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Extended weighted Simpson-like type inequalities for preinvex functions and their use in physical system

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Abstract.: The main aim of this investigation is to establish the weighted Simpson-like type identity and related variants for a mapping for which the power of the absolute of the first derivative is s-preinvex. By considering this identity, numerous novel weighted Simpson's like type and related estimation type results for bounded first order differentiable functions are apprehended. Several notable results can be obtained as consequences for the suitable selection of n and ω. Meanwhile, the results are illustrated with two special functions involving modified Bessel function and q-digamma function to obtain the efficiency and supremacy of the proposed technique for many problems of wave propagation and static potentials.

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1. INTRODUCTION

Hermann Minkowski and Werner Fenche were two outstanding mathematicians who have investigated Some geometric aspects of convex sets and, to a lesser extent, of convex functions before the 1960s. The convexity theory yields a wide range of techniques to study and analysis problems related to both pure and applied mathematics [26, 27, 28, 30, 29, 31, 32, 51, 9, 38, 10, 39]. These techniques have provided many ways to study different problems related to convex analysis. Consequently, during last few years by using these techniques different inequalities has been formed. The idea of convexity is basically adopted from geometry, but it is also involved in many branches of mathematical science.

As a result, there is a wide range of research in this field [11, 2, 52, 40, 41, 42, 43, 1, 44]. At the beginning of the 1960s, the convex analysis was enormously evolved in the works of R. Tyrrell Rockafellar and Jeean-Jacques Morreau who initiated a systematic investigation of this new field. There are several books and articles dedicated to various parts of convex analysis and optimization theory, see [7, 8, 13, 14, 45, 18, 3, 33, 34, **?**, 4, 49, 35, 36, 37]. The following inequality is known as Simpson's integral inequality:

$$\left|\frac{1}{6}\left[\varphi(\zeta_1)+4\varphi\Big(\frac{\zeta_1+\zeta_2}{2}\Big)+\varphi(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int\limits_{\zeta_1}^{\zeta_2}\varphi(z)dz\right|\leq\frac{\|\varphi^{(4)}\|_\infty(\zeta_2-\zeta_1)^4}{2880},\quad(1.\ 1)$$

where $\varphi:[\zeta_1,\zeta_2]\to\mathbb{R}$ is fourth order differentiable function on (ζ_1,ζ_2) with the condition that

$$\|\varphi^{(4)}\|_\infty=\sup_{z\in(\zeta_1,\zeta_2)}\big|\varphi(z)\big|<\infty.$$

Several generalizations and extensions concerning to (1. 1) have been proposed by many researchers, for instance, Set et al. [46], Rashid et al. [25] and Du et al. [50] established certain Simpson's type variants for Riemann-Liouville fractional integrals, which are convex, s-preinvex and extended (s,m)-convex, respectively. Our aim is to derive some novel estimates for weighted Simpsons-like type integral inequalities and related estimation type results for s-preinvex functions. For more details see, [46, 25, 50] and the references therein.

In [12], Craven introduced the "term" for calling this class of functions because of their component characterized as "invariance by convexity". Weir and Mond [48] explored the idea of preinvex functions Hanson [16] contemplated the possibility of differentiable invex in the context of their precise global optimum behavior Noor [24] derived the most captivating Hermite-Hadamard inequality for preinvex functions. In [23], Mohan and Neogy presented a notable condition C.

In [22], Mititelu characterized the idea of invex sets as follows:

Let $\Omega\subset\mathbb{R}$ be a set and a continuous bifunction is $\eta(.,.):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$.

Throughout this investigation, the set Ω is an invex set, unless otherwise it is specified.

2. Notations and Preliminaries

Definition 2.1. *([22]) A set Ω is said to be invex if*

$$\zeta_2+\epsilon\eta(\zeta_1,\zeta_2)\in\Omega,\qquad\zeta_2,\zeta_1\in\Omega,\ \epsilon\in[0,1].$$

The invex set is known to be η-connected set. Observe that, if $\eta(\zeta_1,\zeta_2)=\zeta_1-\zeta_2$, this makes the sense that every convex sets is an invex set, but converse is not true.

Example 2.2. *Let* $\Omega=[\frac{-\pi}{2},0)\cup(0,\frac{\pi}{2}]$

$$\eta(z_1,z_2)=\begin{cases}\cos(z_1-z_2), & z_2\in(0,\pi/2],z_1\in(0,\pi/2];\\ -\cos(z_1-z_2), & z_2\in[-\pi/2,0),z_1\in[-\pi/2,0);\\ \cos z_2, & z_2\in(0,\pi/2],z_1\in[-\pi/2,0);\\ -\cos z_2, & z_2\in[-\pi/2,0),z_1\in(0,\pi/2];\end{cases}$$

then Ω is an invex set in respecting η for $\epsilon \in [0,1]$.

Definition 2.3. *([48]) A mapping $\varphi : \Omega \to \mathbb{R}$ is said to be preinvex if the inequality*

$$\varphi\big(\zeta_2 + \epsilon\eta(\zeta_1, \zeta_2)\big) \leq (1-\epsilon)\varphi(\zeta_2) + \epsilon\varphi(\zeta_1) \tag{2. 2}$$

holds for $\zeta_1, \zeta_2 \in \Omega$, $\epsilon \in [0,1]$.

Example 2.4. *([15]) The mapping $\varphi(z) = \frac{1-|2z-1|}{2}$, $(z \in \mathbb{R})$ in respecting the subsequent*

$$\eta(z_1, z_2) = \begin{cases} 2(z_1 - z_2), & z_2 \geq \frac{1}{2}, z_1 \geq \frac{1}{2}, z_1 \geq z_2, \\ 0, & z_2 > \frac{1}{2}, z_1 \geq \frac{1}{2}, z_1 < z_2, \\ 0, & z_2 < \frac{1}{2}, z_1 \leq \frac{1}{2}, z_1 > z_2, \\ z_1 - z_2, & z_2 \leq \frac{1}{2}, z_1 \leq \frac{1}{2}, z_1 \leq z_2, \\ 1 - z_1 - z_2, & z_2 < \frac{1}{2}, z_1 > \frac{1}{2}, z_1 + z_2 \geq 1, \\ 0, & z_2 < \frac{1}{2}, z_1 > \frac{1}{2}, z_1 + z_2 \leq 1, \\ 0, & z_2 > \frac{1}{2}, z_1 < \frac{1}{2}, z_1 + z_2 \geq 1, \\ 1 - z_1 - z_2, & z_2 > \frac{1}{2}, z_1 < \frac{1}{2}, z_1 + z_2 \leq 1, \end{cases}$$

is preinvex on $\mathbb{R}$ while it is not convex on $\mathbb{R}$.

For recent results on preinvex functions and related speculations, modifications and developments we refer the readers to [5, 6].
In [25], Rashid et al. introduced a class of modified s-preinvex functions, which is a modification of preinvexity.
Recalling the concept of modified s-preinvex function as follows.

Definition 2.5. *([25]) Let $s \in (0,1)$ and a function $\varphi : \Omega \to \mathbb{R}$ is said to be modified s-preinvex if the following inequality*

$$\varphi\big(\zeta_2 + \epsilon\eta(\zeta_1, \zeta_2)\big) \leq (1-\epsilon^s)\varphi(\zeta_2) + \epsilon^s\varphi(\zeta_1) \tag{2. 3}$$

holds for $\zeta_2, \zeta_1 \in \Omega$, $\epsilon \in [0,1]$.

It is clear that if $s = 1$, then modified s-preinvex functions reduces to preinvex functions. Also, for $\eta(\zeta_1, \zeta_2) = \zeta_1 - \zeta_2$, then the modified s-preinvexity collapes with modified s-convex functions and along with the assumption $s = 1$, it coinsides with convex functions. For the concerned outcomes associating with such kind of generalizations, we refer [5, 6, 19] and the references cited in.
Owing to the above mentioned trend, the principal aim of this paper is to derive some novel estimates concerned with the weighted Simpsons-like type integral inequalities. By considering the auxiliary result, we establish certain functions whose absolute value of the first derivative is s-preinvex, the first order differentiability of the proposed function is bounded and also the first-order derivative fulfills the Lipschitz condition. In the application viewpoint, the derived outcomes analyze by the special functions by utilizing modified Bessel function and q-digamma function appears in many diverse scenarios, particularly situations involving cylindrical symmetry.

3. AUXILIARY IDENTITY

This segment dedicated to our main contribution to this paper. The identification is described as follows.

