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New Subclass of Analytic Functions Associated with Fractional $q-$ Differintegral Operator

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Abstract.: In this paper, we have introduced a new subclass $T^{\alpha}_{q,\xi,\delta}$ of univalent and analytic functions defined by fractional $q-$ differintegral operator in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We obtained, among other results, coefficient inequality, convex set, extreme points, growth and distortion theorem, radii of a class of starlikeness, convexity, and neighborhood and Hadamard product for this subclass.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic function of form

$$p(z) = z + \sum_{l=2}^{\infty} a_l z^l \tag{1. 1}$$

defined in $\mathbb{U}$. Let $\mathcal{S}$ be the class of $\mathcal{A}$ and is univalent in $\mathbb{U}$. A function p of $\mathcal{A}$ is called bi-univalent in $\mathbb{U}$ if p and p^{-1} are univalent in $\mathbb{U}$.
Let $\mathcal{A}_m$ be the class of univalent and analytic functions of the type

$$p(z) = z + \sum_{l=m+1}^{\infty} a_l z^l, \ \ m \in \mathbb{N} \tag{1. 2}$$

defined in $\mathbb{U}$. Also, let $\overline{\mathcal{A}_m}$ be the subclass of $\mathcal{A}_m$, which contains analytic and univalent functions expressed in the form

$$p(z) = z - \sum_{l=m+1}^{\infty} a_l z^l, \ \ a_l \geq 0, m \in \mathbb{N}. \tag{1. 3}$$

Purohit and Raina [17] defined "a fractional $q-$ differintegral operator $\Omega_{q,z}^{\alpha}$ for a functions $p(z)$ of the form (1. 2) given by

$$\begin{aligned} \Omega_{q,z}^{\alpha} p(z) &= z + \sum_{l=m+1}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)} a_l z^l, \qquad (1.\ 4) \\ &= z + \frac{\Gamma_q(2-\alpha)}{\Gamma_q(2)} z^l \Omega_{q,z}^{\alpha} p(z), \ \ -\infty < \alpha < 2, m \in \mathbb{N}, 0 < q < 1, z \in \mathbb{U}, \end{aligned}$$

where $\Omega_{q,z}^{\alpha} p(z)$ in (1. 4) represents, respectively a fractional $q-$ integral of $p(z)$ of order α when $-\infty < \alpha < 0$ and a fractional $q-$ derivative of $p(z)$ of order α when $0 \leq \alpha < 2$."
Joshi and Sangle [14] defined and studied the new subclass $D_{\lambda}(\alpha, \beta, \xi; m)$ consisting of analytic functions $p(z) \in \mathcal{A}$ which satisfy the condition

$$\left| \frac{(D^m p(z))' - 1}{2\xi[(D^m p(z))' - \alpha] - [(D^m p(z))' - 1]} \right| < \beta, \ \ \frac{1}{2} \leq \xi \leq 1, 0 \leq \alpha < \frac{1}{2}\xi, 0 < \beta \leq 1, m \in \mathbb{N} \cup 0,$$

where $D^m p(z)$ is the generalized Salagean operator introduced by Al-oboudi [3] and is defined as

$$D^m p(z) = z + \sum_{l=2}^{\infty} [1 + (l-1)\lambda]^m a_l z^l, \ \ \lambda \geq 0, m \in \mathbb{N}_0 = \mathbb{N} \cup 0.$$

Motivated by the above works, we introduce the class of functions involving the operator $\Omega_{q,z}^{\alpha}$ as

$$T_{q,\xi,\delta}^{\alpha} p(z) = \left\{ p(z) \in \overline{\mathcal{A}_m}, \left| \frac{(\Omega_{q,z}^{\alpha} p(z))' - 1}{2\xi[(\Omega_{q,z}^{\alpha} p(z))' - \delta] - [(\Omega_{q,z}^{\alpha} p(z))' - 1]} \right| < \beta \right\} \tag{1. 5}$$

$$-\infty < \alpha < 2, \ \ \frac{1}{2} \leq \xi \leq 1, \ \ 0 \leq \delta \leq \frac{1}{2}\xi, \ \ 0 < \beta \leq 1, \ \ m \in \mathbb{N}, 0 < q < 1, \ \ z \in \mathbb{U}.$$

Recently, several authors defined and studied the new subclass of analytic functions (see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 19, 22]). In the following, we obtain different results for functions of the form (1. 2) and (1. 3) belonging to the class $T_{q,\xi,\delta}^{\alpha}$.

