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Application of Double Shehu Transforms to Caputo Fractional Partial Differential Equations

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Abstract.: Recently, a new generalization of the double Laplace transform and double Sumudu transform, namely, double Shehu transform, has been introduced. This study has derived the double Shehu transform (DHT) formulas for the fractional Caputo operators. We then applied this generalized integral transform to solve fractional partial differential equations involving Caputo derivative.

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1. Introduction

Differential equations of fractional orders are applied in various branches of applied mathematics, physics, finance, and other sciences [25, 19, 28]. Several approaches have been introduced and developed to get approximate or exact solutions of these types of equations including, the Adomian decomposition method (ADM) [1, 2], the homotopy perturbation method [27], variational iteration method [30], differential transform method [21], fractional residual power series method [23]. Among these methods, the integral transformation methods which are rather and popular, thus in the literature, there are various types of integral transforms such as Laplace transform [29, 24, 7, 4], Sumudu transform [12, 6, 5], natural transform [9] and many others. These transforms are widely implemented to get analytical solutions of ordinary, partial differential equations, and integral equations. After the appearance of one dimensional integral transform, the idea was extended to the two dimensional and named the double integral transform [17, 18, 15]. One dimensional Shehu transform [26] is closely related to Sumudu and Laplace transforms. This transform was first defined in 2019 by Shehu Maitama and Weidong Zhao, and they implemented it to solve ordinary and partial differential equations for more about shehu transform and its properties and applications see [22, 14, 13]. Double Shehu transform (DHT) [3] which is a new generalization of double Laplace transform (DLT) and double Sumudu transform (DST), was recently introduced by Alfaqeih and Mısırlı which applied to get the exact solutions of partial differential equations of two variables and integral equations with convolution.

The rest of the article is presented as follows: In section 2, we give some notations about fractional calculus. We then present the definition of one dimensional Shehu transform and the double Shehu transform with related theorems in section 3. In section 4, we derive the formulas of the double Shehu transform for the fractional Caputo operators. In section 5, we implement the double Shehu transform to solve fractional partial differential equations. Section 6 is for the conclusions of this article.

2. Preliminaries

In this segment, some basic notations about fractional calculus and Shehu transforms are presented.

Definition 2.1. [11, 10] *Let $g(x,t)$ be continuous function. Then, the Riemann-Liouville partial fractional integrals are defined by:*

$$ {}_t\mathrm{I}^{\mu} g(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\lambda)^{\mu-1} g(x,\lambda)\, d\lambda, \tag{2. 1}$$

$$ {}_x\mathrm{I}^{\nu} g(x,t) = \frac{1}{\Gamma(\nu)} \int_0^x (x-\theta)^{\nu-1} g(\theta,t)\, d\theta, \tag{2. 2}$$

where $(x,t) \in (0,\infty)\times(0,\infty)$ and $\nu,\mu > -1$.

Definition 2.2. [11] *Let $g(x,t)$ be a continuous function. Then, the Riemann-Liouville partial fractional derivatives are defined by:*

$$ {}_R D_t^{\mu} g(x,t) = \frac{1}{\Gamma(n-\mu)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\lambda)^{n-\mu-1} g(x,\lambda)\, d\lambda, \tag{2. 3}$$

$$ {}_R D_x^{\nu} g(x,t) = \frac{1}{\Gamma(m-\nu)} \left(\frac{d}{dx}\right)^m \int_0^x (x-\theta)^{m-\nu-1} g(\theta,t)\, d\theta. \tag{2. 4}$$

Definition 2.3. [11] *The Caputo partial fractional derivatives of a function $g(x,t)$ are given by:*

$$ D_t^{\mu} g(x,t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\lambda)^{n-\mu-1} g^{(n)}(x,\lambda)\, d\lambda, \tag{2. 5}$$

$$ D_x^{\nu} g(x,t) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-\theta)^{m-\nu-1} g^{(m)}(\theta,t)\, d\theta, \tag{2. 6}$$

where $m-1<\nu<m$, $n-1<\mu<n$ and $n,m\in\mathrm{N}$.
The following are some basic properties of the partial fractional integrals and Caputo derivatives;

(1) ${}_t\mathrm{I}^{\mu}\,{}_x\mathrm{I}^{\nu} g(x,t) = \frac{1}{\Gamma(\nu)\Gamma(\mu)} \int_0^t (t-\lambda)^{\mu-1} (x-\theta)^{\nu-1} g(\theta,\lambda)\, d\lambda d\theta,$

(2) $D_t^{\mu} D_x^{\nu} g(x,t) =$
$$\frac{1}{\Gamma(n-\mu)\Gamma(m-\nu)} \int_0^t (t-\lambda)^{n-\mu-1} (x-\theta)^{m-\nu-1} g^{(n+m)}(\theta,\lambda)\, d\lambda d\theta,$$

(3) ${}_t\mathrm{I}^{\mu} D_t^{\mu} g(x,t) = g(x,t) - \sum_{i=0}^{m} \frac{g^{(i)}(x,0)}{i!} t^i,$