Lemma 3.1. *For $n \in \mathbb{N}$ and let $\varphi, \hbar : \Omega_\eta \to \mathbb{R}$ be two differntiable mappings on Ω_η° (Ω_η° is the interior of Ω_η) with $\zeta_2, \zeta_2 + \eta(\zeta_1, \zeta_2) \in \Omega_\eta$ with $\eta(\zeta_1, \zeta_2) > 0$. $If\ \varphi', \hbar \in \longrightarrow_1$ $([\zeta_2, \zeta_2 + \eta(\zeta_1, \zeta_2)])$ (the Lebesgue integrable functions). Then, for $\omega \in [\zeta_2, \zeta_2 + \eta(\zeta_1, \zeta_2)]$, we have the following identity*

$$\begin{aligned}
&\frac{\varphi(\omega)}{2\eta(\zeta_1,\zeta_2)}\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}\hbar(z)dz+\frac{\varphi\Big(\omega+\frac{\eta(\zeta_2,\omega)}{n+1}\Big)}{2\eta(\zeta_1,\zeta_2)}\int_{\omega}^{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}\hbar(z)dz\\
&+\frac{\varphi\Big(\omega+\frac{\eta(\zeta_1,\omega)}{n+1}\Big)}{2\eta(\zeta_1,\zeta_2)}\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{\omega}\hbar(z)dz-\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}\varphi(z)\hbar(z)dz\\
&=\frac{(\eta(\zeta_1,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}\int_0^1\Phi_1(\epsilon)\varphi'\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\Big)d\epsilon\\
&+\frac{(\eta(\zeta_2,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}\int_0^1\Phi_2(\epsilon)\varphi'\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\Big)d\epsilon,
\end{aligned} \tag{3. 4}$$

where

$$\Phi_1(\epsilon)=\frac{1}{n+1}\int_0^{\epsilon}\hbar\Big(\omega+\frac{1-\theta}{n+1}\eta(\zeta_1,\omega)\Big)d\theta-\frac{1}{2(n+1)}\int_0^1\hbar\Big(\omega+\frac{1-\theta}{n+1}\eta(\zeta_1,\omega)\Big)d\theta \tag{3. 5}$$

and

$$\Phi_2(\epsilon)=\frac{1}{2(n+1)}\int_0^1\hbar\Big(\omega+\frac{1-\theta}{n+1}\eta(\zeta_2,\omega)\Big)d\theta-\frac{1}{(n+1)}\int_0^{\epsilon}\hbar\Big(\omega+\frac{1-\theta}{n+1}\eta(\zeta_2,\omega)\Big)d\theta. \tag{3. 6}$$

Particularly,

$$\begin{aligned}
&\big|\Lambda_{\varphi,\hbar}(n,\eta,\omega;\zeta_1,\zeta_2)\big| \\
&\leq \frac{(\eta(\zeta_1,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\int\limits_0^1\Big|\frac{\epsilon}{n+1}-\frac{1}{2(n+1)}\Big|\varphi'\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\Big)d\epsilon \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\int\limits_0^1\Big|\frac{1}{2(n+1)}-\frac{\epsilon}{n+1}\Big|\varphi'\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\Big)d\epsilon,
\end{aligned} \tag{3. 7}$$

where $\|\hbar\|_{\Omega_\eta,\infty}=\sup_{z\in\Omega}|\hbar(z)|$.

Proof. Integrating by parts and exchanging variable, for $\omega_\eta\in(\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2))$, we have

$$\begin{aligned}
I_1 &= \int\limits_0^1\Phi_1(\epsilon)\varphi'\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\Big)d\epsilon \\
&= -\frac{n+1}{\eta(\zeta_1,\omega)}\Bigg[\frac{1}{n+1}\int\limits_0^\epsilon\hbar\Big(\omega+\frac{1-\theta}{n+1}\eta(\zeta_1,\omega)\Big)d\theta-\frac{1}{2(n+1)}\int\limits_0^1\hbar\Big(\omega+\frac{1-\theta}{n+1}\eta(\zeta_1,\omega)\Big)d\theta\Bigg] \\
&\quad\times\varphi\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\Big)\Big|_0^1+\frac{1}{\eta(\zeta_1,\omega)}\int\limits_0^1\varphi\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\Big)\hbar\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\Big)d\epsilon \\
&= \frac{(n+1)}{2(\eta(\zeta_1,\omega))^2}\Big[\varphi(\omega)+\varphi\Big(\omega+\frac{1}{n+1}\eta(\zeta_1,\omega)\Big)\Big]\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{\omega}\hbar(z)dz \\
&\quad-\frac{n+1}{((\eta(\zeta_1,\omega))^2)}\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{\omega}\varphi(z)\hbar(z)dz
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int\limits_0^1\Phi_2(\epsilon)\varphi'\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\Big)d\epsilon \\
&= \frac{n+1}{\eta(\zeta_2,\omega)}\Bigg[\frac{1}{2(n+1)}\int\limits_0^1\hbar\Big(\omega+\frac{1-\theta}{n+1}\eta(\zeta_2,\omega)\Big)d\theta-\frac{1}{n+1}\int\limits_0^\epsilon\hbar\Big(\omega+\frac{1-\theta}{n+1}\eta(\zeta_2,\omega)\Big)d\theta\Bigg] \\
&\quad\times\varphi\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\Big)\Big|_0^1-\frac{1}{\eta(\zeta_2,\omega)}\int\limits_0^1\varphi\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\Big)\hbar\Big(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\Big)d\epsilon \\
&= \frac{n+1}{2(\eta(\zeta_2,\omega))^2}\Big[\varphi(\omega)+\varphi\Big(\omega+\frac{1}{n+1}\eta(\zeta_2,\omega)\Big)\Big]\int_{\omega}^{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}\hbar(z)dz \\
&\quad-\frac{n+1}{((\eta(\zeta_2,\omega,))^2)}\int_{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}^{\omega}\varphi(z)\hbar(z)dz.
\end{aligned}$$

Taking product on both sides of I_1 and I_2 by $\frac{(\eta(\zeta_1,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}$ and $\frac{(\eta(\zeta_2,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}$, respectively and adding these two identitis, we get the desired identity (3. 8).
This completes the proof of Lemma 3.1.

□

Throughout our investigation, we utilize notations below.

$$\begin{aligned}
&\Lambda_{\varphi,\hbar}\big(n,\eta,\omega;\zeta_1,\zeta_2\big)\\
&:= \frac{\varphi(\omega)}{2\eta(\zeta_1,\zeta_2)}\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}\hbar(z)dz+\frac{\varphi\Big(\omega+\frac{\eta(\zeta_2,\omega)}{n+1}\Big)}{2\eta(\zeta_1,\zeta_2)}\int_{\omega}^{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}\hbar(z)dz\\
&\quad+\frac{\varphi\Big(\omega+\frac{\eta(\zeta_1,\omega)}{n+1}\Big)}{2\eta(\zeta_1,\zeta_2)}\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{\omega}\hbar(z)dz-\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}\varphi(z)\hbar(z)dz,
\end{aligned} \tag{3. 8}$$

where $\omega=\zeta_2+\mu\eta(\zeta_1,\zeta_2)$, for all $\mu\in[0,1]$.

Remark 3.2. *If ones take $\hbar(z)=1$, then we have*

$$\begin{aligned}
&\Lambda_{\varphi}\ \ n,\eta,\omega;\zeta_1,\zeta_2\\
&:= \frac{1}{2(n+1)\eta(\zeta_1,\zeta_2)}\quad \eta(\zeta_2,\omega)-\eta(\zeta_1,\omega)\ \ \varphi(\omega)+\eta(\zeta_2,\omega)\varphi\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1}\quad -\eta(\zeta_1,\omega)\varphi\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1}\\
&-\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{(\omega+\frac{\eta(\zeta_1,\omega)}{n+1})}^{(\omega+\frac{\eta(\zeta_2,\omega)}{n+1})}\varphi(z)\hbar(z)dz.
\end{aligned} \tag{3. 9}$$

4. WEIGHTED SIMPSON-LIKE TYPE INEQUALITIES FOR DIFFERENTIABLE FUNCTIONS

Theorem 4.1. *For some fixed $s\in(0,1]$, $n\in\mathbb{N}$, $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ with $\alpha>1$, $\alpha^{-1}+\beta^{-1}=1$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on Ω_η° such that $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a modified s-preinvex function, then*

$$\begin{aligned}
&\big|\Lambda_{\varphi,\hbar}\big(n,\eta,\omega;\zeta_1,\zeta_2\big)\big|\\
&\le \frac{\|\hbar\|_{\Omega_\eta,\infty}\big(\frac{1}{\beta+1}\big)^{1/\beta}}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\Bigg[(\eta(\zeta_1,\omega))^2\bigg[\Big(1-\frac{1}{(n+1)^s(s+1)}\Big)\Big|\varphi(\omega)\Big|^\alpha+\Big(\frac{1}{(n+1)^s(s+1)}\Big)\Big|\varphi(\zeta_1)\Big|^\alpha\bigg]^{1/\alpha}\\
&\quad+(\eta(\zeta_2,\omega))^2\bigg[\Big(1-\frac{1}{(n+1)^s(s+1)}\Big)\Big|\varphi(\omega)\Big|^\alpha+\Big(\frac{1}{(n+1)^s(s+1)}\Big)\Big|\varphi(\zeta_2)\Big|^\alpha\bigg]^{1/\alpha}\Bigg].
\end{aligned} \tag{4. 10}$$

Proof. From the integral equation given in Lemma 3.1, the Hölder's inequality and using the modified s-preinvexity of $|\varphi'|^\alpha$, we have

$$\begin{aligned}
&\left|\Lambda_{\varphi,\hbar}(n,\eta,\omega;\zeta_1,\zeta_2)\right| \\
&\le \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\int_0^1 \left|1-2\epsilon\right|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right)\right|d\epsilon \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\int_0^1 \left|1-2\epsilon\right|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right)\right|d\epsilon \\
&\le \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\int_0^1 \left|1-2\epsilon\right|^\beta d\epsilon\right)^{1/\beta}\left(\int_0^1 \left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right)\right|^\alpha d\epsilon\right)^{1/\alpha} \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\int_0^1 \left|1-2\epsilon\right|^\beta d\epsilon\right)^{1/\beta}\left(\int_0^1 \left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right)\right|^\alpha d\epsilon\right)^{1/\alpha} \\
&\le \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\int_0^1 \left|1-2\epsilon\right|^\beta d\epsilon\right)^{1/\beta} \\
&\quad\left[\int_0^1\left(\left(1-\left(\frac{1-\epsilon}{n+1}\right)^s\right)\left|\varphi(\omega)\right|^\alpha+\left(\frac{1-\epsilon}{n+1}\right)^s\left|\varphi(\zeta_1)\right|^\alpha\right)d\epsilon\right]^{1/\alpha} \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\int_0^1 \left|1-2\epsilon\right|^\beta d\epsilon\right)^{1/\beta} \\
&\quad\left[\int_0^1\left(\left(1-\left(\frac{1-\epsilon}{n+1}\right)^s\right)\left|\varphi(\omega)\right|^\alpha+\left(\frac{1-\epsilon}{n+1}\right)^s\left|\varphi(\zeta_2)\right|^\alpha\right)d\epsilon\right]^{1/\alpha} \\
&= \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{1}{\beta+1}\right)^{1/\beta} \\
&\quad\left[\left(1-\frac{1}{(n+1)^s(s+1)}\right)\left|\varphi(\omega)\right|^\alpha+\left(\frac{1}{(n+1)^s(s+1)}\right)\left|\varphi(\zeta_1)\right|^\alpha\right]^{1/\alpha} \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{1}{\beta+1}\right)^{1/\beta} \\
&\quad\left[\left(1-\frac{1}{(n+1)^s(s+1)}\right)\left|\varphi(\omega)\right|^\alpha+\left(\frac{1}{(n+1)^s(s+1)}\right)\left|\varphi(\zeta_2)\right|^\alpha\right]^{1/\alpha},
\end{aligned} \tag{4. 11}$$

where we have used the fact that

$$\int_0^1 \Big(1-(\frac{1-\epsilon}{n+1})^s\Big)d\epsilon = \Big(1-\frac{1}{(n+1)^s(s+1)}\Big),\ \int_0^1 \Big(\frac{1-\epsilon}{n+1}\Big)^s d\epsilon = \frac{1}{(n+1)^s(s+1)} \quad (4.\ 12)$$

and

$$\int_0^1 \big|1-2\epsilon\big|^\beta d\epsilon = \frac{1}{\beta+1}. \quad (4.\ 13)$$

This completes the proof. □

Now we will consider Some special cases of Theorem 4.1.

