the IFSs have gained more attention in different areas of research. Different uses of IFSs in different areas of life can be seen [17, 28, 40, 48].

A graph is an easy way of expressing information including an association between entities. The entities are shown by vertices and relations by edges. In different problems, we get incomplete information about the problem. So there is blurriness in the explanation of the entities or their relationship or both. To tackle this form of problem we need to design a fuzzy graph (FG) model. The fundamental idea of FGs was provided by Rosenfeld [37], after 10 years of Zadeh's outstanding paper on FSs. FGs have plenty of uses in different fields like telecommunication, medical diagnosis, and social network but they fail to provide information about the impact of a vertex on an edge. The complement, automorphism groups, of FGs was provided by Sunitha and Vijayakumar [44]. Bhutani et al. [13] described fuzzy end nodes along with various types of properties. Bhutani and Rosenfeld [14] characterized strong arcs in FGs and after them, classification of arcs in FGs such as an α-strong, β-strong, and δ-arcs initiated by Mathew and Sunitha [30]. Size and order analogous to crisp graphs in FGs were examined by Gani and Ahamed [25]. Akram et al. [5] discussed spherical FGs with application to decision-making. Al-Hawary[7] proposed the notion of complete FG. For more detail and impressive work on FGs, we may refer to the reader [8, **?**, 11]. Habib et al. [27] presented q-rung orthopair fuzzy competition graphs with application in the soil ecosystem. Akram [1] has talked about m-polar FGs and their related characteristics. Mordeson and Nair [32] have talked about fuzzy hypergraphs.
Shannon and Atanassov [39] discussed intuitionistic fuzzy relations and intuitionistic fuzzy graphs (IFGs). The notion of how to find the degree, order, and size in IFGs was presented by Gani and Begum [26]. Al-Hawary and Hourani [9] discussed on IFGs. Parvathi and Karunambigai [34] analyzed certain components in IFGs. Parvathi et al. [35] discussed various kinds of operations in IFGs including, complement, union, and composition along with their characteristics. Sahoo and Pal [38] demonstrated some different kinds of products comprising direct product, semi-direct product, and investigated a variety of fascinating properties in IFGs.
As IFSs can consider membership and non-membership degrees but they do not talk about the neutral degree this drawback of unavailability of the neutral degree in IFSs encouraged Cuong and Kreinovich [20] to propose the notion of picture fuzzy sets (PFSs) which is an improved version of FSs and IFSs. PFSs responded to the human's opinions which comprise more than two responses such as, yes, no, refusal, and neutral, and are more effective than IFSs to handle uncertain problems related to real life. Casting a vote is a magnificent example of such a situations because the voters can be split up into four groups such as vote for, vote neutral, vote against, and vote refusal. The PFSs are narrated by three ingredients, the degree of positive membership (P_M), neutral membership (N_M), and negative membership (n_M) of an element such that $0 \leq P_M + N_M + n_M \leq 1$. Since PFSs are appropriate for apprehending imprecise, uncertain, and inconsistent information, therefore they can be applied to many decision-making processes such as financial forecasting and estimating risks in business.
The latest developments of PFSs incorporated: Cuong et. al [21] explained an innovative picture fuzzy negator on PFSs and some De Morgan triples, Wang et al. [47] initiated some operational laws of PFSs and introduced different geometric operations along with their properties. Cuong and Hai [18] have presented some fuzzy logic operators such as conjunctions, disjunctions, and implications for PFSs. Cuong [19] has described a variety of properties of PFS. Different features of compositions of picture fuzzy relations were given by Phong et al. [36] and interlinked these ideas for medical diagnosis. Zuo et al. [51] developed picture fuzzy graphs (PFGs) and discussed their various types. Xiao et al. [49] explained regular PFG along with its applications in communication networks. Akram

et al. [2, 4] talked about the non-identical decision-making model under complex picture fuzzy aggregation operators and discussed trapezoidal picture fuzzy numbers. Akram and Habib [3] introduced q-rung PFGs. For a comprehensive study on PFSs and PFGs we may refer to the reader [6, 15, 16, 22, 41, 42, 43, 45, 46].

As discussed earlier, there was a lack in FGs, these were not helpful to provide detail about the impact of a vertex on an edge therefore, Dinesh [23] introduced the new notion in graph theory named fuzzy incidence graphs (FIGs). FIGs are useful and provide information about the impact of a vertex on an edge. For example, if vertices show different residence societies and edges show roads joining these residence societies, we can have a FG expressing the extent of traffic from one society to another. The society has the maximum number of residents will have maximum ramps in society. So, if c and d are two societies and cd is a road joining them then (c, cd) could express the ramp system from the road cd to the society c. In the case of an unweighted graph, c and d both will have an influence of 1 on cd. In a directed graph, the influence of c on cd represented by (c, cd) is 1 whereas (d, cd) is 0. This idea can be generalized by FIGs. FIGs talk about the influence of a vertex on an edge. Connectivity ideas like cut-vertex, bridge, and incidence cut pair in FIGs given by Mathew and Mordeson [29]. After them, Malik et al. [31] use these said graphs in human trafficking. Fang et al. [24] provided the formula to calculate the connectivity and Wiener index of FIG. Nazeer et al. [33] introduced intuitionistic fuzzy incidence graphs (IFIGs)and presented an application of product on IFIGs in the textile industry.

FIGs are silent about the degree of P_M, N_M and n_M of an element but PFIGs have contained this feature of P_M, N_M and n_M of an element. This motivates us to propose the idea of PFIGs. In graph theory, edge exploration is not prime as all edges are strong [14]. But in PFIGs it is crucial to recognize the nature of edges and no such analysis on edges is present in the literature. Depending on the strength of an edge, we categorize edges into three different kinds namely $\alpha - strong$, $\beta - strong$, and $\delta - edge$. The analysis of kinds of edges investigates the structure of a PFIG so that the notions like picture fuzzy bridge (PFB), picture fuzzy incidence bridge (PFIB), picture fuzzy incidence cutpair (PFICP), picture fuzzy block (PFBL), picture fuzzy incidence block (PFIBL) and CPFIG, etc. can be studied in detail. We also explore the different kinds of degree, order, and size of PFIGs and compare the interrelation among degree, order, and size of PFIGs. Our work will open a new door for researchers for a comprehensive study of PFIGs. The work of this paper is as follows: section 1 carries the fundamental ideas and terminologies of FIGs to comprehend PFIGs. In section 2 we propose the definition of PFIG and its various properties. In section 3 we define different types of degrees in PFIG, order, and size of PFIG. We also define a CPFIG and a complement of PFIG. Section 4 contains an application to overcome the illegal transfer of people from India to America with the help of PFIG. In section 5 a comparison of PFIGs with the previously existing FIGs is provided. Section 6 carries a conclusion and the directions for future work.

Definition 1.1. [51] *Let W be an IFS. Then $W \in X$ is defined as*

$$W = \{x, \mu_W(x), \phi_W(x) \mid x \in X\},$$

where $\mu_W(x), \phi_W(x) \in [0, 1]$ with $0 \leq \mu_W(x) + \phi_W(x) \leq 1$. The $\mu_W(x)$ expresses the membership degree and $\phi_W(x)$ expresses non-membership degree.

Definition 1.2. [51] *Consider W is a PFS, W in X is defined by*

$$W = \{x, \mu_W(x), \phi_W(x), \varphi_W(x) \mid x \in X\},$$

where $\mu_W(x), \phi_W(x), \varphi_W(x) \in [0, 1]$ with $0 \leq \mu_W(x) + \phi_W(x) + \varphi_W(x) \leq 1$. The $\mu_W(x)$ expresses the positive membership degree, $\phi_W(x)$ expresses neutral membership *degree and $\varphi_W(x)$ shows negative membership degree of the element x in W. The refusal membership degree is defined by $\pi_W(x) = 1 - \mu_W(x) - \phi_W(x) - \varphi_W(x)$.*

Definition 1.3. [51] *Let $\dot{G} = (V, E)$ be a graph. A pair $\ddot{G} = (Q, R)$ is said to be a PFG on $\dot{G}$ where $Q = (\mu_Q, \phi_Q, \varphi_Q)$ is a PFS on V and $R = (\mu_R, \phi_R, \varphi_R)$ is a PFS on $E \subseteq V \times V$ such that for each edge $ef \in E$. $\mu_R(e, f) \leq \wedge(\mu_Q(e), \mu_Q(f)), \phi_R(e, f) \leq \wedge(\phi_Q(e), \phi_Q(f)), \varphi_R(e, f) \leq \vee(\varphi_Q(e), \varphi_Q(f))$.*

In this paper minimum and maximum operators are represented by $\wedge$ and $\vee$ respectively.

Definition 1.4. [23] *Let $\dot{G} = (V, E)$ be a graph with non empty vertex set V. Then, $G' = (V, E, I)$ is said to be an incidence graph (IG) where $I \subseteq V \times E$. The members of I are known as pairs or incidence pairs. A FIG of IG, $G' = (V, E, I)$ is $\tilde{G} = (\rho, \varrho, \sigma)$, where ρ, ϱ and σ are fuzzy subset of V, $V \times V$ and I respectively such that $\sigma(e, ef) \leq \wedge\{\rho(e), \varrho(ef)\}$, for all $e \in V, ef \in E$. Two vertices e and f in a FIG are said to be connected if there exists a path of the form $e, (e, ef), ef, (f, ef), f$ between them. Vertex e and an edge ef are connected if there is a path such that $e, (e, ef), ef$ between them.*

Two elements $Q = (\mu_Q, \phi_Q, \varphi_Q)$ and $R = (\mu_R, \phi_R, \varphi_R)$ of PFSs are comparable such that, $(\mu_Q(x), \phi_Q(x), \varphi_Q(x)) < (\mu_R(x), \phi_R(x), \varphi_R(x)) \Rightarrow \mu_Q(x) < \mu_R(x), \phi_Q(x) < \phi_R(x)$ and $\varphi_Q(x) \geq \varphi_R(x)$.