(4) $D_t^{\mu}\, {}_t\mathrm{I}^{\mu} g(x,t) = g(x,t)\,.$

Definition 2.4. *The Mittag-Leffler function is defined by:*
$$E_{\nu,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\nu j+\beta)},\ x,\beta \in \mathrm{C},\, \Re(\nu) > 0. \tag{2. 7}$$

3. SHEHU TRANSFORM

Definition 3.1. [26] *The single Shehu transforms (HT) of a real-valued function* $g(x,t)$ *with respect to the variables* x and t *respectively, are defined by:*
$$H_x(g(x,t):p,u) = \int_0^{\infty} e^{-\frac{px}{u}} g(x,t)\, dx, \tag{3. 8}$$
$$H_t(g(x,t):q,v) = \int_0^{\infty} e^{-\frac{qt}{v}} g(x,t)\, dt. \tag{3. 9}$$

In the following lemmas the Shehu transforms of the fractional integrals and the Caputo fractional derivatives are given.

Lemma 3.1. [14] *Let* $\mu > 0$*, and* $g(x,t)$ *are of exponential order. Then, the single Shehu transform of* ${}_t\mathrm{I}^{\mu} g(x,t)$ *is given by:*
$$H_t({}_t\mathrm{I}^{\mu} g(x,t)) = \left(\frac{q}{v}\right)^{-\mu} H_t(g(x,t))\,. \tag{3. 10}$$

Lemma 3.2. [14] *Let* $\mu > 0,\ m-1 < \mu < m,\ (m \in \mathrm{N})$*, be such that* $g \in C^n(0,\infty)$ *and is of exponential order. Then, the single Shehu transform of Caputo fractional derivative* $D_t^{\mu} g(x,t)$ *is given by:*
$$H_t(D_t^{\mu} g(x,t)) = \left(\frac{q}{v}\right)^{\mu} H_t(g(x,t)) - \sum_{j=0}^{m-1} \left(\frac{q}{v}\right)^{\mu-1-j} \left(\frac{\partial^j}{\partial t^j} g(x,0)\right). \tag{3. 11}$$

Definition 3.2. [3] *The (DHT) of the function* $g(x,t)$ *is defined by the double integral as*
$$H_x H_t(g(x,t)) = G[(p,q),(u,v)] = \int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{px}{u}+\frac{qt}{v}\right)} g(x,t)\, dxdt. \tag{3. 12}$$

In the following lemma we present the double Shehu transforms formulas of the partial derivatives of an arbitrary integer order.

Lemma 3.3. [3] *Let* $g \in C^1[(0,\infty)\times(0,\infty)]$*. Then, the double Shehu transform of partial derivatives* $\frac{\partial^n g}{\partial x^n}, \frac{\partial^m g}{\partial t^m}$*, are given respectively by:*

(1) $H_x H_t\left(\frac{\partial^n g}{\partial x^n}\right) = \left(\frac{p}{u}\right)^n H_x H_t(g(x,t)) - \sum_{j=0}^{n-1} \left(\frac{p}{u}\right)^{n-1-j} H_t\left(\frac{\partial^j}{\partial x^j} g(0,t)\right),$

(2) $H_xH_t\left(\frac{\partial^m g}{\partial t^m}\right) = \left(\frac{q}{v}\right)^m H_xH_t\left(g(x,t)\right) - \sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{m-1-j} H_x\left(\frac{\partial^j}{\partial t^j} g\left(x,0\right)\right),$

(3) $$H_xH_t\left(\frac{\partial^{n+m} g}{\partial x^n t^m}\right) = \left(\frac{p}{u}\right)^n \left(\frac{q}{v}\right)^m H_xH_t\left(g(x,t)\right) - \left(\frac{q}{v}\right)^m \sum_{j=0}^{n-1}\left(\frac{p}{u}\right)^{n-1-j} H_t\left(\frac{\partial^i}{\partial x^i} g\left(0,t\right)\right) - \left(\frac{p}{u}\right)^n \sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{m-1-j} H_x\left(\frac{\partial^j}{\partial t^j} g\left(x,0\right)\right) + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{m-1-j}\left(\frac{p}{u}\right)^{n-1-i}\left(\frac{\partial^{j+i}}{\partial t^j \partial x^i} g\left(0,0\right)\right).$$

Some fundamental properties of double Shehu transform and inverse double Shehu transform

Let a, b be an arbitrary constants, then the following are holds [3]:

(1) $H_xH_t\left(1\right) = \frac{uv}{pq},$

(2) $H_xH_t\left(e^{(ax+bt)}\right) = \frac{uv}{(p-au)(q-bv)},$

(3) $H_xH_t\left(x^n t^m\right) = n!m!\left(\frac{u}{p},\right)^{n+1}\left(\frac{v}{q}\right)^{m+1},\ n, m = 0, 1, 2, \cdots,$

(4) $H_xH_t\left(x^\alpha t^\gamma\right) = \Gamma\left(\alpha+1\right)\left(\frac{u}{p}\right)^{\alpha+1}\Gamma\left(\gamma+1\right)\left(\frac{v}{q}\right)^{\gamma+1},\ \alpha \geq -1, \gamma \geq -1,$

