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Evolutions of the Ruled Surfaces along a Spacelike Space Curve

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Abstract.: In this paper, we work on the ruled surfaces obtained by a quasi normal and quasi binormal vectors along a spacelike space curve in three dimensional Minkowski space. Time evolution equations depending on quasi curvatures are obtained. Studying directional evolutions of both quasi normal and quasi binormal ruled surfaces by using their directrices, we investigate some geometric properties such as inextensibilty, developability, flatness and minimality of these ruled surfaces.

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1. Introduction

Curves, in three dimensional space, are essential tool to model numerous problems in physics. Flows; especially, inextensible flows of a curve(surface) generate the time evolution of a curve(surface). If the arclength of the curve is preserved, the flow of the curve is inextensible and if the instrinsic curvature of the surface is preserved then the flow of the surface is said to be inextensible. Since the motion, in physics, is created by the inextensible curve flows, the evolutions of curves have numerous imperative applications of physics as magnetic spin chains and vortex filaments ([2][10][17][21]).

The motions of inelastic plane curves have been of great interest to many authors in both Euclidean and Minkowski spaces ([7],[9],[14],[15]). Many researchers have studied the surfaces generated by these curves such as translation surfaces in ([1]), special ruled surfaces in ([8],[12]), timelike ruled surfaces in ([24]) and developable surfaces in ([16]). Using Serret Frenet frame, some geometric properties of the surfaces generated from the motion of inextensible curves in $\mathbb{R}^3$ are investigated ([11]). In ([25]), flows of the curve that are inextensible must satisfy a partial differential equation which has curvatures of the curve on a null surface is given.

After quasi-normal vector of a space curve was presented in ([5]), Dede et.al. ([6]) found out quasi frame in 2015 since the well known Serret Frenet frame has not been adequate. One of the advantages of quasi frame is that it is well defined even if the second derivative of the curve is zero unlike Serret-Frenet frame and it avoids unnecessary twist around the tangent vector. Besides, it is easy to calculate unlike Bishop ([3]). Using quasi frame, timelike surfaces of evolution in Minkowski space and the evolutions of the ruled surface along both space curve and timelike curve were studied in ([20],[23],[13]).

In this work, we deal with evolutions of the ruled surfaces generated by spacelike space curve by quasi frame. We get three differential equations depending on quasi curvatures for the quasi frame vectors of the spacelike space curve by using quasi frame equations with respect to arc-length parameter ω and time t. With the help of first and second fundamental forms of these ruled surfaces, we get geometric properties such as curvatures, flatness, inextensibility and minimality of the quasi normal ruled surface (qNRS) and quasi binormal ruled surface (qBRS).

2. Preliminaries

In three dimensional Minkowski space $\mathbb{R}^3_1$, the dot and cross products of two vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are defined as

$$< \alpha, \beta >= \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3$$

and

$$\alpha \wedge \beta = (\alpha_3\beta_2 - \alpha_2\beta_3)u_1 + (\alpha_1\beta_3 - \alpha_3\beta_1)u_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)u_3$$

where $u_1 \wedge u_2 = u_3$, $u_2 \wedge u_3 = -u_1$, $u_3 \wedge u_1 = -u_2$, respectively.

The norm of the vector α is given by

$$\|\alpha\| = \sqrt{|\langle \alpha, \alpha \rangle|}$$

We say that a Lorentzian vector α is spacelike, lightlike or timelike if $\langle \alpha, \alpha \rangle > 0$ or $\alpha = 0$, $\langle \alpha, \alpha \rangle = 0$ and $\alpha \neq 0$, $\langle \alpha, \alpha \rangle < 0$, respectively ([18],[19]).

A ruled surface is a surface generated by the movement of a line on a curve in space. Therefore, it has a parametrization of the form

$$\varphi(\upsilon, \nu) = \omega(\upsilon) + \nu\delta(\upsilon)$$

where ω is called the directrix and δ is the director curve.