2. Picture fuzzy incidence graphs

In this section, we define PFIGs and explain them with different types of examples. Throughout in this paper, we express an IG by $G' = (V, E, I)$ and a PFIG by $\widehat{G} = (K, L, M)$.

Definition 2.1. *A PFIG, $\widehat{G} = (K, L, M)$ of an IG, $G' = (V, E, I)$ is defined as*

(1) *K is a PFS on V.*
(2) *L is a PFS on $E \subseteq V \times V$.*
(3) *M is a PFS on $V \times E$ such that*

$$P_M(e, ef) \leq min\{P_K(e), P_L(ef)\},$$
$$N_M(e, ef) \leq min\{N_K(e), N_L(ef)\},$$
$$n_M(e, ef) \leq max\{n_K(e), n_L(ef)\} \forall e \in V, ef \in E.$$

where $P_M(e, ef), N_M(e, ef), n_M(e, ef) \in [0, 1]$ show the positive membership degree, neutral membership degree and negative membership degree of pairs respectively.

Definition 2.2. *Let $\widehat{G} = (K, L, M)$ be a PFIG. Then its support is $G^* = (K^*, L^*, M^*)$ where*
$K^* =$ support of $K = \{e \in V : P_K(e) > 0, N_K(e) > 0, n_K(e) > 0\}$
$L^* =$ support of $L = \{ef \in E : P_L(ef) > 0, N_L(ef) > 0, n_L(ef) > 0\}$
$M^* =$ support of $M = \{(e, ef) \in I : P_M(e, ef) > 0, N_M(e, ef) > 0, n_M(e, ef) > 0\}$

Definition 2.3. *A PFIG, $\widehat{G} = (K, L, M)$ is a cycle iff, $G^* = (K^*, L^*, M^*)$ is a cycle.*

Definition 2.4. *Let $\widehat{G} = (K, L, M)$ be a PFIG. Then $\widehat{G} = (K, L, M)$ is called to be a picture fuzzy cycle (PFC) iff, $G^* = (K^*, L^*, M^*)$ is a cycle and there is no single $ef \in L^*$*

of the type
$P_L(ef) = \wedge\{P_L(uv) : uv \in L^*\}$.
$N_L(ef) = \wedge\{N_L(uv) : uv \in L^*\}$.
$n_L(ef) = \vee\{n_L(uv) : uv \in L^*\}$.

Definition 2.5. *The PFIG, $\widehat{G} = (K, L, M)$ is a picture fuzzy incidence cycle (PFIC) iff it is a PFC and there exists no single pair $(e, ef) \in M^*$, such that*
$P_L(e, ef) = \wedge\{P_M(u, uv) : (u, uv) \in M^*\}$.
$N_L(e, ef) = \wedge\{N_M(u, uv) : (u, uv) \in M^*\}$.
$n_L(e, ef) = \vee\{n_M(u, uv) : (u, uv) \in M^*\}$.

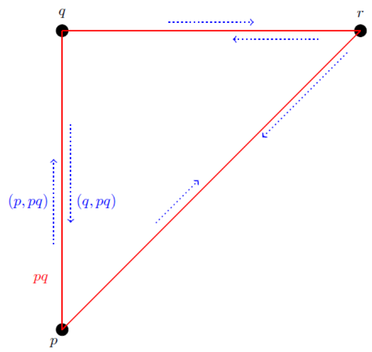

FIGURE 1. Incidence graph

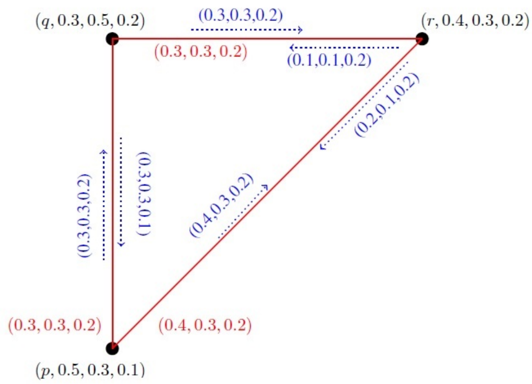

FIGURE 2. PFIG and PFIC

Example 2.6. *An IG and its associated PFIG is shown in Figure 1 and Figure 2 respectively. In Figure 2, $\widehat{G}$ is also a PFC since $G^* = (K^*, L^*, M^*)$ is a PFC.*

$P_L(pq) = \wedge\{P_L(pq), P_L(qr), P_L(rp)\} = 0.3.$
$N_L(pq) = \wedge\{N_L(pq), N_L(qr), N_L(rp)\} = 0.3.$
$n_L(pq) = \vee\{n_L(pq), n_L(qr), n_L(rp)\} = 0.2.$
and
$P_L(qr) = \wedge\{P_L(pq), P_L(qr), P_L(rp)\} = 0.3.$
$N_L(qr) = \wedge\{N_L(pq), N_L(qr), N_L(rp)\} = 0.3.$
$n_L(qr) = \vee\{n_L(pq), n_L(qr), n_L(rp)\} = 0.2.$
So, $\widehat{G}$ is a PFC.
Also, $\widehat{G}$ is a PFIC because it has more than one incidence pair (p, pq) and (q, qr) such that
$P_L(p, pq) = \wedge\{P_M(u, uv) : (u, uv) \in M^*\} = 0.3.$
$N_L(p, pq) = \wedge\{N_M(u, uv) : (u, uv) \in M^*\} = 0.3.$
$n_L(p, pq) = \vee\{n_M(u, uv) : (u, uv) \in M^*\} = 0.2.$
and
$P_L(q, qr) = \wedge\{P_M(u, uv) : (u, uv) \in M^*\} = 0.3.$
$N_L(q, qr) = \wedge\{N_M(u, uv) : (u, uv) \in M^*\} = 0.3.$
$n_L(q, qr) = \vee\{n_M(u, uv) : (u, uv) \in M^*\} = 0.2.$

Now, we are going to initiate the idea of walk, trail, path, connectedness and incidence connectedness (ICN) in PFIG. These ideas will help us to study about PFIG in detail.

Definition 2.7. *If $ef \in L^*$ then ef is named as an edge of the PFIG $\widehat{G} = (K, L, M)$ and if $(e, ef), (f, ef) \in M^*$ then (e, ef) and (f, ef) are said to be pairs of $\widehat{G} = (K, L, M)$.*

Definition 2.8. *A sequence*
$P : t_0, (t_0, t_0t_1), t_0t_1, (t_1, t_0t_1), t_1, (t_1, t_1t_2), t_1t_2, (t_2, t_1t_2), t_2, ..., t_{n-1}, (t_{n-1}t_{n-1}t_n),$
$t_{n-1}t_n, (t_n, t_{n-1}t_n), t_n.$
in $\widehat{G}$ is named as walk. A walk is closed if $t_0 = t_n$. If all the edges in P are different then it is said to be a trail and if all pairs are not same then it is known as an incidence trail. If all vertices are dissimilar then P is called a path. A path P is said to be a cycle if the starting and ending vertex of P is similar. Any two vertices in $\widehat{G}$ are connected if there exists a path between them.

Example 2.9. *In Figure* 2
$P_1 : p, (p, pq), pq, (q, pq), q, (q, qr), qr, (r, qr), p$ *is a closed walk because its starting and ending vertex is same but it is not a path because all vertices are not distinct. P_1 is a trail as well as an incidence trail. $P_2 : p, (p, pq), pq, (q, pq), q$ is a walk, path, trail and an incidence trail.*

Definition 2.10. *Assume $\widehat{G} = (K, L, M)$ is a PFIG. Then, $\widehat{H} = (X, Y, Z)$ is a picture fuzzy incidence subgraph (PFIS) of $\widehat{G}$ if $X \subseteq K, Y \subseteq L$ and $Z \subseteq M$. A PFIS $\widehat{H}$ is called spanning subgraph if $K^* = X^*$.*

Definition 2.11. *Let $\widehat{G} = (K, L, M)$ be a PFIG. Then the strength of path P is represented by $C(P) = (c_1, c_2, c_3)$ where,*
$c_1 = \wedge\{P_L(ef) : ef \in P\}$,
$c_2 = \wedge\{N_L(ef) : ef \in P\}$,
$c_3 = \vee\{n_L(ef) : ef \in P\}$.
In similar way, the incidence strength (I_s) of P in a PFIG $\widehat{G} = (K, L, M)$ is expressed by $I_s(P) = (ic_1, ic_2, ic_3)$ where,

$ic_1 = \wedge\{P_M(e, ef) : (e, ef) \in P\}$,
$ic_2 = \wedge\{N_M(e, ef) : (e, ef) \in P\}$,
$ic_3 = \vee\{n_M(e, ef) : (e, ef) \in P\}$.

Example 2.12. *Let $G' = (V, E, I)$ be an IG given in Figure 3 and its associated PFIG as shown in Figure 4. Then, $P : p, (p, pr), pr, (r, pr), r, (r, rs), rs, (s, rs), s$ is a path in $\widehat{G}$. The $C(P)$ $P = p - r - s$ can be calculated as*
$c_1 = \wedge\{P_L(ps) : ps \in P\} = \wedge\{0.1, 0.2\} = 0.1$,
$c_2 = \wedge\{N_L(ps) : ps \in P\} = \wedge\{0.1, 0.1\} = 0.1$,
$c_3 = \vee\{n_L(ps) : ps \in P\} = \vee\{0.4, 0.3\} = 0.4$.
Therefore, the $C(P)$ $P = p - r - s = (0.1, 0.1, 0.4)$. In the same way, the $I_s(P)$ $P = p - r - s$ can be find as
$ic_1 = \wedge\{P_M(p, s) : (p, pr), (r, pr), (r, rs), (s, rs)\} = \wedge\{0.1, 0.05, 0.2, 0.1\} = 0.05$,
$ic_2 = \wedge\{N_M(p, s) : (p, pr), (r, pr), (r, rs), (s, rs)\} = \wedge\{0.1, 0.1, 0.1, 0.1\} = 0.1$,
$ic_3 = \vee\{n_M(p, s) : (p, pr), (r, pr), (r, rs), (s, rs)\} = \vee\{0.4, 0.4, 0.3, 0.3\} = 0.4$.
Therefore, the $I_s(P)$ $P = p - r - s = (0.05, 0.1, 0.4)$.