2. Main Results

Theorem 2.1. *A function $p(z)$ given by (1.3) is in the class $T^{\alpha}_{q,\xi,\delta}$ if and only if*

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right]a_l \leq 2\beta\xi(1-\delta). \tag{2. 6}$$

Proof. Now, assume that (2. 6) hold. For $|z|=1$, we have

$$\begin{aligned}
&\left|\left(\Omega^{\alpha}_{q,z}p(z)\right)'-1\right|-\beta\left|2\xi\left[\left(\Omega^{\alpha}_{q,z}p(z)\right)'-\delta\right]-\left[\left(\Omega^{\alpha}_{q,z}p(z)\right)'-1\right]\right| \\
= &\left|-\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}\right|-\beta\left|\begin{array}{l}2\xi(1-\delta)-2\xi\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1} \\ +\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}\end{array}\right| \\
\leq &\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right]a_l-2\beta\xi(1-\delta) \;\; (\text{ By (2. 6)}) \\
\leq &\ 0.
\end{aligned}$$

Thus by maximum modulus theorem $p(z)\in T^{\alpha}_{q,\xi,\delta}$.
Conversely, suppose $p(z)$ is given by (1. 3) and belonging in $T^{\alpha}_{q,\xi,\delta}$. Then it follows that

$$\left|\frac{(\Omega^{\alpha}_{q,z}p(z))'-1}{2\xi[(\Omega^{\alpha}_{q,z}p(z))'-\delta]-[(\Omega^{\alpha}_{q,z}p(z))'-1]}\right|<\beta$$

$$\left|\frac{\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}}{2(1-\delta)\xi+2\xi\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}-\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}}\right|<\beta.$$

Since $|Re(z)|\leq|z|$ ($\forall z$),

$$Re\left(\frac{\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}}{2\xi(1-\delta)+2\xi\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}-\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}}\right)<\beta.$$

Choosing the values of z on the real axis so that $\Omega^{\alpha}_{q,\delta}p(z)$ is real. Now let $z\to 1^{-}$ through the real values and from the above inequality, we have

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}<2\beta\xi(1-\delta)-2\beta\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l z^{l-1}(2\xi-1)$$

implies that

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right]a_l \leq 2\beta\xi(1-\delta).$$

□

Equation (2. 6) gives a sharp coefficient bound for

$$p(z) = z - \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}z^l, (l = m+1, m+2, \cdots; m \in \mathbb{N}) \quad (2.\ 7)$$

Corollary 2.2. *Let* $p(z)$ *be function given by* (1.3) *is in the class* $T^{\alpha}_{q,\xi,\delta}$ *then*

$$a_l \leq \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}, (l = m+1, m+2, \cdots; m \in \mathbb{N}).$$

Theorem 2.3. *Let* $-\infty < \alpha < 2, \frac{1}{2} \leq \xi \leq 1, 0 \leq \delta_1 \leq \delta_2 \leq \frac{1}{2}\xi, 0 < \beta \leq 1, 0 < q < 1, m \in \mathbb{N}$, *then*

$$T^{\alpha}_{q,\xi,\delta_2} \subseteq T^{\alpha}_{q,\xi,\delta_1}.$$

Proof. By assumption, we have

$$\frac{2\beta\xi(1-\delta_2)}{\frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)[1+\beta(2\xi-1)]}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}} \leq \frac{2\beta\xi(1-\delta_1)}{\frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)[1+\beta(2\xi-1)]}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}}$$

Now, $p(z) \in T^{\alpha}_{q,\xi,\delta_2}$ implies that

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l \leq \frac{2\beta\xi(1-\delta_2)}{[1+\beta(2\xi-1)]} \leq \frac{2\beta\xi(1-\delta_1)}{[1+\beta(2\xi-1)]}$$

and by Theorem 2.1, we have $p(z) \in T^{\alpha}_{q,\xi,\delta_1}$. □

Theorem 2.4. *The set* $T^{\alpha}_{q,\xi,\delta}$ *is the convex.*

Proof. Let $p_j(z) = z - \sum_{l=m+1}^{\infty} a_{l,j}z^l, (j = 1, 2)$ belong to $T^{\alpha}_{q,\xi,\delta}$ and let $g(z) = \lambda_1 p_1(z) + \lambda_2 p_2(z)$ with λ_1 and λ_2 are non-negative and $\lambda_1 + \lambda_2 = 1$.
Hence, we have

$$g(z) = z - \sum_{l=m+1}^{\infty} \left[\lambda_1 a_{l,1} + \lambda_2 a_{l,2}\right] z^l.$$