Let φ be a surface in Euclidean 3-space, the first fundamental form of the surface φ is given as $I = Ed\upsilon^2 + 2Fd\upsilon d\nu + Gd\nu^2$ where

$$E =< \varphi_\upsilon, \varphi_\upsilon >, F =< \varphi_\upsilon, \varphi_\nu >, G =< \varphi_\nu, \varphi_\nu > . \tag{2. 1}$$

The second fundamental form of φ is defined as $II = ed\upsilon^2 + 2fd\upsilon d\nu + gd\nu^2$ where

$$e =< \varphi_{\upsilon\upsilon}, N >, f =< \varphi_{\upsilon\nu}, N >, g =< \varphi_{\nu\nu}, N > \tag{2. 2}$$

and N is the unit normal of φ. The Gaussian and mean curvatures are written as

$$K = \frac{eg - f^2}{EG - F^2} \text{ and } H = \frac{eG - 2fF + gE}{2\,(EG - F^2)} \tag{2. 3}$$

respectively. A necessary and sufficient condition for a curve to be a flat and minimal is its Gaussian and mean curvatures vanishes identically, respectively ([4]).

On the other hand, in ([16]) a surface evolution $\varphi(\upsilon, \nu, t)$ and its flow $\dfrac{\partial \varphi}{\partial t}$ are inextensible if the partial derivatives of the coefficients of the first fundamental form vanish. That is,

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0. \tag{2. 4}$$

Given a curve $\omega(\upsilon)$, the quasi frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{p}\}$ is given by

$$\mathbf{t} = \frac{\omega'}{\|\omega'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{p}}{\|\mathbf{t} \wedge \mathbf{p}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{2. 5}$$

where $\mathbf{t}$, $\mathbf{n}_q$, $\mathbf{b}_q$ and $\mathbf{p}$ be the unit tangent vector, quasi-normal vector, quasi-binormal vector and standard unit vectors in $\mathbb{R}^3$, respectively.

Let $\omega(\upsilon)$ be a spacelike space curve. In ([22]), there are four cases for the spacelike space curve to examine.

♣ $\mathbf{Case\ 1}$: The derivative equations of the Frenet frame and quasi frame for the spacelike space curve when both tangent vector and normal vector are spacelike while projection vector $\mathbf{p} = (0, 0, 1)$ and quasi-binormal vector are timelike are written as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2. 6}$$

and

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{2. 7}$$

respectively. Then, we have a relation matrix and relation between curvatures in the following form

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \varkappa & \sinh \varkappa \\ 0 & -\sinh \varkappa & -\cosh \varkappa \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{2. 8}$$

$$k_1 = \kappa \cosh \varkappa, \ \ k_2 = \kappa \sinh \varkappa, \ \ k_3 = -d\varkappa - \tau.$$

where $\varkappa$ be the hyperbolic angle between $\mathbf{b}$ and $\mathbf{b}_q$.

♣ $\mathbf{Case\ 2}$: The derivative equations of the Frenet frame and quasi frame for the spacelike space curve when tangent vector and quasi-binormal vector are spacelike while projection vector $\mathbf{p} = (0, 0, 1)$, normal vector and quasi-normal vector are timelike are written as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2. 9}$$

and

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{2. 10}$$

respectively. Then, we have a relation matrix and relation between curvatures in the following form

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sinh\varkappa & -\cosh\varkappa \\ 0 & \cosh\varkappa & \sinh\varkappa \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{2. 11}$$

$$k_1 = \kappa \sinh\varkappa, \quad k_2 = -\kappa\cosh\varkappa, \quad k_3 = d\varkappa + \tau.$$

where $\varkappa$ be the hyperbolic angle between $\mathbf{n}$ and $\mathbf{b}_q$.