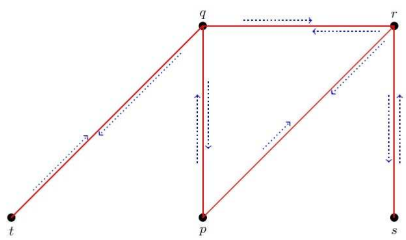

FIGURE 3. An IG

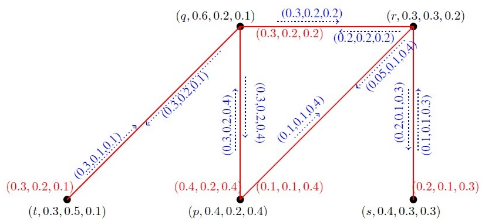

FIGURE 4. A PFIG

Definition 2.13. *In a PFIG, $\widehat{G} = (K, L, M)$ the largest strength of the path from e to f where $e, f \in K^* \cup L^*$ is the largest strength of all the paths from e to f.*

$C^{\infty}(e,f) = \vee\{C(P_1), C(P_2), C(P_3), ...\}$
$C^{\infty}(e,f) = (c_1^{\infty}, c_2^{\infty}, c_3^{\infty})$
$C^{\infty}(e,f) = (\vee(c_{11}, c_{12}, c_{13}, ...), \vee(c_{21}, c_{22}, c_{23}, ...), \wedge(c_{31}, c_{32}, c_{33}, ...))$.
$C^{\infty}(e,f)$ is named as the connectedness between e and f.
In similar way, the largest I_s of the path from e to f where $e, f \in K^* \cup L^*$ is the largest I_s of all paths from e to f.
$I_s^{\infty}(e,f) = \vee\{I_s(P_1), I_s(P_2), I_s(P_3), ...\}$
$I_s^{\infty}(e,f) = (ic_1^{\infty}, ic_2^{\infty}, ic_3^{\infty})$
$I_s^{\infty}(e,f) = (\vee(ic_{11}, ic_{12}, ic_{13}, ...), \vee(ic_{21}, ic_{22}, ic_{23}, ...), \wedge(ic_{31}, ic_{32}, ic_{33}, ...))$.
$I_s^{\infty}(e,f)$ is called an ICN between e and f.

Example 2.14. *A PFIG shown in Figure 4 has two possible paths from t to s.*
$P_1 : t, (t,tq), tq, (q,tq), q, (q,qp), qp, (p,qp), p, (p,pr), pr,$
$(r,pr), r, (r,rs), rs, (s,rs), s$. $P_2 : t, (t,tq), tq, (q,tq), q, (q,qr), qr, (r,qr), r, (r,rs), rs, (s,rs), s$.
Now,
$I_s(P_1) = (c_{11}, c_{21}, c_{31}) = (0.05, 0.1, 0.4)$.
$I_s(P_2) = (c_{12}, c_{22}, c_{32}) = (0.1, 0.1, 0.3)$.
The maximum I_s of the path from t to s is obtained as given below.
$I_s^{\infty}(t,s) = \vee\{I_s(P_1), I_s(P_2)\}$,
$I_s^{\infty}(t,s) = (\vee\{ic_{11}, ic_{12}\}, \vee\{ic_{21}, ic_{22}\}, \wedge\{ic_{31}, ic_{32}\})$,
$I_s^{\infty}(t,s) = (\vee\{0.05, 0.1\}, \vee\{0.1, 0.1\}, \wedge\{0.4, 0.3\})$,
$I_s^{\infty}(t,s) = (0.1, 0.1, 0.3)$.

Now, we are going to elaborate the notion of PFBs, PFIBs picture fuzzy cut-vertices and PFICP in PFIG.

Definition 2.15. *Let $\widehat{G} = (K, L, M)$ be a PFIG. Then an edge gh of $\widehat{G}$ is named as a bridge iff gh is a bridge in $G^* = (K^*, L^*, M^*)$ that is deleting of gh disjoins G^*.*
An edge gh is said to be a PFB if
$C'^{\infty}(e,f) < C^{\infty}(e,f)$ *for some* $e, f \in K^*$,
$(c_1'^{\infty}, c_2'^{\infty}, c_3'^{\infty}) < (c_1^{\infty}, c_2^{\infty}, c_3^{\infty})$
$\Rightarrow c_1'^{\infty} < c_1^{\infty}, c_2'^{\infty} < c_2^{\infty}, c_3'^{\infty} > c_3^{\infty}$.
Here, $C'^{\infty}(e,f)$ and $C^{\infty}(e,f)$ represents the connectedness between e and f in $G = \widehat{G} - \{gh\}$ and $\widehat{G}$ respectively.
An edge gh, is said to be PFIB if
$I_s'^{\infty}(e,f) < I_s^{\infty}(e,f)$ *for some* $e, f \in K^*$,
$(ic_1'^{\infty}, ic_2'^{\infty}, ic_3'^{\infty}) < (ic_1^{\infty}, ic_2^{\infty}, ic_3^{\infty})$
$\Rightarrow ic_1'^{\infty} < ic_1^{\infty}, ic_2'^{\infty} < ic_2^{\infty}, ic_3'^{\infty} > ic_3^{\infty}$.
Here, $I_s'^{\infty}(e,f)$ and $I_s^{\infty}(e,f)$ shows the ICN between e and f in $G = \widehat{G} - \{gh\}$ and $\widehat{G}$ respectively.

Definition 2.16. *Let $\widehat{G} = (K, L, M)$ be a PFIG. Then a vertex $i \neq e \neq f$ is said to be a cut-vertex in $\widehat{G}$ iff it is also a cut-vertex in $G^* = (K^*, L^*, M^*)$ means $G^* - \{i\}$ disconnects graph.*
A vertex i, in a PFIG is named as picture fuzzy cut-vertex if for any pair of vertices the condition given below is satisfied
$C'^{\infty}(e,f) < C^{\infty}(e,f)$ *for some* $e, f \in K^*$.

A vertex i in PFIG, $\widehat{G}$ is said to be picture fuzzy incidence cut-vertex if for any pair of vertices excluding i the given condition is satisfied
$I_s'^{\infty}(e,f) < I_s^{\infty}(e,f)$
Here, $I_s'^{\infty}(e,f)$ and $I_s^{\infty}(e,f)$ show the ICN between e and f of $G = \widehat{G} - i$ and $\widehat{G}$ respectively.

Definition 2.17. *Consider $\widehat{G} = (K, L, M)$ is a PFIG. A pair (e, ef) is named as a cutpair iff it is also a cutpair in $G^* = (K^*, L^*, M^*)$ means after deleting (e, ef) there exists no path between e and ef.*
Assume $\widehat{G} = (K, L, M)$ is a PFIG. A pair (e, ef) is said to be picture fuzzy cutpair if removing of (e, ef) lessen the connectedness between $e, ef \in K^* \cup L^*$, i.e.
$C'^{\infty}(e, ef) < C^{\infty}(e, ef)$,
Here, $C'^{\infty}(e, ef)$ and $C^{\infty}(e, ef)$ show the connectedness between e and ef of $G = \widehat{G} - \{(e, ef)\}$ and $\widehat{G}$ respectively.
A pair (e, ef) is named as PFICP if $I_s'^{\infty}(e, ef) < I_s^{\infty}(e, ef)$ for $e, ef \in K^* \cup L^*$. $I_s'^{\infty}(e, ef)$ and $I_s^{\infty}(e, ef)$ show ICN between e and ef of $G = \widehat{G} - \{(e, ef)\}$ and $\widehat{G}$ respectively.

Example 2.18. *Let $\widehat{G} = (K, L, M)$ be a PFIG as shown in Figure 4. Then qt and rs are bridges because after deleting these two edges the underlying graph will disconnect. qt, rs and pq are PFBs for $t, p \in K^*$, $C'^{\infty}(t, p) < C^{\infty}(t, p)$. Also they are PFIB because $I_s'^{\infty}(t, p) < I_s^{\infty}(t, p)$. The pairs $(t, tq), (q, tq), (r, rs)$ and (s, rs) are cutpairs, picture fuzzy cutpairs and PFICP and $(p, pq), (q, pq), (q, qr), (r, qr)$ are picture fuzzy cutpairs and PFICP but they are not cutpairs.*

Theorem 2.19. *Let $\widehat{G} = (K, L, M)$ be a PFIG. If ef is a PFB then it is not a weakest edge (WE) in any PFIC.*

Proof. Assume ef is a PFB and suppose, to contrary, that ef is a WE of PFIC. Then, this PFIC has a different path P_1, from e to f which not include the edge ef and $C(P_2)$ is less than or equal to $C(P_1)$, where P_2 is a path includes the edge ef. Therefore, deleting of ef from $\widehat{G}$ will not increase or decrease the connectedness between e and f which is a contradiction to our supposition. This proves that ef is not the WE of any PFIC. □

Theorem 2.20. *If (e, ef) is a PFICP, then (e, ef) is not the weakest pair (WP) in any PFIC.*

Proof. Assume (e, ef) is PFICP in $\widehat{G}$. We suppose to contrary that (e, ef) is a WP of a PFIC. Then this PFIC has a different path from e to ef having I_s larger than or equal to the path including (e, ef). Therefore, deletion of (e, ef) does not reduce or enhance the connectedness between e and ef this leads to contradicts our assumption that (e, ef) is a PFICP. This proves that (e, ef) is not a WP in any PFIC. □

Theorem 2.21. *Let $\widehat{G} = (K, L, M)$ be a PFIG. If ef is a PFB in $\widehat{G}$, then*
$C^{\infty}(e, f) = (c_1^{\infty}, c_2^{\infty}, c_3^{\infty}) = (P_L(ef), N_L(ef), n_L(ef))$.