I. If we take $n=1$, then we get a new result for modified s-preinvex functions.

Corollary 4.2. *For some fixed $s\in(0,1]$, $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ with $\alpha>1,\ \alpha^{-1}+\beta^{-1}=1$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on Ω_η° such that $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a modified s-preinvex function, then*

$$\begin{aligned}&\big|\Lambda_{\varphi,\hbar}(1,\eta,\omega;\zeta_1,\zeta_2)\big|\\ &\le \frac{\|\hbar\|_{\Omega_\eta,\infty}\big(\frac{1}{\beta+1}\big)^{1/\beta}}{8\eta(\zeta_1,\zeta_2)}\Bigg[(\eta(\zeta_1,\omega))^2\bigg[\Big(\frac{2^s(s+1)-1}{2^s(s+1)}\Big)\Big|\varphi(\omega)\Big|^\alpha+\frac{1}{2^s(s+1)}\Big|\varphi(\zeta_1)\Big|^\alpha\bigg]^{1/\alpha}\\ &\quad+(\eta(\zeta_2,\omega))^2\bigg[\Big(\frac{2^s(s+1)-1}{2^s(s+1)}\Big)\Big|\varphi(\omega)\Big|^\alpha+\frac{1}{2^s(s+1)}\Big|\varphi(\zeta_2)\Big|^\alpha\bigg]^{1/\alpha}\Bigg]. \quad (4.\ 14)\end{aligned}$$

II. If we take $s=1$, then we obtain a new result for preinvex functions.

Corollary 4.3. *For $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ with $\alpha>1,\ \alpha^{-1}+\beta^{-1}=1$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on Ω_η° such that $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a preinvex function, then*

$$\begin{aligned}&\big|\Lambda_{\varphi,\hbar}(n,\eta,\omega;\zeta_1,\zeta_2)\big|\\ &\le \frac{\|\hbar\|_{\Omega_\eta,\infty}\big(\frac{1}{\beta+1}\big)^{1/\beta}}{8\eta(\zeta_1,\zeta_2)}\Bigg[(\eta(\zeta_1,\omega))^2\bigg[\frac{3}{4}\Big|\varphi(\omega)\Big|^\alpha+\frac{1}{4}\Big|\varphi(\zeta_1)\Big|^\alpha\bigg]^{1/\alpha}\\ &\quad+(\eta(\zeta_2,\omega))^2\bigg[\frac{3}{4}\Big|\varphi(\omega)\Big|^\alpha+\frac{1}{4}\Big|\varphi(\zeta_2)\Big|^\alpha\bigg]^{1/\alpha}\Bigg].\end{aligned}$$

III. If we take $\hbar(z)=1$, along with $\omega=\zeta_1$ or $\omega=\zeta_2$, then we have a new result for s-preinvex functions.

Corollary 4.4. *For $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_1)]$ with $\alpha>1,\ \alpha^{-1}+\beta^{-1}=1$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on Ω_η° such that $\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a modified s-preinvex function, then*

$$\Big|\frac{1}{4}\Big[\varphi(\zeta_2)+2\varphi\Big(\frac{\zeta_2+2\eta(\zeta_1,\zeta_2)}{2}\Big)+\varphi(\zeta_2+2\eta(\zeta_1,\zeta_2))\Big]-\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{\zeta_2}^{\zeta_2+\eta(\zeta_1,\zeta_2)}\varphi(z)dz\Big|$$
$$\leq\frac{(\zeta_1-\zeta_2)\big(\frac{1}{\beta+1}\big)^{1/\beta}}{8.2^{s/\alpha}(s+1)^{1/\alpha}}\Bigg[\Big[\Big|\varphi(\zeta_1)\Big|^{\alpha}+(2^{s+1}-1)\Big|\varphi(\zeta_2)\Big|^{\alpha}\Big]^{1/\alpha}$$
$$+\Big[(2^{s+1}-1)\Big|\varphi(\zeta_1)\Big|^{\alpha}+\Big|\varphi(\zeta_2)\Big|^{\alpha}\Big]^{1/\alpha}\Bigg].$$

Remark 4.5. *If we choose $n=1$, along with $\eta(\zeta_1,\zeta_2)=\zeta_1-\zeta_2$, then Theorem 4.1 reduces to Theorem 2.3 of* [50].

Theorem 4.6. *For some fixed $s\in(0,1]$, $n\in\mathbb{N}$, $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ with $\alpha>1$, $\alpha^{-1}+\beta^{-1}=1$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on Ω_η° such that $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a modified s-preinvex function, then*

$$\Lambda_{\varphi,\hbar}\ \ n,\eta,\omega;\zeta_1,\zeta_2\quad\leq\frac{\|\hbar\|_{\Omega_\eta,\infty}}{2^{2-1/\alpha}(n+1)^2\eta(\zeta_1,\zeta_2)}$$
$$\times\Bigg[(\eta(\zeta_1,\omega))^2\quad\frac{1}{2}-\frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}\quad\varphi(\omega)\ ^{\alpha}+\quad\frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}\quad\varphi(\zeta_1)\ ^{\alpha}\ ^{1/\alpha}$$
$$+(\eta(\zeta_2,\omega))^2\quad\frac{1}{2}-\frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}\quad\varphi(\omega)\ ^{\alpha}+\quad\frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}\quad\varphi(\zeta_2)\ ^{\alpha}\ ^{1/\alpha}\Bigg].$$

Proof. From the integral equation given in Lemma 3.1, the Hölder's inequality and using the modified s-preinvexity of $|\varphi'|^\alpha$, we have

$$\begin{aligned}
&\left|\Lambda_{\varphi,\hbar}(n,\eta,\omega;\zeta_1,\zeta_2)\right| \\
&\leq \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\int_0^1 |1-2\epsilon|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right)\right|d\epsilon \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\int_0^1 |1-2\epsilon|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right)\right|d\epsilon \\
&\leq \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\int_0^1 |1-2\epsilon|d\epsilon\right)^{1-1/\alpha} \\
&\quad\times\left(\int_0^1 |1-2\epsilon|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right)\right|^\alpha d\epsilon\right)^{1/\alpha} \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\int_0^1 |1-2\epsilon|d\epsilon\right)^{1-1/\alpha} \\
&\quad\times\left(\int_0^1 |1-2\epsilon|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right)\right|^\alpha d\epsilon\right)^{1/\alpha} \\
&\leq \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{1}{2}\right)^{1-1/\alpha} \\
&\quad\times\left[\int_0^1 |1-2\epsilon|\left(\left(1-(\frac{1-\epsilon}{n+1})^s\right)\left|\varphi(\omega)\right|^\alpha+\left(\frac{1-\epsilon}{n+1}\right)^s\left|\varphi(\zeta_1)\right|^\alpha\right)d\epsilon\right]^{1/\alpha} \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{1}{2}\right)^{1-1/\alpha} \\
&\quad\times\left[\int_0^1 |1-2\epsilon|\left(\left(1-(\frac{1-\epsilon}{n+1})^s\right)\left|\varphi(\omega)\right|^\alpha+\left(\frac{1-\epsilon}{n+1}\right)^s\left|\varphi(\zeta_2)\right|^\alpha\right)d\epsilon\right]^{1/\alpha} \\
&= \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{1}{2}\right)^{1-1/\alpha}\left[\left(\frac{1}{2}-\frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}\right)\left|\varphi(\omega)\right|^\alpha\right. \\
&\quad\left.+\left(\frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}\right)\left|\varphi(\zeta_1)\right|^\alpha\right]^{1/\alpha} \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{1}{2}\right)^{1-1/\alpha}\left[\left(\frac{1}{2}-\frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}\right)\left|\varphi(\omega)\right|^\alpha\right. \\
&\quad\left.+\left(\frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}\right)\left|\varphi(\zeta_2)\right|^\alpha\right]^{1/\alpha},
\end{aligned}$$

where we have used the fact that

$$\int_0^1 \left|1-2\epsilon\right|\Big(1-\big(\frac{1-\epsilon}{n+1}\big)^s\Big)d\epsilon = \frac{1}{2} - \frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)},$$

$$\int_0^1 \left|1-2\epsilon\right|\Big(\frac{1-\epsilon}{n+1}\Big)^s d\epsilon = \frac{s2^s+1}{2^s(n+1)^s(s+1)(s+2)}$$

and

$$\int_0^1 \left|1-2\epsilon\right|d\epsilon = \frac{1}{2}.$$

This completes the proof. □

Some special cases of Theorem 4.6 are stated as follows.
I. If ones take $n=1$, then we have a new result for modified s-preinvex functions.