(5) $H_xH_t\left(\cos\left(ax+bt\right)\right) = \frac{uv(pq+abuv)}{(p^2+a^2u^2)},$

(6) $H_xH_t\left(\cos\left(ax+bt\right)\right) = \frac{uv(pq+abuv)}{(p^2+a^2u^2)},$

(7) $H_xH_t\left(\sin\left(ax+bt\right)\right) = \frac{uv(bpv+aqu)}{(q^2+b^2v^2)},$

(8) $H_xH_t\left(g\left(ax, bt\right)\right) = \frac{1}{ab} G\left(\left(\frac{p}{a}, \frac{q}{b}\right), (u, v)\right),$

(9) $H_xH_t\left(e^{-(ax+bt)} g\left(x,t\right)\right) = G\left(\left(p+au, q+bv\right), (u,v)\right),$

(10) If $g(x,t) = f(x)h(t)$, then $H_xH_t(f(x)h(t)) = H_x(f(x))H_t(h(t))$,

(11) $H_xH_t[.]$ is a linear operator, that is
$H_xH_t\left[\left(ag+bh\right)(x,t)\right] = aH_xH_t\left[g(x,t)\right] + bH_xH_t\left[h(x,t)\right].$

Theorem 3.4. [14] *The single Shehu transform of* $x^{\beta-1}E_{\nu,\beta}\left(cx^\nu\right)$ *is given by:*

$$H_x\left(x^{\beta-1}E_{\nu,\beta}\left(cx^\nu\right)\right) = \frac{\left(\frac{p}{u}\right)^{\nu-\beta}}{\left(\frac{p}{u}\right)^{\nu} - c},\quad |c| < \left|\left(\frac{p}{u}\right)^{\nu}\right| \tag{3. 13}$$

Theorem 3.5. [3] *Let* $g\left(x,\ t\right),$ and $h\left(x,\ t\right)$ *be of exponential order, having double Shehu transforms* $H^2_{xt}\left(g\left(x,t\right)\right),$ and $H^2_{xt}\left(h\left(x,t\right)\right)$*, respectively. The double Shehu transform of the convolution of* g *and* h

$$\left[g**h\right]\left(x,\ t\right) = \int_0^x\int_0^t g\left(x-\kappa, t-\lambda\right) h\left(\kappa,\lambda\right) d\kappa d\lambda, \tag{3. 14}$$

is given by

$$H_{xt}^{2}\left(\left[g**h\right]\left(x,\ t\right)\right)=H_{xt}^{2}\left(g\left(x,\ t\right)\right)H_{xt}^{2}\left(h\left(x,\ t\right)\right). \tag{3. 15}$$

4. Main Results

In this section, we establish the double Shehu transform of the factional integrals and derivatives.
In the next theorem, we present the (DHT) of the fractional integral w.r.t t.

Theorem 4.1. *Let $\mu>0$, and $g\left(x,t\right)$ is of exponential order. Then, double Shehu transform of the fractional integral ${}_{t}\mathrm{I}^{\mu}g\left(x,t\right)$ is given as follows:*

$$H_xH_t\left({}_{t}\mathrm{I}^{\mu}g\left(x,t\right)\right)=\left(\frac{q}{v}\right)^{-\mu}H_xH_t\left(g\left(x,t\right)\right). \tag{4. 16}$$

Proof. By using the double Shehu transform of the convolution with respect to t, we have

$$\begin{aligned}H_xH_t\left({}_{t}\mathrm{I}^{\mu}g\left(x,t\right)\right)&=H_xH_t\left(\frac{1}{\Gamma\left(\mu\right)}t^{\mu-1}*g\left(x,t\right)\right)\\&=H_x\left(\frac{1}{\Gamma\left(\mu\right)}H_t\left(t^{\mu-1}\right)H_t\left(g\left(x,t\right)\right)\right)\\&=\left(\frac{q}{v}\right)^{-\mu}H_xH_t\left(g\left(x,t\right)\right).\end{aligned}$$

□

Similarly, next theorem, presents the (DHT) of the fractional integral w.r.t x.

Theorem 4.2. *Let $\nu>0$, and $g\left(x,t\right)$ is of exponential order. Then, double Shehu transform of the fractional integral ${}_{x}\mathrm{I}^{\nu}g\left(x,t\right)$ is given as follows:*

$$H_xH_t\left({}_{x}\mathrm{I}^{\nu}g\left(x,t\right)\right)=\left(\frac{p}{u}\right)^{-\nu}H_xH_t\left(g\left(x,t\right)\right). \tag{4. 17}$$

Proof. By using the double Shehu transform of the convolution with respect to t, we have

$$\begin{aligned}H_xH_t\left({}_{x}\mathrm{I}^{\nu}g\left(x,t\right)\right)&=H_xH_t\left(\frac{1}{\Gamma\left(\nu\right)}x^{\nu-1}*g\left(x,t\right)\right)\\&=H_t\left(\frac{1}{\Gamma\left(\nu\right)}H_x\left(x^{\nu-1}\right)H_x\left(g\left(x,t\right)\right)\right)\\&=\left(\frac{p}{u}\right)^{-\nu}H_xH_t\left(g\left(x,t\right)\right).\end{aligned}$$

□

In the following theorem, we introduce the double Shehu transform for the mixed fractional integral.