Proof. Assume $\widehat{G}$ is a PFIG and ef is a PFB in $\widehat{G}$. Suppose to contrary that $C^{\infty}(e, f) > (P_L(ef), N_L(ef), n_L(ef))$. Then, there will be a path P_1 from e to f such that $C(P_1) > (P_L(ef), N_L(ef), n_L(ef))$ and $(P_L(xy), N_L(xy), n_L(xy)) > (P_L(ef), N_L(ef), n_L(ef))$, for all edges on P_1. Now, P_1 including ef makes a PFIC in which ef is the WE, this contradicts the fact that ef is a PFB. Therefore, $C^{\infty}(e, f) = (c_1^{\infty}, c_2^{\infty}, c_3^{\infty})$
$= (P_L(ef), N_L(ef), n_L(ef))$. □

Theorem 2.22. *If (e, ef) is a PFICP in a PFIG, $\widehat{G} = (K, L, M)$, then*
$I_s^{\infty}(e, ef) = (ic_1^{\infty}, ic_2^{\infty}, ic_3^{\infty}) = (P_M(e, ef), N_M(e, ef), n_M(e, ef))$.

Proof. Assume $\widehat{G}$ is a PFIG and (e, ef) is a PFICP in $\widehat{G}$. Suppose to contrary that $I_s^{\infty}(e, ef) > (P_M(e, ef), N_M(e, ef), n_M(e, ef))$. Then, there will be a path P_1 from e to ef such that $C(P_1) > (P_M(e, ef), N_M(e, ef), n_M(e, ef))$ and $(P_M(x, xy), N_M(x, xy), n_M(x, xy)) > (P_M(e, ef), N_M(e, ef), n_M(e, ef))$, for all pairs on P_1. Now, P_1 including (e, ef) forms a PFIC in which (e, ef) is the WP, this contradicts the fact that (e, ef) is a PFICP.
Therefore, $I_s^{\infty}(e, ef) = (ic_1^{\infty}, ic_2^{\infty}, ic_3^{\infty}) = (P_M(e, ef), N_M(e, ef), n_M(e, ef))$. □

Theorem 2.23. *Let $\widehat{G} = (K, L, M)$ be a PFIG and $G^* = (K^*, L^*, M^*)$ is a PFIC. Then an edge ef is a PFB of $\widehat{G}$ iff it is an edge common to two PFICP.*

Proof. Let ef be a PFB of $\widehat{G}$. Then there will vertices e and f with the ef edge lying on each path with largest I_s between e and f. In this way, there will be a unique path P_1 between e and f which carries a ef edge and has also the maximum I_s. Any pair on P_1 will be a PFICP, since the deleting of any of them will disjoin P_1 and decrease the I_s.
Conversely, consider an edge ef common to two PFICP (e, ef) and (f, ef). Thus these two PFICP are not the weakest PFICP of $\widehat{G}$. Now, $G^* = (K^*, L^*, M^*)$ being a PFIC, there exists two paths between any two vertices. Also, the path P_2 from e to f not contains (e, ef) and (f, ef) has low I_s than the path carrying them. This implies the path having largest I_s from e to f is $P_3 : e, (e, ef), ef, (f, ef), f$. Also, $C^{\infty}(e, f) = C(P_3) = (P_L(ef), N_L(ef), n_L(ef))$. This proves that ef is a PFB. □

Definition 2.24. *Let $\widehat{G} = (K, L, M)$ be a PFIG. Then a strong edge is defined as*
$C'^{\infty}(e, f) \leq (P_L(ef), N_L(ef), n_L(ef))$. *Where, $C'^{\infty}(e, f)$ shows the connectedness between e and f in $G = \widehat{G} - \{ef\}$. Particularly, it is named as an α-strong edge if* $C'^{\infty}(e, f) < (P_L(ef), N_L(ef), n_L(ef))$. *And β-strong if*
$C'^{\infty}(e, f) = (P_L(ef), N_L(ef), n_L(ef))$.

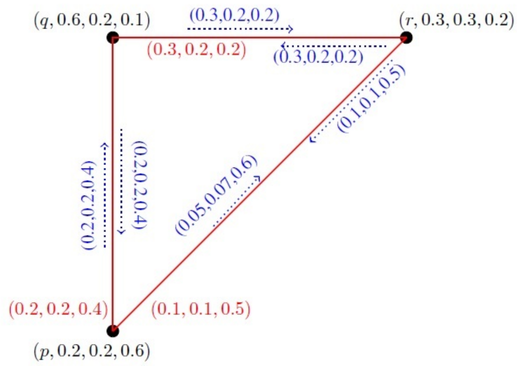

FIGURE 5. PFIG

Example 2.25. *Let $\widehat{G}$ be two graphs as shown in Figure 5 and Figure 6. Then in Figure 5 an edge qr is an $\alpha - strong$ because* $(P_L(qr), N_L(qr), (n_L(qr)) = (0.3, 0.2, 0.2) > C'^{\infty}(qr) = (0.1, 0.1, 0.5)$ *and in Figure 6 an edge pq is $\beta - strong$ because*
$(P_L(pq), N_L(pq), (n_L(pq)) = (0.3, 0.3, 0.2) = C'^{\infty}(pq) = (0.3, 0.3, 0.2)$.

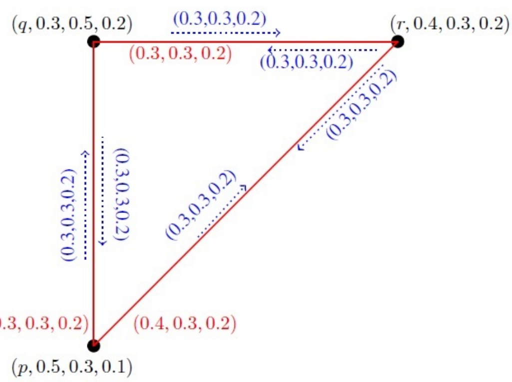

FIGURE 6. PFIG

Definition 2.26. *Assume $\widehat{G}$ is a PFIG. A pair (e, ef) in $\widehat{G}$ is strong pair if* $C'^{\infty}(e, ef) \leq (P_M(e, ef), N_M(e, ef), n_M(e, ef))$.
Where, $C'^{\infty}(e, ef)$ shows the connectedness between e and ef in $G = \widehat{G} - \{(e, ef)\}$. Particularly, it is named as an α-strong pair if $C'^{\infty}(e, ef) < (P_M(e, ef), N_M(e, ef), n_M(e, ef))$. and β-strong if $C'^{\infty}(e, ef) = (P_M(e, ef), N_M(e, ef), n_M(e, ef))$.

Example 2.27. *Let $\widehat{G}$ be two graphs as shown in Figure 5 and Figure 6. Then in Figure 5 pairs (r, qr) and (q, qr) are an $\alpha-strong$ because $(P_M(r, qr), N_M(r, qr), (n_M(r, qr)) = (0.3, 0.2, 0.2) > C'^{\infty}(r, qr) = (0.05, 0.07, 0.4)$ and $(P_M(q, qr), N_M(q, qr), (n_M(q, qr)) = (0.3, 0.2, 0.2) > C'^{\infty}(r, qr) = (0.05, 0.07, 0.4)$ also in Figure 6 pair (p, pq) is $\beta - strong$ because $(P_M(p, pq), N_M(p, pq), (n_L(p, pq)) = (0.3, 0.3, 0.2) = C'^{\infty}(p, pq) = (0.3, 0.3, 0.2)$.*

Remark 2.28. *When a pair is α-strong or β-strong. We simply call it a strong pair.*

Definition 2.29. *Assume $\widehat{G}$ is a PFIG. δ-edge is defined as* $C'^{\infty}(e, f) > (P_L(ef), N_L(ef), n_L(ef))$.
In similar way, δ pair is defined as $C'^{\infty}(e, ef) > (P_M(e, ef), N_M(e, ef), n_M(e, ef))$.

Example 2.30. *Let $\widehat{G}$ be a graph as shown in Figure 5. Then in Figure 5 an edge pr is a $\delta-edge$ because $(P_L(pr), N_L(pr), (n_L(pr)) = (0.1, 0.1, 0.5) < C'^{\infty}(pr) = (0.2, 0.2, 0.4)$ and pair (p, pr) is δ pair because*
$(P_M(p, pr), N_M(p, pr), (n_L(p, pr)) = (0.05, 0.07, 0.6) < C'^{\infty}(p, pr) = (0.1, 0.1, 0.5)$.

Theorem 2.31. *In a PFIG, every PFICP is a strong cutpair.*

Proof. Assume $\widehat{G} = (K, L, M)$ is a PFIG. Let $(e, ef) \in M^*$ be a PFICP. Then, by Definition 2.17. $I_s'^{\infty}(e, ef) < I_s^{\infty}(e, ef)$. Suppose on contrary, (e, ef) is not strong. Then, $I_s'^{\infty}(e, ef) > (P_M(e, ef), N_M(e, ef), n_M(e, ef))$. Assume P is the path from e to ef in $G = \widehat{G} - \{(e, ef)\}$ having maximum I_s. Then, P along with (e, ef) makes a PFIC in $\widehat{G}$. (e, ef) will be the WP in this PFIC but according to Theorem 2.20. it is impossible, because (e, ef) is a PFICP. This means that our supposition is not correct therefore (e, ef) is a strong cutpair. □

Theorem 2.32. *In a PFIG the pair (e, ef) is a PFICP iff it is α-strong.*

Proof. Assume (e, ef) is a PFICP in $\widehat{G}$. Then by definition of a PFICP, $I_s'^{\infty}(e, ef) < I_s^{\infty}(e, ef)$. Then, according to Theorem 2.22. $I_s'^{\infty}(e, ef) < (P_M(e, ef), N_M(e, ef), n_M(e, ef))$. This implies that $P : e, (e, ef), ef$ is the only path from e to ef which has the maximum I_s among all paths. Therefor, all remaining paths from e to ef will have lesser I_s. $I_s'^{\infty}(e, ef) < I_s^{\infty}(e, ef)$. Hence, (e, ef) is a PFICP. □

We are going to explain the idea of block, picture fuzzy block (PFBL) and picture fuzzy incidence block (PFIBL).