♣ **Case 3** : The derivative equations of the Frenet frame and quasi frame for the spacelike space curve when tangent vector, projection vector $\mathbf{p} = (0, 1, 0)$ and quasi-binormal vector are spacelike while normal vector and quasi-normal vector are timelike are written as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2. 12}$$

and

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{2. 13}$$

respectively. Then, we have a relation matrix and relation between curvatures in the following form

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varkappa & \sinh\varkappa \\ 0 & \sinh\varkappa & \cosh\varkappa \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{2. 14}$$

$$k_1 = -\kappa\cosh\varkappa, \quad k_2 = -\kappa\sinh\varkappa, \quad k_3 = d\varkappa + \tau.$$

where $\varkappa$ be the hyperbolic angle between $\mathbf{n}$ and $\mathbf{n}_q$.

♣ **Case 4** : The derivative equations of the Frenet frame and quasi frame for the spacelike space curve when tangent vector, projection vector $\mathbf{p} = (0, 1, 0)$, normal vector, quasi-normal vector are spacelike while quasi-binormal vector is timelike are written as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2. 15}$$

and

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{2. 16}$$

respectively. Then, we have a relation matrix and relation between curvatures in the following form

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh\varkappa & \cosh\varkappa \\ 0 & -\cosh\varkappa & -\sinh\varkappa \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{2. 17}$$

$$k_1 = \kappa\sinh\varkappa, \quad k_2 = -\kappa\cosh\varkappa, \quad k_3 = -d\varkappa - \tau.$$

where $\varkappa$ be the hyperbolic angle between $\mathbf{b}$ and $\mathbf{n}_q$.

3. EVOLUTION OF SPACELIKE SPACE CURVE WITH TIME BY QUASI FRAME

In this part, we obtain time evolution equations depend on quasi curvatures of the evolving curve $\omega(\upsilon,t)$ in order to obtain spacelike space curve with quasi frame. That is, integrating time evolution equations for given $\lambda,\mu,\eta,$ one can find quasi curvatures. Using variations of quasi frame, we get evolving spacelike curve for each cases.

♣ **Case 1** :

Theorem 3.1. *The variation formula for the quasi curvatures with respect to time t of the evolving spacelike curve $\omega(\upsilon,t)$ is written as*

$$\frac{\partial}{\partial t}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 & -\eta & \mu \\ -\eta & 0 & \lambda \\ \mu & -\lambda & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} + \frac{\partial}{\partial \upsilon}\begin{bmatrix} \lambda \\ \mu \\ \eta \end{bmatrix} \tag{3. 18}$$

where the derivation formula of quasi frame with respect to time t is in the form

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 0 & \lambda & -\mu \\ -\lambda & 0 & -\eta \\ -\mu & -\eta & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}.$$

Proof. Using equation (2. 7), and defining

$$q = \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, A = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \lambda & -\mu \\ -\lambda & 0 & -\eta \\ -\mu & -\eta & 0 \end{bmatrix}, \tag{3. 19}$$

we can write

$$\begin{aligned} \frac{\partial q}{\partial \upsilon} &= Aq \\ \frac{\partial q}{\partial t} &= Bq \end{aligned}$$

Applying the compatibility condition

$$\frac{\partial}{\partial t}\left(\frac{\partial q}{\partial \upsilon}\right) = \frac{\partial}{\partial \upsilon}\left(\frac{\partial q}{\partial t}\right),$$

one can find easily

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial \upsilon} + [A,B] = 0 \tag{3. 20}$$

where $[A,B] = AB - BA$ is the Lie bracket of A and B.