Theorem 4.3. *Let $\nu,\mu>0$, and $g\left(x,t\right)$ is of exponential order. Then, double Shehu transform of the fractional integral ${}_{t}\mathrm{I}^{\mu}{}_{x}\mathrm{I}^{\nu}g\left(x,t\right)$ is given as follows:*

$$H_xH_t\left({}_{t}\mathrm{I}^{\mu}{}_{x}\mathrm{I}^{\nu}g\left(x,t\right)\right)=\left(\frac{q}{v}\right)^{-\mu}\left(\frac{p}{u}\right)^{-\nu}H_xH_t\left(g\left(x,t\right)\right). \tag{4. 18}$$

Proof. By using the double Shehu transform of the double convolution, we have

$$\begin{aligned} H_xH_t\left({}_t\mathrm{I}^{\mu}{}_x\mathrm{I}^{\nu}g\left(x,t\right)\right) &= H_xH_t\left(\frac{1}{\Gamma\left(\nu\right)\Gamma\left(\mu\right)}x^{\nu-1}t^{\mu-1}**g\left(x,t\right)\right) \\ &= \frac{1}{\Gamma\left(\nu\right)\Gamma\left(\mu\right)}H_xH_t\left(x^{\nu-1}t^{\mu-1}\right)H_xH_t\left(g\left(x,t\right)\right) \\ &= \left(\frac{q}{v}\right)^{-\mu}\left(\frac{p}{u}\right)^{-\nu}H_xH_t\left(g\left(x,t\right)\right). \end{aligned}$$

□

In the next two theorems, we introduce the double Shehu transform of the Caputo fractional derivative w.r.t t and x, respectively.

Theorem 4.4. *Let* $\mu,\nu>0,\ m-1<\mu<m,\ \mathrm{n}-1<\nu<n\,(m,n\in\mathrm{N})$*, be such that* $g\in C^{k}\left[(0,\infty)\times(0,\infty)\right], k=\max\left\{m,n\right\}$ *and is of exponential order. Then, the double Shehu transforms of Caputo fractional derivatives are given by:*

$$H_xH_t\left(D_t^{\mu}g\left(x,t\right)\right)=\left(\frac{q}{v}\right)^{\mu}H_xH_t\left(g(x,t)\right)-\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j}H_x\left(\frac{\partial^j}{\partial t^j}g\left(x,0\right)\right). \tag{4. 19}$$

Proof. The Caputo fractional derivative with respect to t for the function $g(x,t)$ can be written in the convolution as follows

$$D_t^{\mu}g\left(x,t\right)=\frac{1}{\Gamma\left(m-\mu\right)}t^{m-\mu-1}*\frac{\partial^m g\left(x,t\right)}{\partial t^m}, \tag{4. 20}$$

by applying the double Shehu transform to Eq.(4. 20) we have

$$\begin{aligned} H_xH_t\left(D_t^{\mu}g\left(x,t\right)\right) &= H_xH_t\left(\frac{1}{\Gamma\left(m-\mu\right)}t^{m-\mu-1}*\frac{\partial^m g\left(x,t\right)}{\partial t^m}\right) \\ &= \frac{1}{\Gamma\left(m-\mu\right)}H_xH_t\left(t^{m-\mu-1}\right)H_xH_t\left(\frac{\partial^m g\left(x,t\right)}{\partial t^m}\right) \\ &= \left(\frac{q}{v}\right)^{\mu-m}\left[\begin{array}{l}\left(\frac{q}{v}\right)^{m}H_xH_t\left(g(x,t)\right) \\ -\sum\limits_{j=0}^{m-1}\left(\frac{q}{v}\right)^{m-1-j}H_x\left(\frac{\partial^j}{\partial t^j}g\left(x,0\right)\right)\end{array}\right] \\ &= \left(\frac{q}{v}\right)^{\mu}H_xH_t\left(g(x,t)\right)-\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j}H_x\left(\frac{\partial^j}{\partial t^j}g\left(x,0\right)\right). \end{aligned}$$

□

Theorem 4.5. *Let* $\mu,\nu>0,\ m-1<\mu<m,\ \mathrm{n}-1<\nu<n\,(m,n\in\mathrm{N})$*, be such that* $g\in C^{k}\left[(0,\infty)\times(0,\infty)\right], k=\max\left\{m,n\right\}$ *and is of exponential order. Then, the double Shehu transforms of Caputo fractional derivatives are given by:*

$$H_xH_t\left(D_x^{\nu}g\left(x,t\right)\right)=\left(\frac{p}{u}\right)^{\nu}H_xH_t\left(g(x,t)\right)-\sum_{i=0}^{n-1}\left(\frac{p}{u}\right)^{\nu-1-i}H_t\left(\frac{\partial^i}{\partial x^i}g\left(0,t\right)\right). \tag{4. 21}$$