Corollary 4.7. *For some fixed $s\in(0,1]$, $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ with $\alpha>1$, $\alpha^{-1}+\beta^{-1}=1$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on Ω_η° such that $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a modified s-preinvex function, then*

$$\begin{aligned}\left|\Lambda_{\varphi,\hbar}(1,\eta,\omega;\zeta_1,\zeta_2)\right| &\le \frac{\|\hbar\|_{\Omega_\eta,\infty}}{2^{4-1/\alpha}\eta(\zeta_1,\zeta_2)}\Bigg[(\eta(\zeta_1,\omega))^2\Bigg[\Big(\frac{1}{2}-\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\Big)\left|\varphi(\omega)\right|^\alpha\\ &+\Big(\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\Big)\left|\varphi(\zeta_1)\right|^\alpha\Bigg]^{1/\alpha}+(\eta(\zeta_2,\omega))^2\Bigg[\Big(\frac{1}{2}-\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\Big)\left|\varphi(\omega)\right|^\alpha\\ &+\Big(\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\Big)\left|\varphi(\zeta_2)\right|^\alpha\Bigg]^{1/\alpha}\Bigg].\end{aligned}$$

II. If ones take $n=s=1$, then we have a new result for preinvex functions.

Corollary 4.8. *For $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ with $\alpha>1$, $\alpha^{-1}+\beta^{-1}=1$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on Ω_η° such that $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a preinvex function, then*

$$\begin{aligned}&\left|\Lambda_{\varphi,\hbar}(1,\eta,\omega;\zeta_1,\zeta_2)\right|\\ &\le \frac{\|\hbar\|_{\Omega_\eta,\infty}}{16\eta(\zeta_1,\zeta_2)}\Bigg[(\eta(\zeta_1,\omega))^2\left[\frac{3}{4}\left|\varphi(\omega)\right|^\alpha+\frac{1}{4}\left|\varphi(\zeta_1)\right|^\alpha\right]^{1/\alpha}\\ &\quad+(\eta(\zeta_2,\omega))^2\left[\frac{3}{4}\left|\varphi(\omega)\right|^\alpha+\frac{1}{4}\left|\varphi(\zeta_2)\right|^\alpha\right]^{1/\alpha}\Bigg]^{1/\alpha}\Bigg].\end{aligned}$$

III. If ones take $n=\hbar(z)=1$, along with $\omega=\zeta_1$ or $\omega=\zeta_2$, then we have a new result for s-preinvex functions.

Corollary 4.9. *For $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ with $\alpha>1$, $\alpha^{-1}+\beta^{-1}=1$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on : Ω_η° such that $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a s-preinvex function, then*

$$\frac{1}{4}\Big[\varphi(\zeta_2)+2\varphi\ \ \frac{\zeta_2+2\eta(\zeta_1,\zeta_2)}{2}\ \ +\varphi(\zeta_2+2\eta(\zeta_1,\zeta_2))\Big]-\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{\zeta_2}^{\zeta_2+\eta(\zeta_1,\zeta_2)}\varphi(z)dz$$
$$\leq\frac{(\zeta_2-\zeta_1)}{2^{4-1/\alpha}}\Bigg[\ \ \frac{1}{2}-\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \varphi(\zeta_2)\ ^{\alpha}+\ \frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \varphi(\zeta_1)\ ^{\alpha\ \ 1/\alpha}$$
$$+\ \ \frac{1}{2}-\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \varphi(\zeta_1)\ ^{\alpha}+\ \frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \varphi(\zeta_2)\ ^{\alpha\ \ 1/\alpha}\Bigg]. \tag{4. 15}$$

Remark 4.10. *If we choose* $n=1$, *along with* $\eta(\zeta_1,\zeta_2)=\zeta_1-\zeta_2$, *then Theorem 4.6 reduces to Theorem 2.6 of* [50].

Our next result is the better approach in the literature that can be derived by the Hölder-İşcan integral inequality.

Theorem 4.11. *For some fixed* $s\in(0,1]$, $n\in\mathbb{N}$, $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ *with* $\alpha>1$, $\alpha^{-1}+\beta^{-1}=1$ *and let* $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ *be a differentiable and continuous functions on* Ω_η° *such that* $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. *If* $|\varphi'|^\alpha$ *is a modified* s*-preinvex function, then*

$$\Lambda_{\varphi,\hbar}\ \ n,\eta,\omega;\zeta_1,\zeta_2$$
$$\leq\frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\ \ \frac{2n(\beta+2)+1}{2(n+1)(\beta+1)(\beta+2)}\ ^{1/\beta}$$
$$\times\ \ \frac{2n+1}{2(n+1)}-\frac{n(s+2)+1}{(n+1)^s(s+1)(s+2)}\ \ \varphi(\omega)\ ^{\alpha}+\ \frac{n(s+2)+1}{(n+1)^s(s+1)(s+2)}\ \ \varphi(\zeta_1)\ ^{\alpha\ \ 1/\alpha}$$
$$+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\ \ \frac{1}{2(n+1)(\beta+1)(\beta+2)}\ ^{1/\beta}$$
$$\times\ \ \frac{1}{2(n+1)}-\frac{1}{(n+1)^s(s+2)}\ \ \varphi(\omega)\ ^{\alpha}+\ \frac{1}{(n+1)^s(s+2)}\ \ \varphi(\zeta_2)\ ^{\alpha\ \ 1/\alpha}. \tag{4. 16}$$

Proof. From the integral equation given in Lemma 3.1, the Hölder's inequality and using the modified s-preinvexity of $|\varphi'|^{\alpha}$, we have

$$
\begin{aligned}
&\left|\Lambda_{\varphi,\hbar}(n,\eta,\omega;\zeta_1,\zeta_2)\right|\\
&\le \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\int_0^1 \left|1-2\epsilon\right|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right)\right|d\epsilon\\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\int_0^1 \left|1-2\epsilon\right|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right)\right|d\epsilon\\
&\le \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\int_0^1 \frac{n+\epsilon}{n+1}\left|1-2\epsilon\right|^{\beta}d\epsilon\right)^{1/\beta}\\
&\quad\times\left(\int_0^1 \frac{n+\epsilon}{n+1}\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right)\right|^{\alpha}d\epsilon\right)^{1/\alpha}\\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\int_0^1 \frac{1-\epsilon}{n+1}\left|1-2\epsilon\right|^{\beta}d\epsilon\right)^{1/\beta}\\
&\quad\times\left(\int_0^1 \frac{1-\epsilon}{n+1}\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right)\right|^{\alpha}d\epsilon\right)^{1/\alpha}\\
&\le \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{2n(\beta+2)+1}{2(n+1)(\beta+1)(\beta+2)}\right)^{1/\beta}\\
&\quad\times\left[\int_0^1 \frac{n+\epsilon}{n+1}\left(\left(1-(\frac{1-\epsilon}{n+1})^{s}\right)\left|\varphi(\omega)\right|^{\alpha}+\left(\frac{1-\epsilon}{n+1}\right)^{s}\left|\varphi(\zeta_1)\right|^{\alpha}\right)d\epsilon\right]^{1/\alpha}\\
&+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{1}{2(n+1)(\beta+1)(\beta+2)}\right)^{1/\beta}\\
&\quad\times\left[\int_0^1 \frac{1-\epsilon}{n+1}\left(\left(1-(\frac{1-\epsilon}{n+1})^{s}\right)\left|\varphi(\omega)\right|^{\alpha}+\left(\frac{1-\epsilon}{n+1}\right)^{s}\left|\varphi(\zeta_2)\right|^{\alpha}\right)d\epsilon\right]^{1/\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\eta(\zeta_1,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{2n(\beta+2)+1}{2(n+1)(\beta+1)(\beta+2)}\right)^{1/\beta}\\
&\quad\times\left[\left(\frac{2n+1}{2(n+1)}-\frac{n(s+2)+1}{(n+1)^s(s+1)(s+2)}\right)\left|\varphi(\omega)\right|^{\alpha}+\left(\frac{n(s+2)+1}{(n+1)^s(s+1)(s+2)}\right)\left|\varphi(\zeta_1)\right|^{\alpha}\right]^{1/\alpha}\\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{2(n+1)^2\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\left(\frac{1}{2(n+1)(\beta+1)(\beta+2)}\right)^{1/\beta}\\
&\quad\times\left[\left(\frac{1}{2(n+1)}-\frac{1}{(n+1)^s(s+2)}\right)\left|\varphi(\omega)\right|^{\alpha}+\left(\frac{1}{(n+1)^s(s+2)}\right)\left|\varphi(\zeta_2)\right|^{\alpha}\right]^{1/\alpha},
\end{aligned}
$$

where we have used the fact that

$$\int_0^1 \frac{n+\epsilon}{n+1}(\Big(1-(\frac{1-\epsilon}{n+1})^s\Big)d\epsilon = \Big(\frac{2n+1}{2(n+1)} - \frac{n(s+2)+1}{(n+1)^s(s+1)(s+2)}\Big),$$

$$\int_0^1 \frac{n+\epsilon}{n+1}\Big(\frac{1-\epsilon}{n+1}\Big)^s d\epsilon = \Big(\frac{n(s+2)+1}{(n+1)^s(s+1)(s+2)}\Big),$$

$$\int_0^1 \frac{1-\epsilon}{n+1}\Big(1-(\frac{1-\epsilon}{n+1})^s\Big)d\epsilon = \Big(\frac{1}{2(n+1)} - \frac{1}{(n+1)^s(s+2)}\Big),$$

$$\int_0^1 \Big(\frac{1-\epsilon}{n+1}\Big)^{s+1} d\epsilon = \frac{1}{(n+1)^s(s+2)},$$

$$\int_0^1 \frac{n+\epsilon}{n+1}\big|1-2\epsilon\big|^\beta d\epsilon = \frac{2n(\beta+2)+1}{2(n+1)(\beta+1)(\beta+2)}$$

and

$$\int_0^1 \frac{1-\epsilon}{n+1}\big|1-2\epsilon\big|^\beta d\epsilon = \frac{1}{2(n+1)(\beta+1)(\beta+2)}.$$