The equations (3. 19) and (3. 20) give us the matrix of evolution equations as

$$\begin{bmatrix} 0 & \frac{\partial k_1}{\partial t} - \frac{\partial \lambda}{\partial \upsilon} + k_2\eta - k_3\mu & \frac{-\partial k_2}{\partial t} + \frac{\partial \mu}{\partial \upsilon} - k_1\eta + k_3\lambda \\ \frac{-\partial k_1}{\partial t} + \frac{\partial \lambda}{\partial \upsilon} + k_3\mu - k_2\eta & 0 & \frac{-\partial k_3}{\partial t} + \frac{\partial \eta}{\partial \upsilon} + k_1\mu - k_{2\lambda} \\ \frac{-\partial k_2}{\partial t} + \frac{\partial \mu}{\partial \upsilon} + k_3\lambda - k_1\mu & -\frac{\partial k_3}{\partial t} + \frac{\partial \eta}{\partial \upsilon} - k_2\lambda + k_1\mu & 0 \end{bmatrix} = 0$$

Thus the compatibility condition becomes

$$\begin{aligned}\frac{\partial k_1}{\partial t} &= k_3\mu - k_2\eta + \frac{\partial \lambda}{\partial \upsilon}\\ \frac{\partial k_2}{\partial t} &= k_3\lambda - k_1\eta + \frac{\partial \mu}{\partial \upsilon}\\ \frac{\partial k_3}{\partial t} &= k_1\mu - k_2\lambda + \frac{\partial \eta}{\partial \upsilon}.\end{aligned}$$

With the help of the obtained equations, we prove the theorem. □

Theorem 3.2. *Time evolution of the curve $\omega(\upsilon, t)$ is represented by*

$$\begin{aligned}\lambda &= -\frac{\partial k_2}{\partial \upsilon} - k_1k_3\\ \mu &= -\frac{\partial k_1}{\partial \upsilon} - k_2k_3\\ \eta &= \frac{1}{k_1}\Big(\frac{\partial(-\frac{\partial k_1}{\partial \upsilon} - k_2k_3)}{\partial \upsilon} - \frac{\partial k_2}{\partial t} + k_3(-\frac{\partial k_2}{\partial \upsilon} - k_1k_3)\Big).\end{aligned} \tag{3. 21}$$

Proof. The velocity of the curve ω is given by

$$\frac{\partial \omega}{\partial t} = a\mathbf{t} + b\mathbf{n}_q + c\mathbf{b}_q. \tag{3. 22}$$

Using equation $\frac{\partial}{\partial t}\left(\frac{\partial \omega}{\partial \upsilon}\right) = \frac{\partial}{\partial \upsilon}\left(\frac{\partial \omega}{\partial t}\right)$, we have the following equations

$$\begin{aligned}\lambda &= \frac{\partial b}{\partial \upsilon} + ak_1 - ck_3\\ \mu &= -\frac{\partial c}{\partial \upsilon} + ak_2 + bk_3\\ 0 &= \frac{\partial a}{\partial \upsilon} + bk_1 + ck_2.\end{aligned} \tag{3. 23}$$

From equation (3. 23), and Theorem 1, we obtain

$$\eta = \frac{1}{k_1}\Big(\frac{\partial(-\frac{\partial c}{\partial \upsilon} + ak_2 + bk_3)}{\partial \upsilon} - \frac{\partial k_2}{\partial t} + k_3(-\frac{\partial b}{\partial \upsilon} + ak_1 - ck_3)\Big). \tag{3. 24}$$

For a solution of smoke ring equation, the velocity vector of ω is given by

$$\frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial \upsilon} \wedge \frac{\partial}{\partial \upsilon}\left(\frac{\partial \omega}{\partial \upsilon}\right) = -k_2\mathbf{n}_q + k_1\mathbf{b}_q. \tag{3. 25}$$

The equality of equations (3. 22) and (3. 25) gives

$$a = 0, b = -k_2, c = k_1. \tag{3. 26}$$

Substituting the equation (3. 26) into equations (3. 23) and (3. 24), we have desired equations. □

Case 1 and Case 2 have the same calculation and results.