Proof. The Caputo fractional derivative with respect to x for the function $g(x,t)$ can be written in the convolution as follows

$$D_x^{\nu} g\left(x,t\right) = \frac{1}{\Gamma\left(n-\nu\right)} x^{n-\nu-1} * \frac{\partial^n g\left(x,t\right)}{\partial x^n}, \tag{4. 22}$$

by applying the double Shehu transform to Eq.(4. 22) we have

$$\begin{aligned} H_x H_t\left(D_x^{\nu} g\left(x,t\right)\right) &= H_x H_t\left(\frac{1}{\Gamma\left(n-\nu\right)} x^{n-\nu-1} * \frac{\partial^n g\left(x,t\right)}{\partial x^n}\right) \\ &= \frac{1}{\Gamma\left(n-\nu\right)} H_x H_t\left(x^{n-\nu-1}\right) H_x H_t\left(\frac{\partial^n g\left(x,t\right)}{\partial x^n}\right) \\ &= \left(\frac{p}{u}\right)^{\nu-n}\left[\begin{array}{l} \left(\frac{p}{u}\right)^n H_x H_t\left(g(x,t)\right) \\ -\sum\limits_{j=0}^{n-1}\left(\frac{p}{u}\right)^{n-1-j} H_t\left(\frac{\partial^j}{\partial x^j} g\left(0,t\right)\right) \end{array}\right] \\ &= \left(\frac{p}{u}\right)^{\nu} H_x H_t\left(g(x,t)\right) - \sum_{i=0}^{n-1}\left(\frac{p}{u}\right)^{\nu-1-i} H_t\left(\frac{\partial^i}{\partial x^i} g\left(0,t\right)\right). \end{aligned}$$

□

The last theorem in this section introduces the double Shehu transform of the mixed Caputo fractional derivatives.

Theorem 4.6. *Let* $\mu,\nu>0,\ m-1<\mu<m,\ \mathrm{n}-1<\nu<n\left(m,n\in \mathrm{N}\right)$*, be such that* $g\in C^k\left[(0,\infty)\times(0,\infty)\right], k=\max\left\{m,n\right\}$ *and is of exponential order. Then, the double Shehu transforms of Caputo fractional derivatives are given by:*

$$\begin{aligned} H_x H_t\left(D_x^{\nu} D_t^{\mu} g\left(x,t\right)\right) = &\left(\frac{p}{u}\right)^{\nu}\left(\frac{q}{v}\right)^{\mu} H_x H_t\left(g(x,t)\right) \\ &-\left(\frac{q}{v}\right)^{\mu}\sum_{i=0}^{n-1}\left(\frac{p}{u}\right)^{\nu-1-i} H_t\left(\frac{\partial^i}{\partial x^i} g\left(0,t\right)\right) \\ &-\left(\frac{p}{u}\right)^{\nu}\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j} H_x\left(\frac{\partial^j}{\partial t^j} g\left(x,0\right)\right) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j}\left(\frac{p}{u}\right)^{\nu-1-i}\left(\frac{\partial^{j+i}}{\partial t^j \partial x^i} g\left(0,0\right)\right), \end{aligned} \tag{4. 23}$$

Proof. The Caputo fractional derivative with respect to x,t for the function $g(x,t)$ can be written in the convolution as follows

$$D_x^{\nu} D_t^{\mu} g\left(x,t\right) = \frac{1}{\Gamma\left(n-\nu\right)\Gamma\left(m-\mu\right)} x^{n-\nu-1} t^{m-\mu-1} * \frac{\partial^{n+m} g\left(x,t\right)}{\partial x^n \partial t^m}, \tag{4. 24}$$

by applying the double Shehu transform to Eq.(4. 24) we have

$$H_xH_t\left(D_x^{\nu}D_t^{\mu}g\left(x,t\right)\right)=H_xH_t\left(\frac{1}{\Gamma\left(n-\nu\right)\Gamma\left(m-\mu\right)}x^{n-\nu-1}t^{m-\mu-1}*\frac{\partial^{n+m}g\left(x,t\right)}{\partial x^n\partial t^m}\right)$$
$$=\frac{1}{\Gamma\left(m-\mu\right)}H_xH_t\left(t^{m-\mu-1}\right)\frac{1}{\Gamma\left(n-\nu\right)}H_xH_t\left(x^{n-\nu-1}\right)H_xH_t\left(\frac{\partial^{n+m}g\left(x,t\right)}{\partial x^n\partial t^m}\right),$$

using Theorem (3.5) we have

$$=\left(\frac{p}{u}\right)^{\nu-n}\left(\frac{q}{v}\right)^{\mu-m}\left[\begin{array}{l}\left(\frac{p}{u}\right)^n\left(\frac{q}{v}\right)^m H_xH_t\left(g(x,t)\right)\\ -\left(\frac{q}{v}\right)^m\sum\limits_{j=0}^{n-1}\left(\frac{p}{u}\right)^{n-1-j}H_t\left(\frac{\partial^i}{\partial x^i}g\left(0,t\right)\right)\\ -\left(\frac{p}{u}\right)^n\sum\limits_{j=0}^{m-1}\left(\frac{q}{v}\right)^{m-1-j}H_x\left(\frac{\partial^j}{\partial t^j}g\left(x,0\right)\right)\\ +\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{m-1}\left(\frac{q}{v}\right)^{m-1-j}\left(\frac{p}{u}\right)^{n-1-i}\left(\frac{\partial^{j+i}}{\partial t^j\partial x^i}g\left(0,0\right)\right)\end{array}\right],$$