This completes the proof. □

I. If ones take $n=1$, then we have a new result for modified s-preinvex functions.

Corollary 4.12. *For some fixed $s \in (0,1]$, $\omega \in [\zeta_2, \zeta_2 + \eta(\zeta_1,\zeta_2)]$ with $\alpha > 1$, $\alpha^{-1} + \beta^{-1} = 1$ and let $\varphi, \hbar : \Omega_\eta \to \mathbb{R}$ be a differentiable and continuous functions on : Ω_η° such that $\varphi', \hbar \in L_1[\zeta_2, \zeta_2 + \eta(\zeta_1,\zeta_2)]$. If $|\varphi'|^\alpha$ is a modified s-preinvex function, then*

$$\begin{aligned}
&\big|\Lambda_{\varphi,\hbar}(1,\eta,\omega;\zeta_1,\zeta_2)\big| \\
&\leq \frac{(\eta(\zeta_1,\omega))^2}{8\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\Big(\frac{4(\beta+2)+1}{4(\beta+1)(\beta+2)}\Big)^{1/\beta} \\
&\quad\times\Big[\Big(\frac{3}{4} - \frac{s+3}{2^s(s+1)(s+2)}\Big)\Big|\varphi(\omega)\Big|^\alpha + \Big(\frac{(s+3)}{2^s(s+1)(s+2)}\Big)\Big|\varphi(\zeta_1)\Big|^\alpha\Big]^{1/\alpha} \\
&\quad+\frac{(\eta(\zeta_2,\omega))^2}{8\eta(\zeta_1,\zeta_2)}\|\hbar\|_{\Omega_\eta,\infty}\Big(\frac{1}{4(\beta+1)(\beta+2)}\Big)^{1/\beta} \\
&\quad\times\Big[\Big(\frac{1}{4} - \frac{1}{2^s(s+2)}\Big)\Big|\varphi(\omega)\Big|^\alpha + \Big(\frac{1}{2^s(s+2)}\Big)\Big|\varphi(\zeta_2)\Big|^\alpha\Big]^{1/\alpha}. \qquad (4.\ 17)
\end{aligned}$$

5. Estimation type results

In our coming result, we utilize the boundedness property of φ'.

Theorem 5.1. *For $n \in \mathbb{N}$, $\omega \in [\zeta_2, \zeta_2+\eta(\zeta_1,\zeta_2)]$ and let $\varphi, \hbar : \Omega_\eta \to \mathbb{R}$ be a differentiable and continuous functions on : Ω_η° such that $\varphi', \hbar \in L_1[\zeta_2, \zeta_2+\eta(\zeta_1,\zeta_2)]$. Suppose that there exist constants $q < \mathcal{Q}$ such that $-\infty < q \le \varphi' \le \mathcal{M} < +\infty$. Then*

$$\left|\Lambda_{\varphi,\hbar}(n,\eta,\omega;\zeta_1,\zeta_2) - \frac{\mathcal{Q}+q}{2(n+1)\eta(\zeta_1,\zeta_2)}\left[(\eta(\zeta_1,\omega))^2\int_0^1 \Phi_1(\epsilon)d\epsilon + (\eta(\zeta_2,\omega))^2\int_0^1 \Phi_2(\epsilon)d\epsilon\right|\right.$$
$$\le \frac{(\mathcal{Q}-q)\|\hbar\|_{\Omega_\eta,\infty}}{8(n+1)^2\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^2 + (\eta(\zeta_2,\omega))^2\Big],$$

where $\Phi_1(\epsilon)$ and $\Phi_2(\epsilon)$ are given in Lemma 3.1.

Proof.

$$\begin{aligned}
&\Lambda_{\varphi,\hbar}(n,\eta,\omega;\zeta_1,\zeta_2)\\
&= \frac{(\eta(\zeta_1,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}\int_0^1 \Phi_1(\epsilon)\left[\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right) - \frac{\mathcal{Q}+q}{2} + \frac{\mathcal{Q}+q}{2}\right]d\epsilon\\
&\quad + \frac{(\eta(\zeta_2,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}\int_0^1 \Phi_2(\epsilon)\left[\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right) - \frac{\mathcal{Q}+q}{2} + \frac{\mathcal{Q}+q}{2}\right]d\epsilon\\
&= \frac{1}{(n+1)\eta(\zeta_1,\zeta_2)}\left[(\eta(\zeta_1,\omega))^2\int_0^1 \Phi_1(\epsilon)\left[\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right) - \frac{\mathcal{Q}+q}{2}\right]d\epsilon\right.\\
&\quad \left. + (\eta(\zeta_2,\omega))^2\int_0^1 \Phi_2(\epsilon)\left[\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right) - \frac{\mathcal{Q}+q}{2}\right]d\epsilon\right]\\
&\quad + \frac{\mathcal{Q}+q}{2(n+1)\eta(\zeta_1,\zeta_2)}\left[(\eta(\zeta_1,\omega))^2\int_0^1 \Phi_1(\epsilon)d\epsilon + (\eta(\zeta_2,\omega))^2\int_0^1 \Phi_2(\epsilon)d\epsilon\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\Lambda_{\varphi,\hbar}(n,\eta,\omega;\zeta_1,\zeta_2) - \frac{\mathcal{Q}+q}{2(n+1)\eta(\zeta_1,\zeta_2)}\left[(\eta(\zeta_1,\omega))^2\int_0^1 \Phi_1(\epsilon)d\epsilon + (\eta(\zeta_2,\omega))^2\int_0^1 \Phi_2(\epsilon)d\epsilon\right|\right.\\
&\le \frac{(\eta(\zeta_1,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}\int_0^1 |\Phi_1(\epsilon)|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\right) - \frac{\mathcal{Q}+q}{2}\right|d\epsilon\\
&\quad + \frac{(\eta(\zeta_2,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)}\int_0^1 |\Phi_2(\epsilon)|\left|\varphi'\left(\omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\right) - \frac{\mathcal{Q}+q}{2}\right|d\epsilon.
\end{aligned}$$

Since φ' satisfies $-\infty < q \le \varphi' \le \mathcal{Q} < +\infty$, we have

$$q - \frac{\mathcal{Q}+q}{2} \le \varphi'(z) - \frac{\mathcal{Q}+q}{2} \le \mathcal{Q} - \frac{\mathcal{Q}+q}{2},$$

implies that

$$\left|\varphi'(z)-\frac{\mathcal{Q}+q}{2}\right|\leq\frac{\mathcal{Q}-q}{2}.$$

Hence, we conclude that

$$\begin{aligned}&\left|\Lambda_{\varphi,\hbar}\big(n,\eta,\omega;\zeta_1,\zeta_2\big)-\frac{\mathcal{Q}+q}{2(n+1)\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^2\int_0^1\Phi_1(\epsilon)d\epsilon+(\eta(\zeta_2,\omega))^2\int_0^1\Phi_2(\epsilon)d\epsilon\right|\\&\leq\frac{\mathcal{Q}-q}{2(n+1)\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^2\int_0^1\big|\Phi_1(\epsilon)\big|d\epsilon+(\eta(\zeta_2,\omega))^2\int_0^1\big|\Phi_2(\epsilon)\big|d\epsilon\Big]\\&\leq\frac{(\mathcal{Q}-q)\|\hbar\|_{\Omega_\eta,\infty}}{4(n+1)^2\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^2\int_0^1\big|2\epsilon-1\big|d\epsilon+(\eta(\zeta_2,\omega))^2\int_0^1\big|2\epsilon-1\big|d\epsilon\Big]\\&=\frac{(\mathcal{Q}-q)\|\hbar\|_{\Omega_\eta,\infty}}{8(n+1)^2\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^2+(\eta(\zeta_2,\omega))^2\Big].\end{aligned}$$

This completes the proof. □

Corollary 5.2. *If ones take $\hbar(z)=1$ in Theorem 5.1, then we have a new result*

$$\left|\Lambda_{\varphi}\big(n,\eta,\omega;\zeta_1,\zeta_2\big)\right|\leq\frac{(\mathcal{Q}-q)}{8(n+1)^2\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^2+(\eta(\zeta_2,\omega))^2\Big].$$

Our next result is the further estimation type result governed by the weighted Simpson-like type inequality when φ' satisfies Lipschitz condition.

Theorem 5.3. *For $n\in\mathbb{N}$, $\omega\in[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$ and let $\varphi,\hbar:\Omega_\eta\to\mathbb{R}$ be a differentiable and continuous functions on Ω_η° such that $\varphi',\hbar\in L_1[\zeta_2,\zeta_2+\eta(\zeta_1,\zeta_2)]$. Suppose that φ' satisfies the Lipschitz condition for some $\mathcal{L}>0$. Then*

$$\begin{aligned}&\left|\Lambda_{\varphi,\hbar}\big(n,\eta,\omega;\zeta_1,\zeta_2\big)-\frac{1}{(n+1)\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^2\varphi'\Big(\omega+\frac{\eta(\zeta_1,\omega)}{n+1}\Big)\int_0^1\Phi_1(\epsilon)d\epsilon\right.\\&\quad\left.+(\eta(\zeta_2,\omega))^2\varphi'\Big(\omega+\frac{\eta(\zeta_2,\omega)}{n+1}\Big)\int_0^1\Phi_2(\epsilon)d\epsilon\Big]\right|\\&\leq\frac{\mathcal{L}\|\hbar\|_{\Omega_\eta,\infty}}{8(n+1)^3\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^3+(\eta(\zeta_2,\omega))^3\Big],\end{aligned}\qquad(5.\ 18)$$

where $\Phi_1(\epsilon)$ and $\Phi_2(\epsilon)$ are given in Lemma 3.1.