Definition 2.33. *Let $\widehat{G} = (K, L, M)$ be a PFIG. Then*

(1) *$\widehat{G}$ is named as a block if its underlying graph G^* does not contain a cut-vertices.*
(2) *$\widehat{G}$ is named as a PFBL if $\widehat{G}$ does not contain a picture fuzzy cut-vertices.*
(3) *$\widehat{G}$ is named as a PFIBL if $\widehat{G}$ does not contain a picture fuzzy incidence cut-vertices.*

Example 2.34. *Assume $\widehat{G} = (K, L, M)$ is a graph given in Figure 7. $\widehat{G}$ is a block since its underlying graph G^* does not has a cut-vertex because if we remove any vertex from G^*, it will remain connected i.e. there will be a path between any two vertices and it is also a PFIBL. In $\widehat{G}$, vertex p is a picture fuzzy cut-vertex because for pair of vertices q and r there are two paths namely (i) $P_1 = q - r$ and (ii) $P_2 = q - p - r$ the $C(P_1) = (0.1, 0.1, 0.2)$ and $C(P_2) = \{\wedge(0.2, 0.2), \wedge(0.2, 0.2), \vee(0.1, 0.1) = (0.2, 0.2, 0.1)\}$ then $C^{\infty}(q, r)$ of these two paths is $C^{\infty}(q, r) = \{\vee(0.1, 0.2), \vee(0.1, 0.2), \wedge(0.2, 0.1) = (0.2, 0.2, 0.1)\}$ and $C'^{\infty}(q, r)$ after removing p from graph is $C'^{\infty}(q, r)= (0.1, 0.1, 0.2)$ this implies $C'^{\infty}(q, r)= (0.1, 0.1, 0.2) < C^{\infty}(q, r) = (0.2, 0.2, 0.1)$. Therefore $\widehat{G}$ is not a PFBL.*

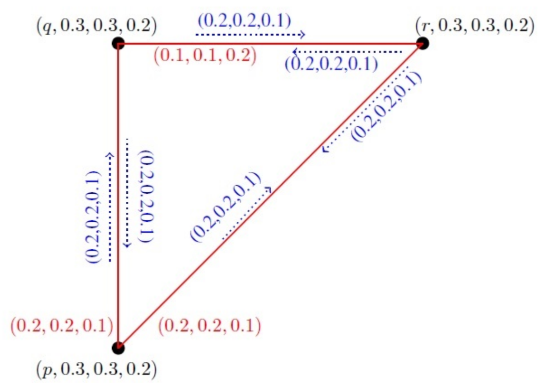

FIGURE 7. A PFIG

Theorem 2.35. *Let $\widehat{G} = (K, L, M)$ be a PFIBL. Then a pair (e, ef) in $\widehat{G}$ such that $(P_M(e, ef), N_M(e, ef), n_M(e, ef)) = (\vee P_M(x, xy), \vee N_M(x, xy), \wedge n_M(x, xy))$, for all $(x, xy) \in M^*$ is a strong pair.*

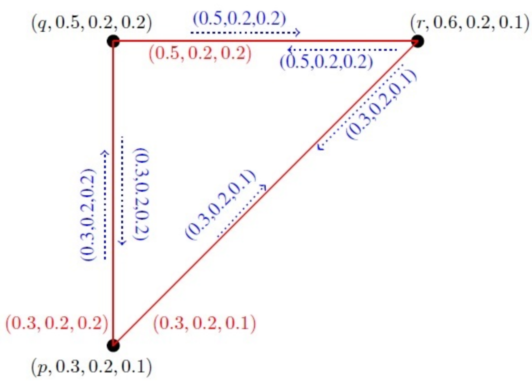

FIGURE 8. CPFIG

Proof. Assume $\widehat{G}$ is a PFIBL. According to the definition of PFIBL there will be no picture fuzzy incidence cut-vertices in $\widehat{G}$. Let $\widehat{G}$ has (e, ef) of the type
$(P_M(e, ef), N_M(e, ef), n_M(e, ef)) = (\vee P_M(x, xy), \vee N_M(x, xy), \wedge n_M(x, xy))$.
We show that (e, ef) is a strong pair by expressing that $(P_M(e, ef), N_M(e, ef), n_M(e, ef)) \geq I_s'^{\infty}(e, ef)$. The value of $(P_M(e, ef), N_M(e, ef), n_M(e, ef))$ will be greater than or equal to the I_s of any path P from e to ef. If (e, ef) is unique in $\widehat{G}$ with
$(P_M(e, ef), N_M(e, ef), n_M(e, ef)) = (\vee P_M(x, xy), \vee N_M(x, xy), \wedge n_M(x, xy))$
then the value of $(P_M(e, ef), N_M(e, ef), n_M(e, ef))$ will be greater than the I_s of all other paths from x to xy in $\widehat{G}$. Therefore, $(P_M(e, ef), N_M(e, ef), n_M(e, ef)) > I_s'^{\infty}(e, ef)$.
This shows that (e, ef) is an α-strong.
If (e, ef) is more than one than the greatest possible value for the I_s of any path $G = \widehat{G} - \{(e, ef)\} = (P_M(e, ef), N_M(e, ef), n_M(e, ef))$. This means there is a path e to ef with $(P_M(e, ef), N_M(e, ef), n_M(e, ef)) = I_s'^{\infty}(e, ef)$. This implies (e, ef) is β-strong. □

3. DEGREE, SIZE AND ORDER IN PFIG

Degree (deg), size and order in PFIGs are discussed here. We define CPFIG and complement of PFIG.

Definition 3.1. *Let $\widehat{G} = (K, L, M)$ be a PFIG. Then deg of a vertex e in $\widehat{G}$ is defined by*
$deg = (deg_P(e), deg_N(e), deg_n(e))$ *where,*
$deg_P(e) = \sum P_M(e, ef)$ *represents positive membership degree of deg of a vertex e.*
$deg_N(e) = \sum N_M(e, ef)$ *represents neutral membership degree of deg of a vertex e.*
$deg_n(e) = \sum n_M(e, ef)$ *represents negative membership degree of deg of a vertex e.*

Definition 3.2. *The minimum deg of $\widehat{G}$ is $\delta(\widehat{G}) = (\delta_P(\widehat{G}), \delta_N(\widehat{G}), \delta_n(\widehat{G}))$ where,*
$\delta_P = \wedge\{deg_P(e) \mid e \in K^*\}$.
$\delta_N = \wedge\{deg_N(e) \mid e \in K^*\}$.

$\delta_n = \wedge\{deg_n(e) \mid e \in K^*\}$.

Definition 3.3. *The maximum* deg *of* $\widehat{G}$ *is* $\Delta(\widehat{G}) = (\Delta_P(\widehat{G}), \Delta_N(\widehat{G}), \Delta_n(\widehat{G}))$ *where,*
$\Delta_P = \vee\{deg_P(e) \mid e \in K^*\}$.
$\Delta_N = \vee\{deg_N(e) \mid e \in K^*\}$.
$\Delta_n = \vee\{deg_n(e) \mid e \in K^*\}$.

Example 3.4. *Assume a graph* $\widehat{G} = (K, L, M)$ *as shown in Figure* 8. *The* deg *for all vertices are* $deg(p) = (P_M(p, pq) + P_M(p, pr), N_M(p, pq) + N_M(p, pr), n_M(p, pq) + n_M(p, pr)) = (0.3+0.3, 0.2+0.2, 0.2+0.1) = (0.6, 0.4, 0.3)$. *In similar way,* $deg(q) = (0.8, 0.4, 0.4)$ *and* $deg(r) = (0.8, 0.4, 0.3)$. *Also,* $\delta(\widehat{G}) = (0.6, 0.4, 0.3)$ *and* $\Delta(\widehat{G}) = (0.8, 0.4, 0.4)$.

Definition 3.5. *Assume* $\widehat{G} = (K, L, M)$ *is a PFIG. Order of* $\widehat{G}$ *is defined as* $O(\widehat{G}) = (O_P(\widehat{G}), O_N(\widehat{G}), O_n(\widehat{G}))$*, where*

- $O_P(\widehat{G}) = \Sigma P_M(e, ef)$ *for all* $(e, ef) \in M^*$.
- $O_N(\widehat{G}) = \Sigma N_M(e, ef)$ *for all* $(e, ef) \in M^*$.
- $O_n(\widehat{G}) = \Sigma n_M(e, ef)$ *for all* $(e, ef) \in M^*$.

In Figure 8 *the* $O(\widehat{G}) = (2.2, 1.2, 1.0)$

Definition 3.6. *Assume* $\widehat{G} = (K, L, M)$ *is a PFIG. Size of* $\widehat{G}$ *is defined as* $S(\widehat{G}) = (S_P(\widehat{G}), S_N(\widehat{G}), S_n(\widehat{G}))$*, where*

- $S_P(\widehat{G}) = \Sigma P_L(ef)$ *for all* $ef \in L^*$.
- $S_N(\widehat{G}) = \Sigma N_L(ef)$ *for all* $ef \in L^*$.
- $S_n(\widehat{G}) = \Sigma n_L(ef)$ *for all* $ef \in L^*$.

In Figure 8 *the* $S(\widehat{G}) = (1.1, 0.6, 0.5)$

Definition 3.7. *An incidence pair of a PFIG* $\widehat{G} = (K, L, M)$ *is named as picture effective incidence pair (PEIP) if*

(1) $P_M(e, ef) = P_K(e) \wedge P_L(ef)$.
(2) $N_M(e, ef) = N_K(e) \wedge N_L(ef)$.
(3) $n_M(e, ef) = n_K(e) \vee n_L(ef)$.

Example 3.8. *Consider a PFIG* $\widehat{G}$*, given in Figure* 9. (p, pq) *and* (s, sp) *are PEIP.*

Definition 3.9. *The effective degree* (deg_E) *of a vertex* e *is defined as* $deg_E(e) = (deg_{EP}(e), deg_{EN}(e), deg_{En}(e))$ *where* $deg_{EP}(e)$ *is sum of the positive membership degrees of the PEIP,* $deg_{EN}(e)$ *is sum of the neutral membership degrees of the PEIP, and* $deg_{En}(e)$ *is sum of the negative membership degrees of the PEIP.*

Definition 3.10. *The lowest* deg_E *of* $\widehat{G}$ *is defined as* $\delta_E(\widehat{G}) = (\delta_{EP}(\widehat{G}), \delta_{EN}(\widehat{G}), \delta_{En}(\widehat{G}))$. *Where,*

- $\delta_{EP}(\widehat{G}) = \wedge\{deg_{EP}(e) \mid e \in K^*\}$.
- $\delta_{EN}(\widehat{G}) = \wedge\{deg_{EN}(e) \mid e \in K^*\}$.
- $\delta_{En}(\widehat{G}) = \wedge\{deg_{En}(e) \mid e \in K^*\}$.