simplifying, we get

$$\begin{aligned}H_xH_t\left(D_x^{\nu}D_t^{\mu}g\left(x,t\right)\right)&=\left(\frac{p}{u}\right)^{\nu}\left(\frac{q}{v}\right)^{\mu}H_xH_t\left(g(x,t)\right)\\&-\left(\frac{q}{v}\right)^{\mu}\sum_{i=0}^{n-1}\left(\frac{p}{u}\right)^{\nu-1-i}H_t\left(\frac{\partial^i}{\partial x^i}g\left(0,t\right)\right)\\&-\left(\frac{p}{u}\right)^{\nu}\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j}H_x\left(\frac{\partial^j}{\partial t^j}g\left(x,0\right)\right)\\&+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j}\left(\frac{p}{u}\right)^{\nu-1-i}\left(\frac{\partial^{j+i}}{\partial t^j\partial x^i}g\left(0,0\right)\right).\end{aligned}$$

□

5. APPLICATIONS

To described the double Shehu transform method, let us consider the linear non-homogeneous fractional partial differential equation in operator form

$$D_t^{\mu}g\left(x,t\right)+D_x^{\nu}g\left(x,t\right)+\sum_{j=0}^{m}A_j\frac{\partial^j g}{\partial t^j}+\sum_{i=0}^{n}B_i\frac{\partial^i g}{\partial x^i}=f\left(x,t\right),\qquad(5.\ 25)$$
$$m-1<\beta\le m,n-1\le\nu\le n,$$

with initial and boundary conditions

$$D_t^j g\left(x,0\right)=\phi_j\left(x\right),D_x^i g\left(0,t\right)=\varphi_i\left(t\right),\qquad(5.\ 26)$$
$$j=0,1,2,...,m-1,i=0,1,2,...n-1.$$

Applying the (DHT) to both sides of Eq.(5. 25) and single (HT) to the initial and boundary conditions Eq.(5. 26), we get

$$
\begin{aligned}
&\left(\frac{q}{v}\right)^{\mu} H_x H_t\left(g(x,t)\right)-\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j} H_x\left(\frac{\partial^j g\left(x,0\right)}{\partial t^j}\right)\\
&+\sum_{j=0}^{m} A_j\left[\left(\frac{q}{v}\right)^{j} H_x H_t\left(g(x,t)\right)-\sum_{k=0}^{j-1}\left(\frac{q}{v}\right)^{k-1-j} H_x\left(\frac{\partial^k g\left(x,0\right)}{\partial t^k}\right)\right]\\
&+\sum_{i=0}^{n} B_i\left[\left(\frac{p}{u}\right)^{i} H_x H_t\left(g(x,t)\right)-\sum_{s=0}^{i-1}\left(\frac{p}{u}\right)^{i-1-s} H_t\left(\frac{\partial^s g\left(0,t\right)}{\partial x^s}\right)\right]\\
&-\sum_{i=0}^{n-1}\left(\frac{p}{u}\right)^{\nu-1-i} H_t\left(\frac{\partial^i g\left(0,t\right)}{\partial x^i}\right)+\left(\frac{p}{u}\right)^{\nu} H_x H_t\left(g(x,t)\right)=H_x H_t\left(f(x,t)\right), \quad (5.\ 27)
\end{aligned}
$$

substituting the single Shehu transforms of the initial and boundary conditions

$$
H_x\left(D_t^j g\left(x,0\right)\right)=\bar{\phi}_j, H_t\left(D_x^i g\left(0,t\right)\right)=\bar{\varphi}_i\ ,
$$

in Eq.(5. 27), we get

$$
\begin{aligned}
&\left(\frac{q}{v}\right)^{\mu} H_x H_t\left(g(x,t)\right)-\sum_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j}\bar{\phi}_j+\left(\frac{p}{u}\right)^{\nu} H_x H_t\left(g(x,t)\right)\\
&-\sum_{i=0}^{n-1}\left(\frac{p}{u}\right)^{\nu-1-i}\bar{\varphi}_i+\sum_{j=0}^{m} A_j\left[\left(\frac{q}{v}\right)^{j} H_x H_t\left(g(x,t)\right)-\sum_{k=0}^{j-1}\left(\frac{q}{v}\right)^{k-1-j}\bar{\phi}_k\right]\\
&+\sum_{i=0}^{n} B_i\left[\left(\frac{p}{u}\right)^{i} H_x H_t\left(g(x,t)\right)-\sum_{s=0}^{i-1}\left(\frac{p}{u}\right)^{i-1-s}\bar{\varphi}_s\right]=H_x H_t\left(f(x,t)\right), \qquad (5.\ 28)
\end{aligned}
$$