♣ **Case 3** :

Theorem 3.3. *The evolution equations for the quasi curvatures of the evolving spacelike curve are written as*

$$\frac{\partial}{\partial t}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 & \eta & -\mu \\ \eta & 0 & -\lambda \\ \mu & -\lambda & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} + \frac{\partial}{\partial \upsilon}\begin{bmatrix} \lambda \\ \mu \\ \eta \end{bmatrix} \tag{3. 27}$$

where the derivation formula of quasi frame with respect to time t is in the form

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 0 & -\lambda & \mu \\ -\lambda & 0 & \eta \\ -\mu & \eta & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}.$$

Proof. The steps to prove the theorem are exactly the same as the previous one.

There is nothing to special for the Case 4; therefore, we omit that part.

4. Inextensible Flow of the Ruled Surfaces Depend on a Spacelike Space Curve

In this part, we work on some geometric properties of evolutions of both qNRS and qBRS by calculating their Gaussian and mean curvatures.

4.1. **Evolution of quasi Normal Ruled Surface.** ♣ **Case 1** : If the curve $\omega(\upsilon)$ moves with the time t, the equation of the qNRS is

$$\varphi(\upsilon,\nu,t) = \omega\left(\upsilon,t\right) + \nu\mathbf{n}_q\left(\upsilon,t\right)$$

First partial derivatives of the surface $\varphi(\upsilon,\nu,t)$ are

$$\begin{aligned} \varphi_\upsilon &= \left(1-\nu k_1\right)\mathbf{t} - \nu k_3\mathbf{b}_q \\ \varphi_\nu &= \mathbf{n}_q. \end{aligned}$$

The normal vector of $\varphi(\upsilon,\nu,t)$ is

$$N = \frac{\varphi_\upsilon \wedge \varphi_\nu}{\|\varphi_\upsilon \wedge \varphi_\nu\|} = \frac{-\nu k_3\mathbf{t} + \left(1-\nu k_1\right)\mathbf{b}_q}{\sqrt{\left|-1+\nu^2\left(k_3^2-k_1^2\right)+2\nu k_1\right|}}.$$

The coefficients of the first fundamental form are calculated by

$$\begin{aligned} E &= \nu^2(k_1^2-k_3^2) - 2\nu k_1 + 1 \\ F &= 0 \\ G &= 1. \end{aligned}$$

Second partial derivatives of the surface $\varphi(\upsilon,\nu,t)$ are

$$\begin{aligned} \varphi_{\upsilon\upsilon} &= \nu(k_2k_3 - \tfrac{\partial k_1}{\partial \upsilon})\mathbf{t} + (\left(1-\nu k_1\right)k_1 + \nu k_3^2)\mathbf{n}_q + ((\nu k_1 - 1)k_2 - \nu\tfrac{\partial k_3}{\partial \upsilon})\mathbf{b}_q \\ \varphi_{\upsilon\nu} &= -k_1\mathbf{t} - k_3\mathbf{b}_q \\ \varphi_{\nu\nu} &= 0. \end{aligned}$$

The coefficients of the second fundamental form are calculated by

$$\begin{aligned} e &= \frac{k_2+\nu(\frac{\partial k_3}{\partial \upsilon}-2k_1k_2)+\nu^2[k_3^2\frac{\partial}{\partial \upsilon}(\frac{k_1}{k_3})+k_2(k_1^2-k_3^2)]}{\sqrt{\left|-1+\nu^2\left(k_3^2-k_1^2\right)+2\nu k_1\right|}} \\ f &= \frac{k_3}{\sqrt{\left|-1+\nu^2\left(k_3^2-k_1^2\right)+2\nu k_1\right|}} \\ g &= 0. \end{aligned}$$

One can find Gaussian and mean curvatures

$$K=\frac{f^2}{E}=\frac{k_3^2}{\left(\nu^2\left(k_1^2-k_3^2\right)-2\nu k_1+1\right)^2}$$

and

$$H=\frac{e}{2E}=\frac{k_2+\nu(\frac{\partial k_3}{\partial \upsilon}-2k_1k_2)+\nu^2[k_3^2\frac{\partial}{\partial \upsilon}(\frac{k_1}{k_3})+k_2(k_1^2-k_3^2)]}{2\left(\nu^2\left(k_3^2-k_1^2\right)+2\nu k_1-1\right)^{3/2}}$$

respectively.