Also, The highest deg_E *of* $\widehat{G}$ *is defined as* $\Delta_E(\widehat{G}) = (\Delta_{EP}(\widehat{G}), \Delta_{EN}(\widehat{G}), \Delta_{En}(\widehat{G}))$. *Where,*

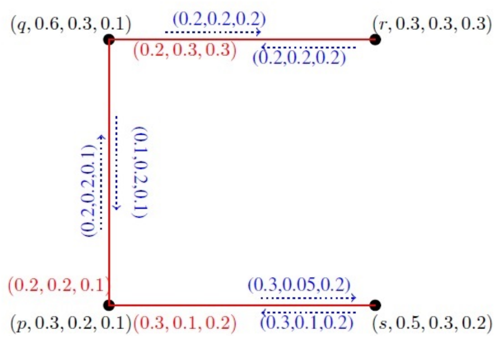

FIGURE 9. A PFIG

- $\Delta_{EP}(\widehat{G}) = \vee\{deg_{EP}(e) \mid e \in K^*\}$.
- $\Delta_{EN}(\widehat{G}) = \vee\{deg_{EN}(e) \mid e \in K^*\}$.
- $\Delta_{En}(\widehat{G}) = \vee\{deg_{En}(e) \mid e \in K^*\}$.

Example 3.11. *Consider a PFIG $\widehat{G}$ shown in Figure 9. In this graph, the deg_E of all vertices are $deg_E(p) = (0.2, 0.2, 0.1)$ and $deg_E(s) = (0.3, 0.1, 0.2)$. Also, $\delta_E(\widehat{G}) = (0.2, 0.1, 0.1)$ and $\Delta_E(\widehat{G}) = (0.3, 0.2, 0.2)$.*

Definition 3.12. *The complement $\overline{G}$ of a PFIG $\widehat{G}$ given in Figure 10 is provided in Figure 11 and is defined as*
$\overline{K} = K$, $\overline{P_K}(v_i) = P_K(v_i), \overline{N_K}(v_i) = N_K(v_i), \overline{n_K}(v_i) = n_K(v_i)$ *for all* $v_j \in K^*$,
$\overline{P_L}(v_i, v_j) = (P_K(v_i) \wedge P_K(v_j)) - P_L(v_i, v_j), \overline{N_L}(v_i, v_j) = (N_K(v_i) \wedge N_K(v_j)) - N_L(v_i, v_j), \overline{n_L}(v_i, v_j) = (n_K(v_i) \vee n_K(v_j)) - n_L(v_i, v_j)$ *for all* $v_iv_j \in L^*$,
$\overline{P_M}(v_i, v_iv_j) = (P_K(v_i) \wedge P_K(v_j)) - P_M(v_i, v_iv_j), \overline{N_M}(v_i, v_iv_j) = (N_K(v_i) \wedge N_K(v_j)) - N_M(v_i, v_iv_j)$, *and* $\overline{n_M}(v_i, v_iv_j) = (n_K(v_i) \vee n_K(v_j)) - n_M(v_i, v_iv_j)$, *for all* $v_i \in K^*, v_iv_j \in L^*$.

Theorem 3.13. *Let $\widehat{G} = (K, L, M)$ be a PFIG and $\overline{G}$ is its complement. Then complement of $\overline{G}$ is always equal to $\widehat{G}$. i.e. $\overline{\overline{G}} = \widehat{G}$.*

Proof. Suppose $\widehat{G}$ is a PFIG. Then by definition of $\overline{G}$, we will get a PFIG having same number of vertices as in $\widehat{G}$ i.e. $K = \overline{K}$ but different P_M, N_M, and n_M. Again by applying definition of complement on $\overline{G}$ we get,
$\overline{\overline{K}} = K$, $\overline{\overline{P_K}}(v_i) = P_K(v_i), \overline{\overline{N_K}}(v_i) = N_K(v_i), \overline{\overline{n_K}}(v_i) = n_K(v_i)$ for all $v_j \in K^*$
$\overline{\overline{P_L}}(v_i, v_j) = P_L(v_i, v_j), \overline{\overline{N_L}}(v_i, v_j) = N_L(v_i, v_j), \overline{\overline{n_L}}(v_i, v_j) = n_L(v_i, v_j)$ for all $v_iv_j \in L^*$,
$\overline{\overline{P_M}}(v_i, v_iv_j) = P_M(v_i, v_iv_j), \overline{\overline{N_M}}(v_i, v_iv_j) = N_M(v_i, v_iv_j)$, and $\overline{\overline{n_M}}(v_i, v_iv_j) = n_M(v_i, v_iv_j)$, for all $v_i \in K^*, v_iv_j \in L^*$. Hence $\overline{\overline{G}} = \widehat{G}$. □

Example 3.14. *Assume $\widehat{G}$ is a PFIG as shown in Figure 10 then from Figure 12 it can be seen that $\overline{\overline{G}} = \widehat{G}$.*

Definition 3.15. *A strong PFG is called strong PFIG if*

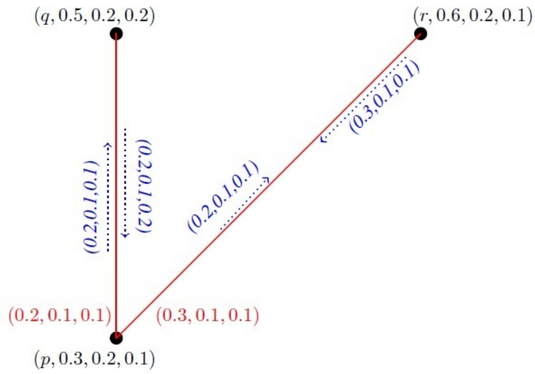

FIGURE 10. PFIG $\widehat{G}$

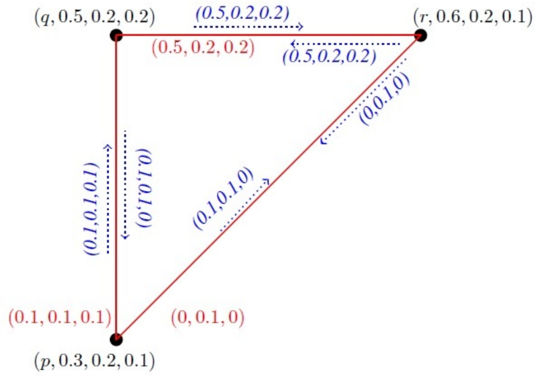

FIGURE 11. Complement of PFIG $\widehat{G}$ of Figure 10

- $P_M(e, ef) = P_K(e) \wedge P_L(ef)$ *for all* $e \in K^*, ef \in L^*$.
- $N_M(e, ef) = N_K(e) \wedge N_L(ef)$ *for all* $e \in K^*, ef \in L^*$.
- $n_M(e, ef) = n_K(e) \vee n_L(ef)$ *for all* $e \in K^*, ef \in L^*$.

It is given in Figure 13*.*

Definition 3.16. *A complete PFG is said to be CPFIG if*

- $P_M(e, ef) = P_K(e) \wedge P_L(ef)$ *for all* $e, f \in K^*$.
- $N_M(e, ef) = N_K(e) \wedge N_L(ef)$ *for all* $e, f \in K^*$.
- $n_M(e, ef) = n_K(e) \vee n_L(ef)$ *for all* $e, f \in K^*$.

CPFIG is shown in Figure 8*.*

Remark 3.17. *Every CPFIG is strong PFIG but converse is not true.*

Proposition 3.18. *The complement of a CPFIG $\widehat{G}$ is a PFIG having isolated vertices.*

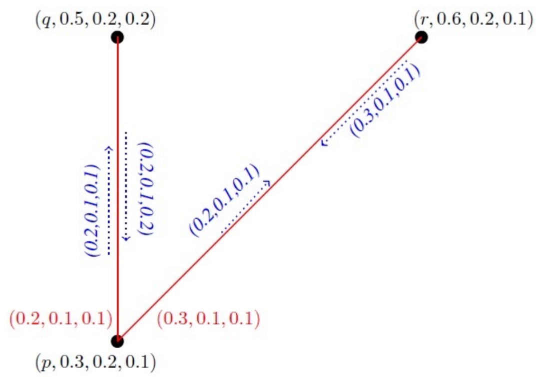

FIGURE 12. $\overline{\overline{G}} = \widehat{G}$

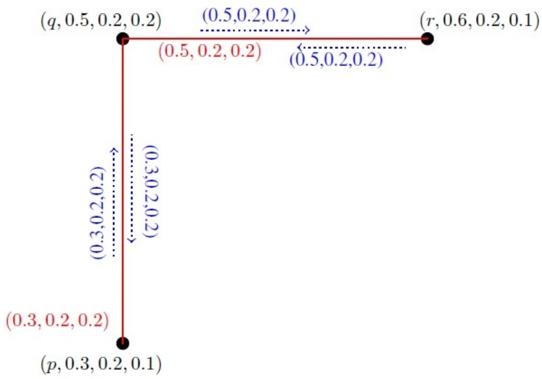

FIGURE 13. Strong PFIG

Proof. Assume $\widehat{G} = (K, L, M)$ is a CPFIG. So,
$P_L(v_i, v_j) = P_K(v_i) \wedge P_K(v_j)$ for all $v_i, v_j \in K^*$.
$N_L(v_i, v_j) = N_K(v_i) \wedge N_K(v_j)$ for all $v_i, v_j \in K^*$.
$n_L(v_i, v_j) = n_K(v_i) \vee n_K(v_j)$ for all $v_i, v_j \in K^*$. and
$P_M(v_i, v_iv_j) = P_K(v_i) \wedge P_L(v_iv_j)$ for all $v_i, v_j \in K^*$.
$N_M(v_i, v_iv_j) = N_K(v_i) \wedge N_L(v_iv_j)$ for all $v_i, v_j \in K^*$.
$n_M(v_i, v_iv_j) = n_K(v_i) \vee n_L(v_iv_j)$ for all $v_i, v_j \in K^*$.
Hence in $\overline{G}$,
$\overline{P_L}(v_i, v_j) = \wedge(P_K(v_i) \wedge P_K(v_j)) - P_L(v_i, v_j) = 0$, for all $i, j, ..., n.$,
$\overline{N_L}(v_i, v_j) = \wedge(N_K(v_i) \wedge N_K(v_j)) - N_L(v_i, v_j) = 0$, for all $i, j, ..., n.$,
$\overline{n_L}(v_i, v_j) = \wedge(n_K(v_i) \vee n_K(v_j)) - n_L(v_i, v_j) = 0$, for all $i, j, ..., n.$,
$\overline{P_M}(v_i, v_iv_j) = (P_K(v_i) \wedge P_K(v_j)) - P_M(v_i, v_iv_j) = 0$, for all $i, j, ..., n.$,
$\overline{N_M}(v_i, v_iv_j) = (N_K(v_i) \wedge N_K(v_j)) - N_M(v_i, v_iv_j) = 0$, for all $i, j, ..., n.$,