We prove that $g(z) \in T^{\alpha}_{q,\xi,\delta}$.
Now,

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right](\lambda_1 a_{l,1} + \lambda_2 a_{l,2})$$
$$= \lambda_1 \sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right]a_{l,1}$$
$$+ \lambda_2 \sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right]a_{l,2}$$
$$\leq 2\lambda_1\beta\xi(1-\delta) + 2\lambda_2\beta\xi(1-\delta)$$
$$= (\lambda_1+\lambda_2)2\beta\xi(1-\delta) = 2\beta\xi(1-\delta)$$

and by Theorem 2.1, we have $g(z) \in T^{\alpha}_{q,\xi,\delta}$. □

3. Extreme Points

The extreme points for the class $T^{\alpha}_{q,\xi,\delta}$ are proposed in this section.

Theorem 3.1. *Let* $p_m(z) = z$ *and*

$$p_l(z) = z - \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}z^l \quad (l = m+1, m+2, \cdots; m \in \mathbb{N}), \tag{3. 8}$$

where $-\infty < \alpha < 2, \frac{1}{2} \leq \xi \leq 1, 0 \leq \delta \leq \frac{1}{2}\xi, 0 < \beta \leq 1, 0 < q < 1$. *Then*

$$p(z) \in T^{\alpha}_{q,\xi,\delta} \;\textit{ if and only if }\; p(z) = \sum_{l=m+1}^{\infty} \lambda_l p_l(z),$$

where $\lambda_l \geq 0$ *and* $\sum_{l=m+1}^{\infty} \lambda_l = 1$ *or* $1 = \lambda_m + \sum_{l=m+1}^{\infty} \lambda_l$.

Proof. Let $p(z) = \sum_{l=m+1}^{\infty} \lambda_l p_l(z)$, where $\lambda_l \geq 0$ and $\sum_{l=m+1}^{\infty} \lambda_l = 1$. To prove that $p(z) \in T^{\alpha}_{q,\xi,\delta}$. We can write

$$\begin{aligned} p(z) &= \sum_{l=m+1}^{\infty} \lambda_l \left[z - \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}\right] z^l \\ &= z - \sum_{l=m+1}^{\infty} \lambda_l \left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}\right] z^l. \end{aligned}$$

Now,

$$\sum_{l=m+1}^{\infty} \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]} \times l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right]\lambda_l$$

$$= 2\beta\xi(1-\delta)\sum_{l=m+1}^{\infty} \lambda_l \leq 2\beta\xi(1-\delta).$$

By Theorem 2.1, we have $p(z) \in T^{\alpha}_{q,\xi,\delta}$.
Conversely, let $p(z)$ given by (1. 3) be in the class $T^{\alpha}_{q,\xi,\delta}$, then

$$a_l \leq \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]} \;, (l = m+1, m+2, \cdots; m \in \mathbb{N}).$$

Putting

$$\lambda_l = \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]} \;\text{ and }\; 1 = \lambda_m + \sum_{l=m+1}^{\infty} \lambda_l\,,$$

we have

$$p(z) = \lambda_m p_m + \sum_{l=m+1}^{\infty} \lambda_l p_l(z) \;\; .$$

Hence the proof. □

4. GROWTH AND DISTORTION THEOREMS

Theorem 4.1. *Let* $p(z)$ *be a function in the class* $T^{\alpha}_{q,\xi,\delta}$, *then*

$$r - r^{m+1}\left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}\right] \le |p(z)| \le r + r^{m+1}\left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}\right], \text{ for } |z| \le r < 1. \tag{4. 9}$$

Proof. By Theorem (2.1), for any function $p(z) \in T^{\alpha}_{q,\xi,\delta}$, we have

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right] a_l \le 2\beta\xi(1-\delta)$$

$$\sum_{l=m+1}^{\infty} a_l \le \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]} .$$