Proof. By means of Lemma 3.1, we have

$$\begin{aligned}
&\Lambda_{\varphi,\hbar}\ \ n,\eta,\omega;\zeta_1,\zeta_2 \\
&= \frac{(\eta(\zeta_1,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)} \int_0^1 \Phi_1(\epsilon)\ \ \varphi'\ \ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\ \ -\varphi'\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1}\ \ +\varphi'\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1}\ \ d\epsilon \\
&\quad + \frac{(\eta(\zeta_2,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)} \int_0^1 \Phi_2(\epsilon)\ \ \varphi'\ \ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\ \ -\varphi'\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1}\ \ +\varphi'\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1}\ \ d\epsilon \\
&= \frac{1}{(n+1)\eta(\zeta_1,\zeta_2)} \Bigg[(\eta(\zeta_1,\omega))^2 \int_0^1 \Phi_1(\epsilon)\ \ \varphi'\ \ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\ \ -\varphi'\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1}\ \ d\epsilon \\
&\quad + (\eta(\zeta_2,\omega))^2 \int_0^1 \Phi_2(\epsilon)\ \ \varphi'\ \ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\ \ -\varphi'\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1}\ \ d\epsilon \Bigg] \\
&\quad + \frac{1}{(n+1)\eta(\zeta_1,\zeta_2)} \Big[(\eta(\zeta_1,\omega))^2 \varphi'\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1}\ \ \int_0^1 \Phi_1(\epsilon)d\epsilon + (\eta(\zeta_2,\omega))^2 \varphi'\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1}\ \ \int_0^1 \Phi_2(\epsilon)d\epsilon \Big].
\end{aligned}$$

Thus

$$\begin{aligned}
&\Lambda_{\varphi,\hbar}\ \ n,\eta,\omega;\zeta_1,\zeta_2\ \ - \frac{1}{(n+1)\eta(\zeta_1,\zeta_2)} \Big[(\eta(\zeta_1,\omega))^2 \varphi'\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1}\ \ \int_0^1 \Phi_1(\epsilon)d\epsilon \\
&\quad + (\eta(\zeta_2,\omega))^2 \varphi'\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1}\ \ \int_0^1 \Phi_2(\epsilon)d\epsilon \Big] \\
&\le \frac{(\eta(\zeta_1,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)} \int_0^1\ \ \Phi_1(\epsilon)\ \ \varphi'\ \ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\ \ -\varphi'\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1}\ \ d\epsilon \\
&\quad + \frac{(\eta(\zeta_2,\omega))^2}{(n+1)\eta(\zeta_1,\zeta_2)} \int_0^1\ \ \Phi_2(\epsilon)\ \ \varphi'\ \ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\ \ -\varphi'\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1}\ \ d\epsilon.
\end{aligned}$$

Since φ' satisfies Lipschitz condition for some $\mathcal{L} > 0$, we have

$$\begin{aligned}
\varphi'\ \ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)\ \ -\varphi'\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1} \qquad &\le \mathcal{L}\ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_1,\omega)-\omega-\frac{\eta(\zeta_1,\omega)}{n+1} \\
&= \frac{\eta(\zeta_1,\omega)}{n+1}\mathcal{L}\ \epsilon
\end{aligned}$$

and

$$\begin{aligned}
\varphi'\ \ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)\ \ -\varphi'\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1} \qquad &\le \mathcal{L}\ \omega+\frac{1-\epsilon}{n+1}\eta(\zeta_2,\omega)-\omega-\frac{\eta(\zeta_2,\omega)}{n+1} \\
&= \frac{\eta(\zeta_2,\omega)}{n+1}\mathcal{L}\ \epsilon\ .
\end{aligned}$$

It follows that

$$\Lambda_{\varphi,\hbar}\ \ n,\eta,\omega;\zeta_1,\zeta_2\ \ -\frac{1}{(n+1)\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^2\varphi'\ \ \omega+\frac{\eta(\zeta_1,\omega)}{n+1}\ \ \int_0^1\Phi_1(\epsilon)d\epsilon$$

$$+(\eta(\zeta_2,\omega))^2\varphi'\ \ \omega+\frac{\eta(\zeta_2,\omega)}{n+1}\ \ \int_0^1\Phi_2(\epsilon)d\epsilon\Big]$$

$$\leq\frac{(\eta(\zeta_1,\omega))^3\mathcal{L}}{(n+1)^2\eta(\zeta_1,\zeta_2)}\int_0^1\epsilon\ \Phi_1(\epsilon)\ d\epsilon+\frac{(\eta(\zeta_2,\omega))^3\mathcal{L}}{(n+1)^2\eta(\zeta_1,\zeta_2)}\int_0^1\epsilon\ \Phi_2(\epsilon)\ d\epsilon$$

$$=\frac{\mathcal{L}\|\hbar\|_{\Omega_\eta,\infty}}{2(n+1)^3\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^3+(\eta(\zeta_2,\omega))^3\Big]\int_0^1\ \ \epsilon-2\epsilon^2\ \ d\epsilon$$

$$=\frac{\mathcal{L}\|\hbar\|_{\Omega_\eta,\infty}}{8(n+1)^3\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^3+(\eta(\zeta_2,\omega))^3\Big]. \tag{5. 19}$$

This completes the proof. □

Corollary 5.4. *If ones take $\hbar(z)=1$, then we have*

$$\Big|\Lambda_\varphi\big(n,\eta,\omega;\zeta_1,\zeta_2\big)\Big|\leq\frac{\mathcal{L}}{8(n+1)^3\eta(\zeta_1,\zeta_2)}\Big[(\eta(\zeta_1,\omega))^3+(\eta(\zeta_2,\omega))^3\Big], \tag{5. 20}$$

6. Inequalities for special functions

This segment inaugurating some applications to the estimations of modified Bessel function and the q-digamma for weighted Simpson type inequalities for differentiable s-preinvex functions by use of our results.

6.1. Modified Bessel functions.

Recalling the series representation of the first kind modified Bessel function ([47])

$$\mathcal{U}_\rho(z)=\sum_{n\geq 0}\frac{(z/2)^{\rho+2n}}{n!\Gamma(\rho+n+1)},\quad \forall z\in\mathbb{R},$$

analogously, the second kind modified Bessel function $\mathcal{K}_\rho$ is defined as

$$\mathcal{K}_\rho(z)=\frac{\pi}{2}\frac{\mathcal{U}_{-\rho}(z)+\mathcal{U}_\rho(z)}{\sin\rho\pi}.$$

Let us assume the function $\mathcal{I}_\rho:\mathbb{R}\to[1,\infty)$, defined by

$$\mathcal{I}_\rho(z)=2^\rho\Gamma(\rho+1)z^{-\rho}\mathcal{U}_\rho(z),$$

where $\Gamma(.)$ is the gamma function.

Proposition 6.2. *For $\rho>-1$ and $\zeta_2>\zeta_1>0$, then*

$$
\begin{aligned}
&\Big|\frac{1}{4}\Big[\mathcal{I}_\rho(\zeta_2)+2\mathcal{I}_\rho\Big(\frac{\zeta_2+2\eta(\zeta_1,\zeta_2)}{2}\Big)+\mathcal{I}_\rho(\zeta_2+2\eta(\zeta_1,\zeta_2))\Big] \\
&\quad -\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{\zeta_2}^{\zeta_2+\eta(\zeta_1,\zeta_2)}\mathcal{I}_\rho(z)dz\Big| \\
&\le \frac{(\zeta_1-\zeta_2)\big(\frac{1}{\beta+1}\big)^{1/\beta}}{8.2^{s/\alpha}(s+1)^{1/\alpha}(\rho+1)}\Bigg[\Big[\zeta_1\Big|\mathcal{I}_{\rho+1}(\zeta_1)\Big|^\alpha+\zeta_2(2^{s+1}-1)\Big|\mathcal{I}_{\rho+1}(\zeta_2)\Big|^\alpha\Big]^{1/\alpha} \\
&\quad +\Big[\zeta_1(2^{s+1}-1)\Big|\mathcal{I}_{\rho+1}(\zeta_1)\Big|^\alpha+\zeta_2\Big|\mathcal{I}_{\rho+1}(\zeta_2)\Big|^\alpha\Big]^{1/\alpha}\Bigg]. \qquad (6.\ 21)
\end{aligned}
$$

Choosing $\rho=-1/2$, *then*

$$
\begin{aligned}
&\Big|\frac{1}{4}\Big[\cosh(\zeta_2)+2\cosh\Big(\frac{\zeta_2+2\eta(\zeta_1,\zeta_2)}{2}\Big)+\cosh(\zeta_2+2\eta(\zeta_1,\zeta_2))\Big] \\
&\quad -\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{\zeta_2}^{\zeta_2+\eta(\zeta_1,\zeta_2)}\cosh(z)dz\Big| \\
&\le \frac{(\zeta_1-\zeta_2)\big(\frac{1}{\beta+1}\big)^{1/\beta}}{4.2^{s/\alpha}(s+1)^{1/\alpha}}\Bigg[\Big[\Big|\sinh(\zeta_1)\Big|^\alpha+(2^{s+1}-1)\Big|\sinh(\zeta_2)\Big|^\alpha\Big]^{1/\alpha} \\
&\quad +\Big[(2^{s+1}-1)\Big|\sinh(\zeta_1)\Big|^\alpha+\Big|\sinh(\zeta_2)\Big|^\alpha\Big]^{1/\alpha}\Bigg]. \qquad (6.\ 22)
\end{aligned}
$$

Proof. Applying inequality (4. 15) to the mapping $\hbar(z)=\mathcal{I}_\rho(z)$, $z>0$ and $\mathcal{I}'_\rho(z)=\frac{z}{\rho+1}\omega_{\rho+1}$. Now we have used the fact that $\omega_{-1/2}(z)=\cosh z$ and $\omega_{1/2}(z)=\frac{\sin h(z)}{z}$, then we the inequality (6. 27) reduces to (6. 22). □