and
$\overline{n_M}(v_i, v_iv_j) = (n_K(v_i) \vee n_K(v_j)) - n_M(v_i, v_iv_j) = 0$, for all $i, j, ..., n$.
Thus $\overline{P_L}(v_i, v_j), \overline{N_L}(v_i, v_j), \overline{n_L}(v_i, v_j) = (0, 0, 0)$.
and $\overline{P_M}(v_i, v_iv_j), \overline{N_M}(v_i, v_iv_j), \overline{n_M}(v_i, v_iv_j) = (0, 0, 0)$.
Hence $\overline{G}$ has only isolated vertices if $\widehat{G}$ is a CPFIG. □

4. Control of illegal transportation of people with the help of PFIG

Poverty, unemployment, lack of health facilities, lack of education facilities, and lack of jobs are common issues in India. That is why many people are not happy to live in India. Due to these reasons, the people of India try to move illegally to America to find asylum for a healthier lifestyle. Many agents of different companies help people to move to America illegally for their earning. In 2018, according to a report provided by US border patrol, almost 8997 people of India were arrested while attempting to cross the US border illegally [52]. Indian people use different land routes to enter illegally America like India, UAE, Russia, Nicaragua, and Mexico. Here we are presenting a Mathematical model of this situation.
Let $\widehat{G} = (K, L, M)$ be a PFIG as given in Figure 14.
Here, set K is showing the set of different countries, set L is representing the legal travel of people from one country to another country and set M is expressing the illegal movement of people from one country to another country. The assignments of numbers to the vertices, edges, and pairs are indicating the P_M, N_M, and n_M respectively. The P_M of the vertices (countries) is representing the percentage of those people who have a plan to go to $America$, the N_M of the countries are indicating the percentage of those people who are confused whether to go to $America$ or not and the n_M of the countries are expressing the percentage of those people who are not willing to move to $America$. In the case of country $(India)$, 0.3 percent people want to move to $America$, 0.5 percent are confused whether to go to $America$ or not and 0.2 percent people do not agree to go to $America$. The $P_M = 0.2$ of the countries $India$ to UAE or UAE to $India$ is showing the percentage of those people who are successful to move from $India$ to UAE or UAE to $India$, the $N_M = 0.3$ of these two countries stating the percentage of those people who are unable to think clearly whether to move or not from $India$ to UAE or UAE to $India$ and the $n_M = 0.4$ of the countries is the percentage of those people who are not willing to move from $India$ to UAE or UAE to $India$. In the same way, we can see the percentage of people from $India$ to $Russia$ or $Russia$ to $India$, $Russia$ to UAE or UAE to $Russia$, UAE to $Nicaragua$ or $Nicaragua$ to UAE and $Nicaragua$ to $Mexico$ or $Mexico$ to $Nicaragua$ in Figure 14. In Figure 14, the $P_M = 0.05$ of pair $(India, (India, UAE)$ is the percentage of those people who successfully move from $India$ to UAE, the $N_M = 0.03$ of pair $(India, (India, UAE)$ is the percentage of those people who are puzzled to move from $India$ to UAE and the $n_M = 0.39$ of pair $(India, (India, UAE)$ is the percentage of those people who are disagree to go from $India$ to UAE. Similarly, the $P_M = 0.1$ of pair $(UAE, (India, UAE)$ is the percentage of those people who successfully move from UAE to $India$, the $N_M = 0.02$ of pair $(UAE, (India, UAE)$ is the percentage of those people who are undecidable to move from UAE to $India$ and the $n_M = 0.31$ of pair $(UAE, (India, UAE)$ is the percentage of those people who are showing their refusal to go from UAE to $India$. In the same way, the $P_M = 0.12$ of $(India, (India, Russia)$ is the percentage of those people who successfully move from $India$ to $Russia$, the $N_M = 0.21$ of $(India, (India, Russia)$ is the percentage of those people who are puzzled to move from $India$ to $Russia$ and the

$n_M = 0.42$ of $(India, (India, Russia)$ is the percentage of those people who are disagree to go from $India$ to $Russia$. Similarly, the $P_M = 0.2$ of $(Russia, (India, Russia)$ is the percentage of those people who successfully move from $Russia$ to $India$, the $N_M = 0.29$ of $(Russia, (India, Russia)$ is the percentage of those people who are undecidable to move from $Russia$ to $India$ and the $n_M = 0.31$ of $(Russia, (India, Russia)$ is the percentage of those people who are showing their refusal to go from $Russia$ to $India$. In the same manner, we can see all the percentages of the remaining countries in Figure 14.
$K = \{(India, 0.3, 0.5, 0.2), (UAE, 0.2, 0.3, 0.4),$
$(Russia, 0.2, 0.3, 0.5), (Nicaragua, 0.3, 0.1, 0.3), (Mexico, 0.4, 0.2, 0.2)\}$ set K expressing the countries.
$L = \{((India, UAE)0.2, 0.3, 0.4), ((India, Russia)0.2, 0.3, 0.5),$
$((UAE, Russia)0.2, 0.3, 0.5), ((UAE, Nicaragua)0.2, 0.1, 0.4),$
$((Nicaragua, Mexico)0.3, 0.1, 0.3))\}$ set L expressing the legal travel of people from one country to another country.
$M = \{((India, (India, UAE)), 0.05, 0.3, 0.39), (UAE, (India, UAE), 0.1, 0.2, 0.31),$
$(India, (India, Russia)), 0.12, 0.21, 0.42), (Russia, (India, Russia)0.2, 0.29, 0.4),$
$(UAE, (UAE, Russia)), 0.2, 0.15, 0.5), (Russia, (UAE, Russia)0.2, 0.17, 0.48),$
$(UAE, (UAE, Nicaragua)0.1, 0.06, 0.33), (Nicaragua, (UAE, Nicaragua)0.19, 0.1, 0.32),$
$(Nicaragua, (Nicaragua, Mexico)0.23, 0.1, 0.21),$
$(Mexico, (Nicaragua, Mexico)0.21, 0.02, 0.3))\}$. Set M shows the illegaly transfer of people from one country to another country.
From Figure 14 it can be seen that the pairs $((India), (India, UAE)), ((UAE), (India, UAE)), ((UAE), (UAE, Russia))$ and $((India), (India, Russia))$ are the PFICPs. So, the government of these countries must make some severe rules to overcome the illegal transportation of people. We show our suggested method in the Algorithm 1 given below. Algorithm 1 will be beneficial to us to find PFICP easily. Below we are presenting the important Steps of our algorithm to find the PFICP.

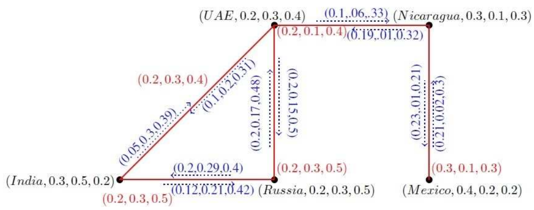

FIGURE 14. A phenomenon of illegal migration from India to America

Algorithm 1: Steps to find PFICP.

Step 1. Input the vertex set V and edge set $E \subseteq V \times V$.
Step 2. Make the PFS K on V.
Step 3. Make the PFS L on $E \subseteq V \times V$.
Step 4. Make the PFS M on $V \times E$.
Step 5. Compute the $I_s(v_i, v_j)$ of all paths from v_i to v_j, such that

$$ic_1 = \wedge\{P_M(v_i, v_iv_{i+1}) : (v_i, v_iv_{i+1}) \in I\},$$
$$ic_2 = \wedge\{N_M(v_i, v_iv_{i+1}) : (v_i, v_iv_{i+1}) \in I\},$$
$$ic_3 = \vee\{n_M(v_i, v_iv_{i+1}) : (v_i, v_iv_{i+1}) \in I\}.$$

Step 6. Compute the $I_s^{\infty}(v_i, v_j)$ of paths from v_i to v_j.
Step 7. Delete (v_i, v_iv_{i+1}) from I.
Step 8. Repeat step 6 and 7 to compute $I_s'^{\infty}(v_i, v_j)$ from v_i to v_j.
Step 9. Compare the two maximum I_s.
Step 10. If $I_s'^{\infty}(v_i, v_j) < I_s^{\infty}(v_i, v_j)$, then (v_i, v_iv_j) is the desired PFICP.