Therefore,

$$\begin{aligned}|p(z)| &\ge |z| - \sum_{l=m+1}^{\infty} a_l|z|^l \ge |z| - |z|^{m+1}\sum_{l=m+1}^{\infty} a_l \\ &\ge |z| - |z|^{m+1}\left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}\right] \\ &\ge r - r^{m+1}\left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}\right].\end{aligned}$$

Similarly, we can prove

$$|p(z)| \le r + r^{m+1}\left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}\right].$$

□

Theorem 4.2. *Let* $p(z)$ *be a function in the class* $T^{\alpha}_{q,\xi,\delta}$, *then*

$$1 - r^{m}\left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}\right] \le |p'(z)| \le 1 + r^{m}\left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}\right], \text{ for } |z| \le r < 1. \tag{4. 10}$$

Proof. By Theorem (2.1), for any function $p(z) \in T^{\alpha}_{q,\xi,\delta}$, we have

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left[1+\beta(2\xi-1)\right] a_l \le 2\beta\xi(1-\delta)$$

$$\sum_{l=m+1}^{\infty} l a_l \le \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]} .$$

Therefore,

$$|p'(z)| \geq 1 - \sum_{l=m+1}^{\infty} la_l |z|^{l-1} \geq 1 - |z|^m \sum_{l=m+1}^{\infty} la_l$$
$$\geq 1 - |z|^m \left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]} \right]$$
$$\geq 1 - r^m \left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]} \right].$$

Similarly, we can prove

$$|p'(z)| \leq 1 + r^m \left[\frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]} \right].$$

□

5. Radius Properties For Class $T^{\alpha}_{q,\xi,\delta}$

Theorem 5.1. *Let $p(z)$ be a function in the class $T^{\alpha}_{q,\xi,\delta}$, then $p(z)$ is starlike of order $\zeta \ (0 \leq \zeta < 1)$ in $|z| \leq r_1$, where*

$$r_1 = \inf \left\{ \left[\frac{l(1-\zeta)\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[(2\xi-1)\beta+1\right]}{(l-\zeta)\Gamma_q(2)\Gamma_q(l+1-\alpha)[2\beta\xi(1-\delta)]} \right]^{\frac{1}{l-1}} \right\}, (l = m+1, m+2, \cdots; m \in \mathbb{N}). \tag{5. 11}$$

Proof. It suffices to establish that $\left| \frac{zp'(z)}{p(z)} - 1 \right| < 1 - \zeta$. That is,

$$\left| \frac{zp'(z)}{p(z)} - 1 \right| = \left| \frac{\sum_{l=m+1}^{\infty}(1-l)a_l z^l}{z - \sum_{l=m+1}^{\infty} a_l z^l} \right| \leq \frac{\sum_{l=m+1}^{\infty}(l-1)a_l|z|^{l-1}}{1 - \sum_{l=m+1}^{\infty} a_l|z|^{l-1}} < 1 - \zeta$$

To prove the theorem, we must show that

$$\sum_{l=m+1}^{\infty} (l-\zeta) a_l |z|^{l-1} < 1 - \zeta$$

By using Theorem 2.1, we get

$$|z|^{l-1} \leq \left[\frac{l(1-\zeta)\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}{(l-\zeta)\Gamma_q(2)\Gamma_q(l+1-\alpha)[2\beta\xi(1-\delta)]} \right]$$

equivalently,

$$|z| \leq \left[\frac{l(1-\zeta)\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[(2\xi-1)\beta+1\right]}{(l-\zeta)\Gamma_q(2)\Gamma_q(l+1-\alpha)[2\beta\xi(1-\delta)]} \right]^{\frac{1}{l-1}} ; |z| < r_1 \ .$$

Thus,

$$r_1 = \inf \left\{ \left[\frac{l(1-\zeta)\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[(2\xi-1)\beta+1\right]}{(l-\zeta)\Gamma_q(2)\Gamma_q(l+1-\alpha)[2\beta\xi(1-\delta)]} \right]^{\frac{1}{l-1}} \right\}, (l = m+1, m+2, \cdots; m \in \mathbb{N}).$$