The qNRS is minimal if and only if

$$k_2=\nu(2k_1k_2-\frac{\partial k_3}{\partial \upsilon})-\nu^2[k_3^2\frac{\partial}{\partial \upsilon}(\frac{k_1}{k_3})+k_2(k_1^2-k_3^2)].$$

The qNRS is both flat and developability if and only if

$$k_3=0.$$

Using (2. 4), if the qNRS is inextensible, we have

$$\frac{\partial k_1}{\partial t}=\frac{\nu}{2}(\frac{\partial}{\partial t}(k_1^2-k_3^2)).$$

♣ **Case 3** : If the curve $\omega(\upsilon)$ moves with the time t, the equation of the qNRS is

$$\varphi(\upsilon,\nu,t)=\omega\left(\upsilon,t\right)+\nu\mathbf{n}_q\left(\upsilon,t\right)$$

First partial derivatives of the surface $\varphi(\upsilon,\nu,t)$ are

$$\begin{aligned} \varphi_\upsilon &= \left(1-\nu k_1\right)\mathbf{t}+\nu k_3\mathbf{b}_q \\ \varphi_\nu &= \mathbf{n}_q. \end{aligned}$$

The normal vector of $\varphi(\upsilon,\nu,t)$ is

$$N=\frac{\nu k_3\mathbf{t}+\left(\nu k_1-1\right)\mathbf{b}_q}{\sqrt{1+\nu^2\left(k_1^2+k_3^2\right)-2\nu k_1}}.$$

The coefficients of the first fundamental form are calculated by

$$\begin{aligned} E &= 1+\nu^2\left(k_1^2+k_3^2\right)-2\nu k_1 \\ F &= 0 \\ G &= -1. \end{aligned}$$

Second partial derivatives of the surface $\varphi(\upsilon,\nu,t)$ are

$$\begin{aligned}\varphi_{\upsilon\upsilon} &= -\nu(\tfrac{\partial k_1}{\partial\upsilon}+k_2k_3)\mathbf{t}+((\nu k_1-1)\,k_1+\nu k_3^2)\mathbf{n}_q+((1-\nu k_1)k_2+\nu\tfrac{\partial k_3}{\partial\upsilon})\mathbf{b}_q\\ \varphi_{\upsilon\nu} &= -k_1\mathbf{t}+k_3\mathbf{b}_q\\ \varphi_{\nu\nu} &= 0.\end{aligned}$$

The coefficients of the second fundamental form are calculated by

$$\begin{aligned}e &= \frac{-k_2-\nu(\frac{\partial k_3}{\partial\upsilon}-2k_1k_2)+\nu^2[k_1^2\frac{\partial}{\partial\upsilon}(\frac{k_3}{k_1})-k_2(k_1^2+k_3^2)]}{\sqrt{1+\nu^2\,(k_1^2+k_3^2)-2\nu k_1}}\\ f &= \frac{-k_3}{\sqrt{1+\nu^2\,(k_1^2+k_3^2)-2\nu k_1}}\\ g &= 0.\end{aligned}$$

One can find Gaussian and mean curvatures

$$K=\frac{k_3^2}{\left(1+\nu^2\,(k_1^2+k_3^2)-2\nu k_1\right)^2}$$

and

$$H=\frac{-k_2-\nu(\frac{\partial k_3}{\partial\upsilon}-2k_1k_2)+\nu^2[k_1^2\frac{\partial}{\partial\upsilon}(\frac{k_3}{k_1})-k_2(k_1^2+k_3^2)]}{2\left(1+\nu^2\,(k_1^2+k_3^2)-2\nu k_1\right)^{3/2}}$$

respectively.