4.1. **Algorithm applied on the case study of Figure** 14**.** In Figure 14 we have vertex set $V = \{India, UAE, Russia, Nicaragua, Mexico\}$ expressing different countries, $E = \{(India, UAE), (India, Russia), (UAE, Russia), (UAE, Nicaragua),$ $(Nicaragua, Mexico)\}$ showing legal traveling among these countries and $I = \{(India, (India, UAE)), (UAE, (India, UAE)), (India, (India, Russia)),$ $(Russia, (India, Russia)), (UAE, (UAE, Russia)),$ $(Russia, (UAE, Russia)), (UAE, (UAE, Nicaragua)), (Nicaragua, (UAE, Nicaragua)),$ $(Nicaragua, (Nicaragua, Mexico)), (Mexico, (Nicaragua, Mexico))\}$ expressing illegal transfer of people among these countries.

From Figure 14, it can be seen that there are two possible paths from India to UAE, first is India to UAE and second is from India to Russia and Russia to UAE. Then by Step (5) we calculate I_s of first path $I_s(India, UAE) = I_s(UAE, India) = (ic_1, ic_2, ic_3)$ where
$ic_1 = \wedge\{(India, (India, UAE)), (UAE, (India, UAE))\} = (0.05, 0.1) = 0.05$
$ic_2 = \wedge\{(India, (India, UAE)), (UAE, (India, UAE))\} = (0.3, 0.2) = 0.2$
$ic_3 = \vee\{(India, (India, UAE)), (UAE, (India, UAE))\} = (0.39, 0.31) = 0.39$
Now, the $I_s(India, UAE) = (0.05, 0.2, 0.31)$. In similar way, the I_s from India to Russia and Russia to UAE is given by $I_s(India, UAE) = I_s(UAE, India) = (ic_1, ic_2, ic_3)$ where,
$ic_1 = \wedge\{(India, (India, Russia)), (Russia, (India, Russia)),$
$(Russia, (Russia, UAE)), (UAE, (Russia, UAE))\} = (0.12, 0.2, 0.2, 0.2) = 0.12$
$ic_2 = \wedge\{(India, (India, Russia)), (Russia, (India, Russia)),$
$(Russia, (Russia, UAE)), (UAE, (Russia, UAE))\} = (0.21, 0.29, 0.17, 0.15) = 0.15$
$ic_3 = \vee\{(India, (India, Russia)), (Russia, (India, Russia)),$
$(Russia, (Russia, UAE)), (UAE, (Russia, UAE))\} = (0.42, 0.4, 0.48, 0.5) = 0.5$
Therefore, $I_s(India, UAE) = I_s(UAE, India) = (0.12, 0.15, 0.5)$.
Now by Step (6) the greatest I_s is provided by,
$I_s^{\infty}(India, UAE) = I_s^{\infty}(UAE, India)\{\vee(0.05, 0.12), \vee(0.2, 0.15), \wedge(0.39, 0.5)\}$
$= (0.12, 0.2, 0.39)$.
Similarly the remaining greatest I_s are given below.
$I_s^{\infty}(India, Russia) = (0.12, 0.21, 0.42) = I_s^{\infty}(Russia, India)$,
$I_s^{\infty}(India, Nicaragua) = (0.1, 0.06, 0.39) = I_s^{\infty}(Nicaragua, India)$,
$I_s^{\infty}(India, Mexico) = (0.1, 0.02, 0.39) = I_s^{\infty}(Mexico, India)$,
$I_s^{\infty}(UAE, Russia) = (0.2, 0.2, 0.42) = I_s^{\infty}(Russia, UAE)$,
$I_s^{\infty}(UAE, Nicaragua) = (0.1, 0.06, 0.33) = I_s^{\infty}(Nicaragua, UAE)$,
$I_s^{\infty}(UAE, Mexico) = (0.1, 0.02, 0.33) = I_s^{\infty}(Mexico, UAE)$,
$I_s^{\infty}(Russia, Nicaragua) = (0.1, 0.06, 0.42) = I_s^{\infty}(Nicaragua, Russia)$,
$I_s^{\infty}(Russia, Mexico) = (0.1, 0.02, 0.42) = (Mexico, Russia)$ and
$I_s^{\infty}(Nicaragua, Mexico) = (0.21, 0.02, 0.3) = I_s^{\infty}(Mexico, Nicaragua)$.
Now by Step (7), one by one we remove pairs from I and compute $I_s'^{\infty}$. After deleting, $(India, (India, UAE))$ from I we get $I_s'^{\infty}(India, UAE) = (0.12, 0.15, 0.5)$. Then by

Step (9) we compare
$I_s'^{\infty}(India, UAE) = (0.12, 0.15, 0.5) < I_s^{\infty}(India, UAE) = (0.2, 0.21, 0.39)$.
This shows that $(India, (India, UAE))$ is a PFICP. Similarly, the remaining PFICPs are $(UAE, (India, UAE)), (India, (India, Russia))$ and $(UAE, (UAE, Russia))$.

5. Comparative analysis

A FIG is shown in Figure 15. There are two paths from India to UAE namely, $P_1 = India - UAE$ and $P_2 = India - Russia - UAE$. The $I_s(P_1) = I_s(India, UAE) = \wedge(0.05, 0.1) = 0.05$ and $I_s(P_2) = I_s(India, UAE) = \wedge(0.12, 0.2, 0.2, 0.2) = 0.12$. Now, the I_s between these two countries is given by $I_s^{\infty}(India, UAE) = \vee(0.05, 0.12) = 0.12 = I_s^{\infty}(UAE, India)$. In a similar manner, the remaining greatest I_s are
$I_s^{\infty}(India, Russia) = 0.12 = I_s^{\infty}(Russia, India)$,
$I_s^{\infty}(India, Nicaragua) = 0.1 = I_s^{\infty}(India, Nicaragua)$,
$I_s^{\infty}(India, Mexico) = 0.1 = I_s^{\infty}(Mexico, India)$,
$I_s^{\infty}(UAE, Russia) = 0.2 = I_s^{\infty}(Russia, UAE)$,
$I_s^{\infty}(UAE, Nicaragua) = 0.1 = I_s^{\infty}(Nicaragua, UAE)$,
$I_s^{\infty}(UAE, Mexico) = 0.1 = I_s^{\infty}(Mexico, UAE)$,
$I_s^{\infty}(Russia, Nicaragua) = 0.1 = I_s^{\infty}(Nicaragua, Russia)$,
$I_s^{\infty}(Russia, Mexico) = 0.1 = I_s^{\infty}(Mexico, Russia)$ and
$I_s^{\infty}(Nicaragua, Mexico) = 0.21 = I_s^{\infty}(Mexico, Nicaragua)$.
In Figure 13, $(India, (India, Russia))$ is a cutpair because after deleting it from FIG the $I_s'^{\infty}(India, Russia) = 0.05 < I_s^{\infty}(India, Russia) = 0.12$. In the same way, $(UAE, (UAE, Russia), (Russia, (UAE, Russia)), (Nicargua, (Nicaragua, Mexico)), (Mexico, (Nicaragua, Mexico)), (UAE, (UAE, Nicaragua)), (Nicaragua, (UAE, Nicaragua))$ are all cutpairs. So, in the case of FIGs, almost government of each country will have to work and make some ordinance, laws, and strategies to lessen the illegal transfer of people from one country to another but after adding the N_M and n_M in FIG the result will be PFIG which is a generalization of FIG in which PFICPs will be different from these cutpairs. In Figure 14, after removing $(India, (India, UAE))$ from the PFIG the $I_s'^{\infty}(India, UAE) = (0.12, 0.15, 0.5) < I_s^{\infty}(India, UAE) = (0.2, 0.21, 0.39)$ therefore, $(India, (India, UAE))$ is a $PFICP$. In the same way, $(UAE, (India, UAE))$, $(India, (India, Russia))$ and $(UAE, (UAE, Russia))$ are PFICPs. So, in the case of PFIGs the government of only two countries UAE and $India$ will have to make some strict rules and policies to overcome the illegal transfer of people among all these countries. Therefore, PFIGs are more instrumental, favorable, and productive than FIGs. Also, if we change the values of P_M, N_M, and n_M we will receive different PFICPs. In Figure 16, after changing the values of N_M, and n_M the PFICPs are $(India, (India, Russia))$, $(UAE, (UAE, Russia))$ and $(Russia, (UAE, Russia))$ which are all not same from the previous ones. This shows that changing of values of P_M, N_M, and n_M or changing the values of N_M, and n_M will change the PFICPs. So, this will affect the results of the overall network of the countries. Also, PFICP in one network of countries may or may not remain PFICP in another network of countries after changing the P_M degree, N_M, and n_M.

6. Conclusion

Graph theory is a handy tool to analyze various kinds of mathematical structures, but they fail to talk about the influence of vertices on the edges. This deficiency was cause to introduce FIGs because FIGs are convenient, reliable, and beneficial for this purpose. The

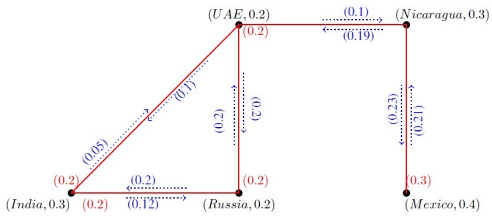

FIGURE 15. FIG

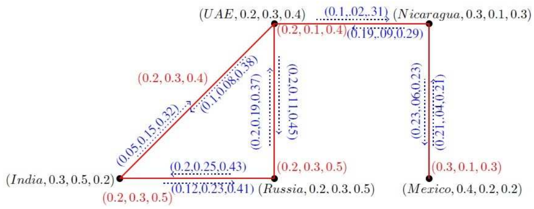

FIGURE 16. PFIG

PFIGs are a generalization of FIGs that can be applied as a gimmick for building mathematical models with unpredictable information. In this article, several properties with crucial results of PFIGs are being discussed. The study of order, size, and different kinds of degrees in PFIGs is also explored. We present a comparison between PFIGs and FIGs and draw some exclusive results. An application of control of illegal transportation of people from India to America has also been discussed. The results discussed in this paper may be used to study different PFIGs invariants. Our objective is to extend our research work to interval-valued PFIGs, soft PFIGs, bipolar PFIGs, threshold PFIGs, competition PFIGs, regular PFIGs, q-rung PFIGs, and application of PFIGs to human trafficking. Further work on these ideas will be reported in upcoming papers.

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