□

Theorem 5.2. *Let $p(z)$ be a function in the class $T^{\alpha}_{q,\xi,\delta}$, then $p(z)$ is convex of order ζ $(0 \leq \zeta < 1)$ in $|z| < r_2$, where*

$$r_2 = \inf \left\{ \left[\frac{(1-\zeta)\Gamma_q(2-\alpha)\Gamma_q(l+1)\,[1+\beta(2\xi-1)]}{(l-\zeta)\Gamma_q(2)\Gamma_q(l+1-\alpha)[2\beta\xi(1-\delta)]} \right]^{\frac{1}{l-1}} \right\}, (l = m+1, m+2, \cdots; m \in \mathbb{N}). \tag{5. 12}$$

Proof. It suffices to establish that $\left|\frac{zp''(z)}{p'(z)}\right| < 1-\zeta$. That is,

$$\left|\frac{zp''(z)}{p'(z)}\right| = \left| \frac{-\sum_{l=m+1}^{\infty}(l-1)la_l z^{l-1}}{1-\sum_{l=m+1}^{\infty} la_l z^{l-1}} \right| \leq \frac{\sum_{l=m+1}^{\infty} l(l-1)a_l|z|^{l-1}}{1-\sum_{l=m+1}^{\infty} la_l|z|^{l-1}} < 1-\zeta$$

To prove the theorem, we must show that

$$\sum_{l=m+1}^{\infty} l(l-\zeta)a_l|z|^{l-1} < 1-\zeta$$

By using Theorem 2.1, we get

$$|z|^{l-1} \leq \left[\frac{(1-\zeta)\Gamma_q(2-\alpha)\Gamma_q(l+1)\,[1+\beta(2\xi-1)]}{(l-\zeta)\Gamma_q(2)\Gamma_q(l+1-\alpha)[2\beta\xi(1-\delta)]} \right].$$

Equivalently,

$$|z| \leq \left[\frac{(1-\zeta)\Gamma_q(2-\alpha)\Gamma_q(l+1)\,[1+\beta(2\xi-1)]}{(l-\zeta)\Gamma_q(2)\Gamma_q(l+1-\alpha)[2\beta\xi(1-\delta)]} \right]^{\frac{1}{l-1}}; |z| < r_2 \ .$$

Thus,

$$r_2 = \inf \left\{ \left[\frac{(1-\zeta)\Gamma_q(2-\alpha)\Gamma_q(l+1)\,[(2\xi-1)\beta+1]}{(l-\zeta)\Gamma_q(2)\Gamma_q(l+1-\alpha)[2\beta\xi(1-\delta)]} \right]^{\frac{1}{l-1}} \right\}, (l = m+1, m+2, \cdots; m \in \mathbb{N}).$$

□

6. Neighbourhood And Hadamard Product Properties

Definition 6.1 ([20, 21]). *"Let $\gamma \geq 0$ and $p(z) \in \overline{\mathcal{A}_m}$, the γ neighborhood of a function $p(z)$ defined by*

$$N_\gamma(p) = \left\{ g \in \overline{\mathcal{A}_m} : g(z) = z - \sum_{l=m+1}^{\infty} b_l z^l \ \textit{and} \ \sum_{l=m+1}^{\infty} l|a_l - b_l| \leq \gamma \right\}. \tag{6. 13}$$

For the identity function $e(z) = z$, we have

$$N_\gamma(e) = \left\{ g \in \overline{\mathcal{A}_m} : g(z) = z - \sum_{l=m+1}^{\infty} b_l z^l \ \text{and} \ \sum_{l=m+1}^{\infty} l|b_l| \leq \gamma \right\}." \tag{6. 14}$$

Theorem 6.2. *Let*

$$\gamma = \frac{\Gamma_q(2)\Gamma_q(m+2-\alpha)[2\beta\xi(1-\delta)]}{\Gamma_q(2-\alpha)\Gamma_q(m+2)\,[1+\beta(2\xi-1)]} \ \textit{then} \ T^{\alpha}_{q,\xi,\delta} \subset N_\gamma(e). \tag{6. 15}$$

Proof. Let $p(z) \in T^{\alpha}_{q,\xi,\delta}$ then by Theorem 2.1, we have

$$\frac{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}{\Gamma_q(2)\Gamma_q(m+2-\alpha)}\sum_{l=m+1}^{\infty} la_l \leq \sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}{\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l$$
$$\leq 2\beta\xi(1-\delta) \ .$$