Proposition 6.3. *For* $\rho>-1$ *and* $\zeta_2>\zeta_1>0$, *then*

$$
\begin{aligned}
&\frac{1}{4}\Big[\mathcal{I}_\rho(\zeta_2)+2\mathcal{I}_\rho\ \ \frac{\zeta_2+2\eta(\zeta_1,\zeta_2)}{2}\ \ +\mathcal{I}_\rho(\zeta_2+2\eta(\zeta_1,\zeta_2))\Big]-\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{\zeta_2}^{\zeta_2+\eta(\zeta_1,\zeta_2)}\mathcal{I}_\rho(z)dz \\
&\le \frac{(\zeta_2-\zeta_1)}{2^{4-1/\alpha}(\rho+1)}\Big[\ \zeta_2\ \ \frac{1}{2}-\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \mathcal{I}_{\rho+1}(\zeta_2)\ ^\alpha+\zeta_1\ \ \frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \mathcal{I}_{\rho+1}(\zeta_1)\ ^{\alpha\ \ 1/\alpha} \\
&\quad +\ \zeta_1\ \ \frac{1}{2}-\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \mathcal{I}_{\rho+1}(\zeta_1)\ ^\alpha+\zeta_2\ \ \frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \mathcal{I}_{\rho+1}(\zeta_2)\ ^{\alpha\ \ 1/\alpha}\Big]. \qquad (6.\ 23)
\end{aligned}
$$

Choosing $\rho=-1/2$, *then*

$$
\begin{aligned}
&\frac{1}{4}\Big[\cosh(\zeta_2)+2\cosh\ \ \frac{\zeta_2+2\eta(\zeta_1,\zeta_2)}{2}\ \ +\cosh(\zeta_2+2\eta(\zeta_1,\zeta_2))\Big]-\frac{1}{\eta(\zeta_1,\zeta_2)}\int_{\zeta_2}^{\zeta_2+\eta(\zeta_1,\zeta_2)}\cosh(z)dz \\
&\le \frac{(\zeta_2-\zeta_1)}{2^{3-1/\alpha}}\Big[\ \ \frac{1}{2}-\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \sinh(\zeta_2)\ ^\alpha+\ \frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \sinh(\zeta_1)\ ^{\alpha\ \ 1/\alpha} \\
&\quad +\ \ \frac{1}{2}-\frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \sinh(\zeta_1)\ ^\alpha+\zeta_2\ \ \frac{s2^s+1}{2^{2s}(s+1)(s+2)}\ \ \sinh(\zeta_2)\ ^{\alpha\ \ 1/\alpha}\Big]. \qquad (6.\ 24)
\end{aligned}
$$

Proof. Applying inequality (4. 15) to the mapping $\hbar(z) = \mathcal{I}_\rho(z),\ z > 0$ and $\mathcal{I}'_\rho(z) = \frac{z}{\rho+1}\omega_{\rho+1}$. Now we have used the fact that $\omega_{-1/2}(z) = \cosh z$ and $\omega_{1/2}(z) = \frac{\sin h(z)}{z}$, then we the inequality (6. 28) reduces to (6. 24). □

6.4. q-**digamma function.** For $q \in (0,1)$, the q-analogue of the digamma function is denoted by $\check{\phi}_q$ is stated as follows

$$\begin{aligned} \check{\phi}_q(z) \qquad &= -\ln(1-q) + \ln q \sum_{\mathcal{K}=0}^{\infty} \frac{q^{\mathcal{K}+z}}{1-q^{\mathcal{K}+z}} \\ &= -\ln(1-q) + \ln q \sum_{\mathcal{K}=0}^{\infty} \frac{q^{-\mathcal{K}z}}{1-q^{\mathcal{K}}}. \end{aligned} \tag{6. 25}$$

For $q > 1$ and $z > 0$, the q-digamma function $\check{\phi}_q$ is stated as follows

$$\begin{aligned} \check{\phi}_q(z) \qquad &= -\ln(1-q) + \ln q \Big[z - \frac{1}{2} - \sum_{\mathcal{K}=0}^{\infty} \frac{q^{-(\mathcal{K}+z)}}{1-q^{-(\mathcal{K}+z)}} \Big] \\ &= -\ln(1-q) + \ln q \Big[z - \frac{1}{2} - \sum_{\mathcal{K}=0}^{\infty} \frac{q^{-\mathcal{K}z}}{1-q^{-\mathcal{K}z}} \Big]. \end{aligned} \tag{6. 26}$$

However, in [47], it has been observed that $\lim_{q\to 1^+} \check{\phi}_q(z) = \lim_{q \to 1^-} \check{\phi}_q(z) = \check{\phi}(z)$.

Proposition 6.5. *Let* $q \in (0,1),\ \zeta_2 > \zeta_1 > 0$, *then the following inequality holds holds*

$$\begin{aligned} &\Big| \frac{1}{4}\Big[\check{\phi}_q{}'(\zeta_2) + 2\check{\phi}_q{}'\Big(\frac{\zeta_2 + 2\eta(\zeta_1,\zeta_2)}{2}\Big) + \check{\phi}_q{}'(\zeta_2 + 2\eta(\zeta_1,\zeta_2))\Big] - \Big(\frac{\check{\phi}_q(l_2) - \check{\phi}_q{}'(l_1)}{l_2 - l_1}\Big)\Big| \\ &\leq \frac{(\zeta_1 - \zeta_2)\big(\frac{1}{\beta+1}\big)^{1/\beta}}{8.2^{s/\alpha}(s+1)^{1/\alpha}} \Bigg[\Big[\Big|\phi_q^{\check{(2)}}(\zeta_1)\Big|^\alpha + (2^{s+1}-1)\Big|\phi_q^{\check{(2)}}(\zeta_2)\Big|^\alpha \Big]^{1/\alpha} \\ &\quad + \Big[(2^{s+1}-1)\Big|\phi_q^{\check{(2)}}(\zeta_1)\Big|^\alpha + \Big|\phi_q^{\check{(2)}}(\zeta_2)\Big|^\alpha \Big]^{1/\alpha} \Bigg], \end{aligned} \tag{6. 27}$$

for all $z > 0$.

Proof. Taking into account the definition of function $\check{\phi}_q$, we conlude that a completely monotone function $z \to \check{\phi}_q{}'(z)$ defined on $(0,\infty)$ and also is s-preinvex on $(0,\infty)$. Therefore, applying inequality (4. 15), we get the immediate consequences. □

Proposition 6.6. *Let* $q \in (0,1),\ \zeta_2 > \zeta_1 > 0$, *then the following inequality holds holds*

$$\begin{aligned} &\frac{1}{4}\Big[\check{\phi}_q{}'(\zeta_2) + 2\check{\phi}_q{}' \ \frac{\zeta_2 + 2\eta(\zeta_1,\zeta_2)}{2} \ + \check{\phi}_q{}'(\zeta_2 + 2\eta(\zeta_1,\zeta_2))\Big] - \ \frac{\check{\phi}_q(l_2) - \check{\phi}_q{}'(l_1)}{l_2 - l_1} \\ &\leq \frac{(\zeta_2 - \zeta_1)}{2^{4-1/\alpha}} \Bigg[\ \frac{1}{2} - \frac{s2^s + 1}{2^{2s}(s+1)(s+2)} \ \phi_q^{\check{(2)}}(\zeta_2) \ ^\alpha + \ \frac{s2^s + 1}{2^{2s}(s+1)(s+2)} \ \phi_q^{\check{(2)}}(\zeta_1) \ ^\alpha \ ^{1/\alpha} \\ &\quad + \ \frac{1}{2} - \frac{s2^s + 1}{2^{2s}(s+1)(s+2)} \ \phi_q^{\check{(2)}}(\zeta_1) \ ^\alpha + \ \frac{s2^s + 1}{2^{2s}(s+1)(s+2)} \ \phi_q^{\check{(2)}}(\zeta_2) \ ^\alpha \ ^{1/\alpha} \Bigg], \end{aligned} \tag{6. 28}$$

for all $z > 0$.

Proof. Taking into account the definition of function $\breve{\phi}_q$, we conlude that a completely monotone function $z \rightarrow \breve{\phi}_q{}'(z)$ defined on $(0, \infty)$ and also is s-preinvex on $(0, \infty)$. Therefore, applying inequality (4. 15), we get the immediate consequences. □

7. CONCLUSION

A novel idea of the weighted identity of Simpson-like type, we established numerous generalizations of Simpson-like type integral inequalities for modified s-preinvex mappings. Moreover, several intriguing results are computed, for instance, we applied the explored outcomes to two special functions including modified Bessel function and q-digamma function. With these strategies and concept established in this investigation, it is important to additionally investigate the weighted inequalities of Simpson-like type. we may use these results in inequality theory, machine learning, weather forecasting, quantum calculus, robotics and optimizations.

REFERENCES

[1] T. Abdeljawad, S. Rashid, H. Khan and Y. M. Chu, *On new fractional integral inequalities for p-convexity within interval–valued functions,* Adv. Diff. Equs. **2020**(2020), 1–17.

[2] T. Abdeljawad, S. Rashid, Z. Hammouch and Y. M. Chu, *Some new local fractional inequalities associated with generalized (s,m)-convex functions and applications,* Adv. Diff. Equs. **2020**(2020) 1–27.

[3] M. A. Alqudah, R. Ashraf, S. Rashid, J. Singh, Z. Hammouch and T. Abdeljawad, *Novel numerical investigations of fuxxy Cauchy reaction–diffusion models via generalized fuzzy fractional derivative operators,* Fract Fractional, **5**(2021) 151.

[4] M. Al-Qurashi, S. Rashid and F. Jarad, *A computational study of a stochastic fractal-fractional hepatitis B virus infection incorporating delayed immune reactions via the exponential decay,* Math. Biosci Eng. **19**(2022) 12950–12980

[5] T. Antczak, *Mean value in invexity analysis,* Nonlinear Anal. **60,** (2005)1473–1484, 2005.

[6] A. Barani, A. G. Ghazanfari and S. S.Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex,* J. Inequal. Appl.**2012,** (2012).