simplifying,

$$
H_x H_t\left(g(x,t)\right) = \frac{\begin{array}{c}\sum\limits_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j}\bar{\phi}_j+\sum\limits_{i=0}^{n-1}\left(\frac{p}{u}\right)^{\nu-1-i}\bar{\varphi}_i\\ -\sum\limits_{j=0}^{m} A_j\sum\limits_{k=0}^{j-1}\left(\frac{q}{v}\right)^{k-1-j}\bar{\phi}_k-\sum\limits_{i=0}^{n} B_i\sum\limits_{s=0}^{i-1}\left(\frac{p}{u}\right)^{i-1-s}\bar{\varphi}_s\end{array}}{\left[\left(\frac{q}{v}\right)^{\mu}+\left(\frac{p}{u}\right)^{\nu}+\sum\limits_{j=0}^{m} A_j\left(\frac{q}{v}\right)^{j}+\sum\limits_{i=0}^{n} B_i\left(\frac{p}{u}\right)^{i}\right]}, \quad (5.\ 29)
$$

taking the inverse of (DHT), we get

$$g(x,t)=H_x{}^{-1}H_t{}^{-1}\left[\frac{\begin{array}{c}\sum\limits_{j=0}^{m-1}\left(\frac{q}{v}\right)^{\mu-1-j}\bar{\phi}_j+\sum\limits_{i=0}^{n-1}\left(\frac{p}{u}\right)^{\nu-1-i}\bar{\varphi}_i\\-\sum\limits_{j=0}^{m}A_j\sum\limits_{k=0}^{j-1}\left(\frac{q}{v}\right)^{k-1-j}\bar{\phi}_k-\sum\limits_{i=0}^{n}B_i\sum\limits_{s=0}^{i-1}\left(\frac{p}{u}\right)^{i-1-s}\bar{\varphi}_s\end{array}}{\left[\left(\frac{q}{v}\right)^{\mu}+\left(\frac{p}{u}\right)^{\nu}+\sum\limits_{j=0}^{m}A_j\left(\frac{q}{v}\right)^{j}+\sum\limits_{i=0}^{n}B_i\left(\frac{p}{u}\right)^{i}\right]}\right], \tag{5. 30}$$

which is the solution of Eq.(5. 25) subject to the initial and boundary conditions given in Eq.(5. 26).

Example 5.1. *Consider the following time fractional heat partial differential equation*

$$D_t^{\beta}g\left(x,t\right)=\frac{1}{\pi^2}\frac{\partial^2 g\left(x,t\right)}{\partial x^2},\ 0<\beta\leq 1,\ x,t>0, \tag{5. 31}$$

with initial and boundary conditions

$$g\left(0,t\ \right)=0,\ \frac{\partial}{\partial x}g\left(0,t\ \right)=\pi E_{\beta}\left(-t^{\beta}\right),g\left(x,\ 0\right)=\sin\left(\pi x\right). \tag{5. 32}$$

Applying the (DHT) to both sides of Eq.(5. 31) *and single (HT) to the initial and boundary conditions* Eq.(5. 32)*, we get*

$$\begin{aligned}&\left(\frac{q}{v}\right)^{\beta}\bar{G}\left((p,q),(u,v)\right)-\left(\frac{q}{v}\right)^{\beta-1}H_x\left(g\left(x,0\right)\right)\\&=\frac{1}{\pi^2}\left[\left(\frac{p}{u}\right)^{2}\bar{G}\left((p,q),(u,v)\right)-\left(\frac{p}{u}\right)H_t\left(g\left(0,t\right)\right)-H_t\left(\frac{\partial}{\partial x}g\left(0,t\right)\right)\right],\end{aligned} \tag{5. 33}$$

substituting the single (HT) of initial and boundary conditions

$$H_t\left(g\left(0,t\right)\right)=0,\ \ H_t\left(\frac{\partial}{\partial x}g\left(0,t\right)\right)=\pi\frac{\left(\frac{q}{v}\right)^{\beta-1}}{\left(\frac{q}{v}\right)^{\beta}+1},H_x\left(g\left(x,0\right)\right)=\frac{\pi}{\pi^2+\left(\frac{p}{u}\right)^2},$$

in Eq.(5. 33)*, we get*

$$\left(\frac{q}{v}\right)^{\beta}\bar{G}\left((p,q),(u,v)\right)-\left(\frac{q}{v}\right)^{\beta-1}\frac{\pi}{\pi^2+\left(\frac{p}{u}\right)^2}$$

$$=\frac{1}{\pi^2}\left[\left(\frac{p}{u}\right)^{2}\bar{G}\left((p,q),(u,v)\right)-\pi\frac{\left(\frac{q}{v}\right)^{\beta-1}}{\left(\frac{q}{v}\right)^{\beta}+1}\right],$$

simplifying, we obtain

$$\bar{G}\left((p,q),(u,v)\right)=\pi\frac{\left(\frac{q}{v}\right)^{\beta-1}}{\left(\left(\frac{q}{v}\right)^{\beta}+1\right)\left(\pi^2+\left(\frac{p}{u}\right)^2\right)},$$

taking the inverse of (DHT), we get

$$g(x,t) = H_{xt}^{-2}\left[\frac{\left(\frac{q}{v}\right)^{\beta-1}}{\left(\left(\frac{q}{v}\right)^{\beta}+1\right)}\frac{\pi}{\left(\pi^2+\left(\frac{p}{u}\right)^2\right)}\right] = E_{\beta}\left(-t^{\beta}\right)\sin(\pi x),$$

which agrees with the solution already obtained in [8].