Therefore,

$$\sum_{l=m+1}^{\infty} la_l \leq \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}$$

from (6. 15), we get

$$\sum_{l=m+1}^{\infty} la_l \leq \gamma \ .$$

Therefore, $p(z) \in N_{\gamma}(e)$. □

Definition 6.3. *The function* $p(z) \in \overline{\mathcal{A}_m}$ *is called as a member of the subclass* $T^{\alpha}_{q,\xi,\delta,\eta}$, *if there is function* $h \in T^{\alpha}_{q,\xi,\delta}$ *such that*

$$\left|\frac{p(z)}{h(z)}-1\right| \leq 1-\eta \ \ (0 \leq \eta < 1, \ z \in \mathbb{U}). \tag{6. 16}$$

Theorem 6.4. *Let* $h \in T^{\alpha}_{q,\xi,\delta}$ *and* $\eta = 1-\gamma d$. *Then* $N_{\gamma}(h) \subset T^{\alpha}_{q,\xi,\delta,\eta}$, *where* $-\infty < \alpha < 2, \frac{1}{2} \leq \xi \leq 1, 0 \leq \delta \leq \frac{1}{2}\xi, 0 < \beta \leq 1, 0 < q < 1, 0 \leq \eta < 1, m \in \mathbb{N}$ *and*

$$d = \frac{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1\right]-2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}. \tag{6. 17}$$

Proof. Let $p(z) \in N_{\gamma}(h)$ then by definition (6.1), we have $\sum_{l=m+1}^{\infty} l|a_l-b_l| \leq \gamma$. Therefore,

$$\sum_{l=m+1}^{\infty} |a_l-b_l| \leq \frac{\gamma}{m+1}. \tag{6. 18}$$

Since $h \in T^{\alpha}_{q,\xi,\delta}$, we have

$$\sum_{l=m+1}^{\infty} b_l \leq \frac{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1\right]} \ .$$

Now,

$$\left|\frac{p(z)}{h(z)}-1\right| \leq \frac{\sum_{l=m+1}^{\infty}|a_l-b_l|}{1-\sum_{l=m+1}^{\infty}b_l}$$
$$\leq \gamma\left[\frac{\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1)\right]}{(m+1)\Gamma_q(2-\alpha)\Gamma_q(m+2)\left[1+\beta(2\xi-1\right]-2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(m+2-\alpha)}\right]$$
$$= \gamma d = 1-\eta \ .$$

Considering definition (6.3), we have $p(z) \in T^{\alpha}_{q,\xi,\delta,\eta}$. □

Theorem 6.5. *Let*

$$g(z) = z - \sum_{l=m+1}^{\infty} b_l z^l \text{ and } p(z) = z - \sum_{l=m+1}^{\infty} a_l z^l \ (a_l, b_l \geq 0),$$

be in the class $T^{\alpha}_{q,\xi,\delta_1}$. Then the Hadamard product $h(z) = z - \sum_{l=m+1}^{\infty} a_l b_l z^l$ is in the subclass $T^{\alpha}_{q,\xi,\delta_2}$, where

$$\delta_2 \leq \frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[(2\xi-1)\beta+1\right] - 2\beta\xi(1-\delta_1)^2\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+(2\xi-1)\beta\right]}.$$

Proof. By Theorem 2.1, we get

$$\sum_{l=m+1}^{\infty} \frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\delta_1)\Gamma_q(2)\Gamma_q(l+1-\alpha)} a_l \leq 1$$

$$\text{and} \quad \sum_{l=m+1}^{\infty} \frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\delta_1)\Gamma_q(2)\Gamma_q(l+1-\alpha)} b_l \leq 1 \ .$$

We have only to find the largest δ_2 such that

$$\sum_{l=m+1}^{\infty} \frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\delta_2)\Gamma_q(2)\Gamma_q(l+1-\alpha)} a_l b_l \leq 1 \ .$$

By Cauchy-Schwarz inequality, we have

$$\sum_{l=m+1}^{\infty} \frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\delta_1)\Gamma_q(2)\Gamma_q(l+1-\alpha)} \sqrt{a_l b_l} \leq 1,$$

we need only to show that

$$\frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\delta_2)\Gamma_q(2)\Gamma_q(l+1-\alpha)} a_l b_l \leq \frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\delta_1)\Gamma_q(2)\Gamma_q(l+1-\alpha)} \sqrt{a_l b_l} \ . \tag{6. 19}$$