[7] S. Boyd and L. Vandenberghe, *Convex Optimization,* Cambridge University Press, (2004). DOI: 10.1017/CBO9780511804441-xi.

[8] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization,* Springer, (2000). DOI: 10.1007/978-1-4757-9859-3-xi.

[9] S. B. Chen, S. Rashid, M. A. Noor, R. Ashraf and Y. M. Chu, *A new approach on farctional calculus and probability density function,* AIMS Mathematics, **5**(2020) 7041-7055.

[10] S. B. Chen, S. Rashid, Z. Hammouch, M. A. Noor, R. Ashraf and Y. M. Chu, *Integral inequalities via Raina's fractional integral operators with respect to a monotone function,* Adv. Diff. Equs. **2020**(2020) 1–20.

[11] S. B. Chen, S. Rashid, M. A. Noor, Z. Hammouch and Y. M. Chu, *New fractional approaches for n-polynomial p-convexity with applications in special function theory,* Adv. Diff. Equs. **2020**(2020) 1–13.

[12] B. D. Craven, *Duality for generalized convex fractional programs. In Generalized Convexity in Optimization and Economics; Schaible,* S., Ziemba, T., Eds.; Academic Press: San Diego, CA, USA, (1981) 473–489.

[13] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,* Appl. Math. Lett.**11,** (1998)91–95.

[14] S. S. Dragomir, R. P. Agarwal and P. Cerone, *On Simpson's inequality and applications,* J. Inequal. Appl.**5,** (2000) 533–579.

[15] T. Du, H. Wang, M. A. Latif and Y. Zhang, *Estimation type results associated to k-fractional integral inequalities with applications,* J. King. Saud, Univ-Sci., (2018). https://doi.org/10.1016/j.jksus.2018.09.010

[16] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions,* J. Math. Anal. Appl. **80,** (1981) 545–550.

[17] F. A. Idris, A. L. Buhari and T. U. Adamu, *Bessel functions and their applications: solution to schrödinger equation in a cylindrical function of the second kind and Hankel functions,* Inter. J. Novel Research in Physics Chemistry & Mathematics, **3,** (2016) 17–31.

[18] F. Jarad, T. Abdeljawad, S. Rashid and Z. hammouch, *More properties of the proportional fractional integrals and derivatives of a function with respect to another function,* Adv. Diff. Eqs. **2020**(2020) 1–16.

[19] Y. Khurshid, M. Adil Khan,Y. -M. Chu and Z. A. Khan, *Hermite-Hadamard-Fejér inequalities for conformable fractional integrals via preinvex functions,* J. Funct. Spaces,**2019,** (2019).

[20] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula,* Appl. Math. Comput.**147,** (2004) 137–146 2004.

[21] M. A. Khan, Y. -M. Chu, T. U. Khan and J. Khan, *Some new inequalities of Hermite Hadamard type for s-convex functions with applications,* Open math. **15**, (2017) 1414–1430, 2017.

[22] S. Mititelu, *Invex sets,* Stud. Cercet. Stiint. Univ. Bacau Ser. Mat. **46,** (1994) 529–532.

[23] S. R. Mohan and S. K. Neogy, *On invex sets and preinvex functions,* J. Math. Anal. Appl. **189,** (1995) 901–908

[24] M. A. Noor, *Hermite-Hadamard integral inequalities for log-preinvex functions,* J. Math. Anal. Approx. Theory,**2,** (20074) 126–131, 2007.

[25] S. Rashid, A. O. Akdemir, F. Jarad, M. A. Noor and K. I. Noor, *Simpson's type integral inequalities for k-fractional integrals and their applications,* AIMS Mathematics, **4,** No. 4 (2019) 1087–1100.

[26] S. Rashid, T. Abdeljawad, F. Jarad and M. A. Noor, *Some estimates for generalized Riemann-Liouville fractional integrals of exponentially convex functions and their applications,* Mathematics, **7**(2019) 807.

[27] S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom and Y. M. Chu, *Inequalities by means of generalized proportional fractional integral operators with respect to another function,* Mathematics, **7**(2019) 1225.

[28] S. Rashid, M. A. Noor and K. I. Noor, *New estimates for exponentially convex functions via conformable fractional operator,* Fract. Fractional, **3**(2019) 19.

[29] S. Rashid, F. Jarad and Y. M. Chu, *A note on reverse Minkowski inequality via generalized proportional fractional integral operator with respect to anoher function,* Math. Prob. Eng. **2020**(2020).

[30] S. Rashid, R. Ashraf, M. A. Noor, K. I. Noor and Y. M. Chu, *New weighted generalization for differentiable exponentially convex mapping with application,* emphAIMS Mathematics, **5**(2020) 3525-3547.

[31] S. Rashid, Z. Hammouch, R. Ashraf, D. Baleanu and K. S. Nisar, *New quantum estimates in the settings of fractional calculus theory,* Adv. Diff. Equs. **2020**(2020) 1–17.

[32] S. Rashid, I. I?can, D. Baleanu and Y. M. Chu, *Generation of new fractional inequalities via n-polynomial s-type convexity with applications,* Adv. Diff. Equs, **2020**(2020) 1–20.

[33] S. Rashid, K. T. Kubra and J. L. G. Guirao, *Construction of an approximate analytical solution for multi-dimensional fractional Zakharov-Kuznetsov equation via Aboodh Adomian decomposition method,* Symmetry, **13**(2021) 1542.

[34] S. Rashid, A. G. Ahmad, F. Jarad and A. Alsaadi, *Nonlinear fractional differential equations and their existence via fixed point theory concerning to Hilfer generalized proportional fractional derivative,* AIMS Mathematics, **8**(2022), 382–403.bibitemSA26S. Rashid, S. I. Butt, Z. Hammouch, E. Bonyah. An Efficient Method for Solving Fractional Black-Scholes Model with Index and Exponential Decay Kernels, J. Fun Space. **2022**(2022).

[35] S. Rashid, R. Ashraf, Q. Ul-Ain Asif and F. Jarad, *Novel dynamics of a stochastic fractal-fractional immune effector response to viral infection via latently infectious tissues,* Math. Bioscis Eng. **19**(2022) 11563–11594.

[36] S. Rashid, F. Jarad and A. K. Alsharidi, *Numerical investigation of fractional-order cholera epidemic model with transmission dynamics via fractal-fractional operator technique,* Chaos Solit. Fract. **162**(2022).

[37] S. Rashid, B. Kanwal, F. Jarad and S. K. Elagan, *A peculiar application of the fractal–fractional derivative in the dynamics of a nonlinear scabies model,* Res. Phy. **38**(2022), 105634.

[38] S. Rashid, Z. Hammouch, H. Aydi, A. G. Ahmad and A. M. Alsharif, *New computations of the time-fractional Fisher's model via generalized farctional integral operators by means of the Elzaki transform,* Fract. Fractional, **5**(2021) 94.

[39] S. Rashid, H. Kalsoom, Z. Hammouch, R. Ashraf, D. Baleanu and Y. M. Chu, *New multi-parameterized estimates having pth-order differentiability in fractional calculus for predominating h-convex functions in Hilbert space,* Symmetry,**12**(2021) 222.

[40] S. Rashid, A. Khalid, S. Sultana, Z. Hammouch, R. Shah and A. M. Alsharif, *A novel analytical view of time-fractional Korteweg-De Vries equations via a new integral transform,* Symmetry, **13**(2021) 1254.

[41] S. Rashid, S. Sultana, Z. Hammouch, F. Jarad and Y. S. Hamed, *Nove aspects of discrete dynamical type inequalities within fractional operators having generalized h-discrete Mittag-Leffler kernel and application,* Chaos. Solit. Fract, **115**(2021), 111204.

[42] S. Rashid, Y. M. Chu, J. Singh and D. Kumar, *A unifying computational framework for novel estimates involving discrete fractional calculus approaches,* Alexandria Eng. J, **60**(2021) 2677-2685.

[43] S. Rashid, M. A. Noor and K. I. Noor, *Fractional exponentially m-convex functions and inequalities,* Inter. Anala. Appl. **17**(2019) 464-478.

[44] S. Rashid, R. Ashraf, A. O. Akdemir, M. A. Alqudah, T. Abdeljawad and M. S. Mohamed, *Analytic fuzzy formulation of a time-fractional Fornberg-Whitham model with power and Mittag-Leffler kernels* Fract. Fractional, **5**(2021), 113.

[45] S. Rashid, Z. Hammouch, D. Baleanu and Y. M. Chu, *New generalizations in the sense of the weighted non-singular fractional integral operator*, Fractals,**28**(2020) 2040003.

[46] E. Set, A. O. Akdemir and M. E.Özdemir, *Simpson type integral inequalities for convex functions via Riemann-Liouville integrals,* Filomat,**31,** (2017) 4415–4420

[47] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, (1944).

[48] T. Weir and B. Mond, *Preinvex functions in multi objective optimization*, J. Math. Anal. Appl. **136,** (1988) 29–38

[49] S. -W. Yao, S. Rashid, M. Inc and E. Elattar, *On fuzzy numerical model dealing with the control of glucose in insulin therapies for diabetes via non-singular kernel in the fuzzy sense,* AIMS Mathematics. **7**(2022) 17913–17941.

[50] T. Zhu, P. Wang and T. Du, *Some estimates on the weighted Simpon-like type integral inequalities and their application*, J. Nonlinear. Funct. Anal. **2020,** Article ID 17, 2020.

[51] S. S. Zhou, S. Rashid, F. Jarad, H. Kalsoom and Y. M. Chu, *New estimates considering the generalized proportional Hadamard fractional integral operator,* Adv. Diff. Equs. **2020**(2020) 1-15.

[52] S. S. Zhou, S. Rashid, M. A. Noor, K. I. Noor, F. Safdar and Y. M. Chu, *New Hermite-Hadamard type inequalities for exponentially convex functions and applications, AIMS Mathematics,* **5**(2020) 6874-6901.