Example 5.2. *Consider the following homogeneous time fractional partial differential Telegraph equation*

$$D_t^{\beta} g(x,t) + D_t^{\delta} g(x,t) - \frac{\partial^2 g(x,t)}{\partial x^2} + g(x,t) = 0, 1 < \beta \le 2, \frac{1}{2} < \delta \le 1, \tag{5. 34}$$

with respect to the initial and boundary conditions

$$\begin{aligned} &g(x,\ 0) = 0\ , g_t(x,\ 0) = e^{x}, \\ &g(0,t\ ) = tE_{\beta-\delta,2}\left(-t^{\beta-\delta}\right),\ g_x(0,t\ ) = tE_{\beta-\delta,2}\left(-t^{\beta-\delta}\right). \end{aligned} \tag{5. 35}$$

Applying the (DHT) to both side of equation Eq.(5. 34) *and single (HT) to the initial and boundary conditions* Eq.(5. 35)*, we get*

$$\begin{aligned} &\left(\frac{q}{v}\right)^{\beta}\bar{G}((p,q),(u,v)) - \left(\frac{q}{v}\right)^{\beta\ -1} H_x(g(x,0)) - \left(\frac{q}{v}\right)^{\beta\ -2} H_x(g_x(x,0)) \\ &+ \left(\frac{q}{v}\right)^{\delta}\bar{G}((p,q),(u,v)) - \left(\frac{q}{v}\right)^{\delta-1} H_x(g(x,0)) - \left(\frac{p}{u}\right)^{2}\bar{G}((p,q),(u,v)) \\ &+ \left(\frac{p}{u}\right) H_t(g(0,t)) + H_t\left(\frac{\partial}{\partial x} g(0,t)\right) + \bar{G}((p,q),(u,v)) = 0, \end{aligned} \tag{5. 36}$$

substituting the single (HT) of initial and boundary conditions,

$$\begin{aligned} H_x(g(x,0)) &= 0,\ H_x(g_t(x,0)) = \frac{1}{\left(\frac{p}{u}\right)-1}, \\ H_t(g(0,t)) &= H_t(g_x(0,t)) = \frac{\left(\frac{q}{v}\right)^{\beta-\delta-2}}{\left(\frac{q}{v}\right)^{\beta-\delta}+1}, \end{aligned}$$

in Eq.(5. 36)*, we get*

$$\begin{aligned} &\left[\left(\frac{q}{v}\right)^{\beta} + \left(\frac{q}{v}\right)^{\delta} - \left(\frac{p}{u}\right)^{2} + 1\right]\bar{G}((p,q),(u,v)) \\ &= \frac{\left(\frac{q}{v}\right)^{\beta-2}}{\left(\frac{p}{u}\right)-1} - \frac{\left(\frac{q}{v}\right)^{\beta-\delta-2}}{\left(\frac{q}{v}\right)^{\beta-\delta}+1} - \frac{\left(\frac{q}{v}\right)^{\beta-\delta-2}\left(\frac{p}{u}\right)}{\left(\frac{q}{v}\right)^{\beta-\delta}+1}, \end{aligned}$$

simplifying, we obtain

$$\bar{G}((p,q),(u,v)) = \frac{\left(\frac{q}{v}\right)^{\beta-\delta-2}}{\left(\left(\frac{q}{v}\right)^{\beta-\delta}+1\right)}\frac{1}{\left(\left(\frac{p}{u}\right)-1\right)},$$

taking the inverse of (DHT), we get

$$g(x,t) = H_{xt}^{-2}\left[\frac{\left(\frac{q}{v}\right)^{\beta-\delta-2}}{\left(\left(\frac{q}{v}\right)^{\beta-\delta}+1\right)}\frac{1}{\left(\left(\frac{p}{u}\right)-1\right)}\right] = te^{x}E_{\beta-\delta,2}\left(-t^{\beta-\delta}\right),$$

which is in full agreement with the results obtained in [20, 16].

6. Conclusions

In fractional calculus, developing new techniques for solving differential equations involving fractional derivatives is always interesting. In this study, we have established the double Shehu transform formulas for the fractional partial integrals and derivatives in the sense of Caputo fractional derivative. Then, we have proved some theorems related to this new transform. Furthermore, we have applied the double Shehu transform to solve some linear partial fractional differential equations. It is worthwhile to mention that (DHT) can be coupled with some other methods to solve non-linear (PDEs) arising in applied mathematics, applied physics, and engineering, which will be discussed in subsequent articles.

7. Authors Contribution

The authors contributed equally to this work.

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