The qNRS is minimal if and only if

$$k_2=\nu(2k_1k_2-\frac{\partial k_3}{\partial\upsilon})+\nu^2[k_1^2\frac{\partial}{\partial\upsilon}(\frac{k_3}{k_1})-k_2(k_1^2+k_3^2)].$$

The qNRS is both flat and developability if and only if

$$k_3=0.$$

Using (2. 4), if the qNRS is inextensible, we have

$$\frac{\partial k_1}{\partial t}=\frac{\nu}{2}(\frac{\partial}{\partial t}(k_1^2+k_3^2)).$$

4.2. **Evolution of quasi Binormal Ruled Surface.** ♣ **Case 1** : If the curve $\omega(\upsilon)$ moves with the time t, the equation of qBRS is

$$\varphi(\upsilon,\nu,t)=\omega\,(\upsilon,t)+\upsilon\mathbf{b}_q\,(\upsilon,t)$$

First partial derivatives of the surface $\varphi(\upsilon,\nu,t)$ are

$$\begin{aligned}\varphi_\upsilon &= (1-\nu k_2)\,\mathbf{t}-\nu k_3\mathbf{n}_q\\ \varphi_\nu &= \mathbf{b}_q.\end{aligned}$$

The normal vector of $\varphi(\upsilon,\nu,t)$ is

$$N=\frac{\nu k_3\mathbf{t}+(1-\nu k_2)\,\mathbf{n}_q}{\sqrt{1+\nu^2\,(k_2^2+k_3^2)-2\nu k_2}}.$$

The coefficients of the first fundamental form are calculated by

$$\begin{aligned} E &= \nu^2(k_2^2+k_3^2)-2\nu k_2+1 \\ F &= 0 \\ G &= -1. \end{aligned}$$

Second partial derivatives of the surface $\varphi(\upsilon,\nu,t)$ are

$$\begin{aligned} \varphi_{\upsilon\upsilon} &= \nu(k_1k_3-\tfrac{\partial k_2}{\partial \upsilon})\mathbf{t}+((1-\nu k_2)\,k_1-\nu\tfrac{\partial k_3}{\partial \upsilon})\mathbf{n}_q+((\nu k_2-1)k_2+\nu k_3^2)\mathbf{b}_q \\ \varphi_{\upsilon\nu} &= -k_2\mathbf{t}-k_3\mathbf{n}_q \\ \varphi_{\nu\nu} &= 0. \end{aligned}$$

The coefficients of the second fundamental form are calculated by

$$\begin{aligned} e &= \frac{k_1-\nu(\frac{\partial k_3}{\partial \upsilon}+2k_1k_2)+\nu^2[k_2^2\frac{\partial}{\partial \upsilon}(\frac{k_3}{k_2})+k_1(k_2^2+k_3^2)]}{\sqrt{1+\nu^2\,(k_2^2+k_3^2)-2\nu k_2}} \\ f &= \frac{-k_3}{\sqrt{1+\nu^2\,(k_2^2+k_3^2)-2\nu k_2}} \\ g &= 0. \end{aligned}$$

One can find Gaussian and mean curvatures

$$K=\frac{k_3^2}{\left(\nu^2\,(k_1^2-k_3^2)-2\nu k_1+1\right)^2}$$

and

$$H=\frac{k_1-\nu(\frac{\partial k_3}{\partial \upsilon}+2k_1k_2)+\nu^2[k_2^2\frac{\partial}{\partial \upsilon}(\frac{k_3}{k_2})+k_1(k_2^2+k_3^2)]}{2\left(\nu^2(k_2^2+k_3^2)-2\nu k_2+1\right)^{3/2}}$$

respectively.