Equivalently,

$$\sqrt{a_l b_l} \leq \frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[(2\xi-1)\,\beta+1\right]}{2\beta\xi(1-\delta_1)\Gamma_q(2)\Gamma_q(l+1-\alpha)} \frac{2\beta\xi(1-\delta_2)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1\right]}$$
$$\leq \frac{(1-\delta_2)}{(1-\delta_1)}$$

But from (6. 19),we have

$$\sqrt{a_l b_l} \leq \frac{2\beta\xi(1-\delta_1)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1\right]} \ .$$

Consequently, we need to prove that

$$\sqrt{a_l b_l} \leq \frac{2\beta\xi(1-\delta_1)\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]} \leq \frac{(1-\delta_2)}{(1-\delta_1)},$$

or equivalently, that

$$\delta_2 \leq \frac{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1\right] - 2\beta\xi(1-\delta_1)^2\Gamma_q(2)\Gamma_q(l+1-\alpha)}{l\Gamma_q(2-\alpha)\Gamma_q(l+1)\left[1+\beta(2\xi-1)\right]}.$$

□

Theorem 6.6. *Let $p(z)$ be a function in the class $T^{\alpha}_{q,\xi,\delta}$ and $d\,(d>-1)$ any real number, then the function*

$$H(z)=\frac{d+1}{z^d}\int_0^z t^{d-1}p(t)dt \ \ (d>-1),$$

in the class $T^{\alpha}_{q,\xi,\delta}$.

Proof. Since $p(z)\in\overline{\mathcal{A}_m}$,

$$H(z)=\frac{d+1}{z^d}\int_0^z t^{d-1}p(t)dt=\frac{d+1}{z^d}\int_0^z\left(t^d-\sum_{l=m+1}^{\infty}a_l t^{l+d-1}\right)dt$$
$$=z-\sum_{l=m+1}^{\infty}a_l\left(\frac{d+1}{l+d}\right)z^l.$$

Now,

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)\,[1+\beta(2\xi-1]}{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left(\frac{d+1}{l+d}\right)a_l$$
$$\leq\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)\,[1+\beta(2\xi-1]}{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l\leq 1.$$

Since $\left(\frac{d+1}{l+d}\right)\leq 1$ and by theorem (2.1), we get $H(z)\in T^{\alpha}_{q,\xi,\delta}$. □

Theorem 6.7. *Let $p(z)$ be a function in the class $T^{\alpha}_{q,\xi,\delta}$ and*

$$F_\eta(z)=z(1-\eta)+\eta\int_0^z\frac{p(\varphi)}{\varphi}d\varphi \ \ (\eta\geq 0, z\in\mathbb{U}).$$

Then $F_\eta(z)$ is in the class $T^{\alpha}_{q,\xi,\delta}$, if $0\leq\eta\leq(m+1)$.

Proof. Since $p(z)\in\overline{\mathcal{A}_m}$,

$$F_\eta(z)=(1-\eta)z+\eta\int_0^z\left(\frac{\varphi-\sum_{l=m+1}^{\infty}a_l\varphi^l}{\varphi}\right)d\varphi$$
$$=z-\sum_{l=m+1}^{\infty}a_l\left(\frac{\eta}{l}\right)z^l.$$

Now,

$$\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)\,[1+\beta(2\xi-1]}{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}\left(\frac{\eta}{l}\right)a_l$$
$$\leq\sum_{l=m+1}^{\infty} l\frac{\Gamma_q(2-\alpha)\Gamma_q(l+1)\,[1+\beta(2\xi-1]}{2\beta\xi(1-\delta)\Gamma_q(2)\Gamma_q(l+1-\alpha)}a_l\leq 1.$$

Since $\left(\frac{\eta}{l}\right)\leq 1$ and by theorem (2.1), we get $F_\eta(z)\in T^{\alpha}_{q,\xi,\delta}$. □

7. Conclusions

In this paper, we introduced and investigated some properties of the subclass of analytic functions with $q-$ differintegral operator. Based on this work further useful study on different subclasses of analytic functions associated with fractional $q-$ differintegral operator can be established.

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