The qBRS is minimal if and only if

$$k_1=\nu(\frac{\partial k_3}{\partial \upsilon}+2k_1k_2)-\nu^2[k_2^2\frac{\partial}{\partial \upsilon}(\frac{k_3}{k_2})+k_1(k_2^2+k_3^2)].$$

The qBRS is both flat and developability if and only if

$$k_3=0.$$

Using (2. 4), if the qBRS is inextensible, we have

$$\frac{\partial k_2}{\partial t}=\frac{\nu}{2}(\frac{\partial}{\partial t}(k_2^2+k_3^2)).$$

♣ **Case 3** : If the curve $\omega(\upsilon)$ moves with the time t, the equation of qBRS is

$$\varphi(\upsilon,\nu,t)=\omega\,(\upsilon,t)+\nu\mathbf{b}_q\,(\upsilon,t)$$

First partial derivatives of the surface $\varphi(\upsilon,\nu,t)$ are

$$\begin{aligned} \varphi_\upsilon &= (1-\nu k_2)\,\mathbf{t}+\nu k_3\mathbf{n}_q \\ \varphi_\nu &= \mathbf{b}_q. \end{aligned}$$

The normal vector of $\varphi(\upsilon,\nu,t)$ is

$$N=\frac{-\nu k_3\mathbf{t}+(\nu k_2-1)\,\mathbf{n}_q}{\sqrt{\nu^2\left(k_3^2-k_2^2\right)+2\nu k_2-1}}.$$

The coefficients of the first fundamental form are calculated by

$$\begin{aligned}E&=\nu^2(k_2^2-k_3^2)-2\nu k_2+1\\F&=0\\G&=1.\end{aligned}$$

Second partial derivatives of the surface $\varphi(\upsilon,\nu,t)$ are

$$\begin{aligned}\varphi_{\upsilon\upsilon}&=-\nu(k_1k_3+\tfrac{\partial k_2}{\partial\upsilon})\mathbf{t}+((\nu k_2-1)\,k_1+\nu\tfrac{\partial k_3}{\partial\upsilon})\mathbf{n}_q+((1-\nu k_2)k_2+\nu k_3^2)\mathbf{b}_q\\\varphi_{\upsilon\nu}&=-k_2\mathbf{t}+k_3\mathbf{n}_q\\\varphi_{\nu\nu}&=0.\end{aligned}$$

The coefficients of the second fundamental form are calculated by

$$\begin{aligned}e&=\frac{-k_1+\nu(\frac{\partial k_3}{\partial\upsilon}+2k_1k_2)+\nu^2[k_3^2\frac{\partial}{\partial\upsilon}(\frac{k_2}{k_3})+k_1(k_3^2-k_2^2)]}{\sqrt{\nu^2\left(k_3^2-k_2^2\right)+2\nu k_2-1}}\\f&=\frac{k_3}{\sqrt{\nu^2\left(k_3^2-k_2^2\right)+2\nu k_2-1}}\\g&=0.\end{aligned}$$

One can find Gaussian and mean curvatures

$$K=\frac{-k_3^2}{\left(\nu^2\left(k_3^2-k_2^2\right)+2\nu k_2-1\right)^2}$$

and

$$H=\frac{-k_1+\nu(\frac{\partial k_3}{\partial\upsilon}+2k_1k_2)+\nu^2[k_3^2\frac{\partial}{\partial\upsilon}(\frac{k_2}{k_3})+k_1(k_3^2-k_2^2)]}{2\left(\nu^2\left(k_3^2-k_2^2\right)+2\nu k_2-1\right)^{3/2}}$$

respectively.

The qBRS is minimal if and only if

$$k_1=\nu(\frac{\partial k_3}{\partial\upsilon}+2k_1k_2)+\nu^2[k_3^2\frac{\partial}{\partial\upsilon}(\frac{k_2}{k_3})+k_1(k_3^2-k_2^2)].$$

The qBRS is both flat and developability if and only if

$$k_3=0.$$

Using (2. 4), if the qBRS is inextensible, we have

$$\frac{\partial k_2}{\partial t}=\frac{\nu}{2}(\frac{\partial}{\partial t}(k_2^2-k_3^2)).$$

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