Punjab University Journal of Mathematics (2022),54(9),565-573
https://doi.org/10.52280/pujm.2022.540901

New Analytical Method and Application for Solve the Nonlinear Wave Equation

Ram Dayal Pankaj
Department of Mathematics and statistics,
J.N.V.University,Jodhpur,Rajasthan ,India,
Email: drrdpankaj@yahoo.com

Received: 21 Jun, 2021 / Accepted: 24 August, 2022 / Published online: 29 September, 2022

Abstract.:The proposed analytical approach is a sophisticated grouping of Elzaki Transform and Homotopy Analysis Method (HAM), called Homotopy Analysis Elzaki Transform (HAET) method, for solving Non-linear Wave System which is combination of two wave equations (Telegraph and Klein Gordon (K-G) Type Equations). This study will be exemplifying to well-becoming character of HAET method. The obtained solution is graphically sketched..

AMS (MOS) Subject Classification Codes: 35M10; 65Z05; 65Z06; 35A35.
Key Words: Telegraph and K-G Type Equation; Homotopy Analysis Elzaki Transforms (HAET) method.

1. INTRODUCTION

Engineering, astronomy, physics and many more fields are capable of using integral transforms. The Laplace transform is varsity in use and application among all the transforms. Ordinary and partial differential equations are solved by integral transforms easily. In recent time Tariq M. Elzaki put a new transform known as Elzaki transform [1]-[4] which is modified transform of Sumudu and Laplace transforms [5]-[10].Elzaki Transform is derived from the classical Fourier Integral Transform. The method is capable to reduce the size of computational work. the solution by this method is found in the form of a convergent series with easily computed components. Nonlinear partial differential equations are solved by this technique.
Briefly speaking about Homotopy Analysis Approach, one constructs a continuous mapping of an initial guessed approximation to the exact solution of the equations considered. An auxiliary linear operator is chosen for constructing such a continuous mapping, and an auxiliary parameter is used to ensure the convergence of the solution series. The method enjoys great freedom in choosing initial approximations and auxiliary linear operators. By means of this kind of freedom, a complicated nonlinear problem can be transformed into an infinite number of simpler, linear sub-problems, which is the advantage of the method in this computer age.

Almost, in all the scientific and engineering fields nonlinear wave phenomena appears in one or other ways. This phenomenon can be well explained by the nonlinear evolution equations (NLEEs)[[11]- [14]]. the powerful and efficient method to find exact solutions of nonlinear equations is still a challenge in the field of physical sciences.The wave propagations and waves interaction can also be explained by these solutions. The Telegraph and Klein Gordon (K-G) Type Equations are a set of nonlinear evolution equations (NLEEs) may be written as

$$\theta_{tt} - \theta_{xx} + \theta + \alpha|\theta|^2\theta + \beta\,\theta_t = 0 \tag{1. 1}$$

Where, $\theta(x,t)$ is complex function with *t* time and *x* space variable. If $\alpha = 0, \beta = 1$ it describes telegraph Equation [15] and $\alpha = -1, \beta = 0$ its nonlinear Klein Gordon (NKG) Type Equation[16]. Many powerful methods [[17]- [20],[28]] were used to investigate the solutions of non-linear partial differential system or NLEEs. In this task we proposed a new analytic method for solving the model equation. Solitary wave solution is also discussed to explain nonlinear wave phenomena arising in NLEEs.The advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series solution of nonlinear partial differential equations (NPDE) which will be most effective straight forward method. The following strategy is pursued in the rest of this work. outlines Basic Idea and common steps for Homotopy analysis Elzaki transform (HAET) method in Section 2. Applications of this method to consider equation are presented in Section 3. Essence are eventually drawn in Section 4.

2. Homotopy Analysis Elzaki Transform (HAET) Method

We combined the Homotopy analysis method (HAM) and Elzaki transform for new approach to obtain convergent series solutions of cubic nonlinear problem on the basis of the Homotopy derivative concept. This method advantages the others in the way that it eliminates the need of linearization, perturbation or any other transformation. It also reduces the mountainous computation work previously required by other methods.
We consider a second order Non-linear partial differential equation

$$\frac{\partial^2\theta}{\partial t^2} + \frac{\partial^2\theta}{\partial x^2} + D\left(\theta(t,x)\right) = 0 \tag{2. 2}$$

Where D is a nonlinear operator. The initial conditions are also as $\theta(0,x) = g(x), \theta_t(0,x) = h(x)$ Applying the Elzaki transform E [[5, 6, 7]] to eq. (2. 2) we obtain

$$E\left\{\theta(t,x)\right\} - v^2 g\left(x\right) - v^3 h(x) + v^2\left\{E\left[\frac{\partial^2\left\{\theta(t,x)\right\}}{\partial x^2} + D\left\{\theta(t,x)\right\}\right]\right\} = 0 \tag{2. 3}$$

where v is Elzaki's parameter in real and complex.Its Non-linear equation. Now we interpolate the HAM [19],[21]-[24] in eq. (2. 3), So nonlinear equation may be written as

$$\begin{aligned} D\left[\left\{\psi\left(x,t;\gamma\right)\right\}\right] &= E\left\{\psi\left(x,t;\gamma\right)\right\} - v^2 g\left(x\right) + v^3 h(x) \\ &\quad - v^2\left\{E\left[D\left\{\psi\left(x,t;\gamma\right)\right\} + \frac{\partial^2}{\partial x^2}\left\{\psi\left(x,t;\gamma\right)\right\}\right]\right\} = 0 \end{aligned} \tag{2. 4}$$

where D is a nonlinear operator on unknown function $\psi(x,t;\gamma)$. let $\hbar \neq 0$ and $\Re$ are auxiliary parameter and linear operator respectively with $\gamma \in [0,1]$. We use it to construct Homotopy [[25]- [27]], such that

$$(1-\gamma)\Re[\psi(x,t;\gamma)-\theta_0(t,x)] = \gamma\hbar D[\psi(x,t;\gamma)] \tag{2. 5}$$

Where $\psi\left(x,t;0\right) = \theta_0\left(t,x\right) = g(x)$ and if $\hbar \neq 0$,$\gamma = 1$, we get solution $\psi\left(x,t;1\right) = \theta\left(t,x\right)$. Expansion of the function $\psi\left(x,t;\gamma\right)$ in a power of γ , Using Taylor's theorem

$$\psi\left(x,t;\gamma\right) = \psi\left(x,t;0\right) + \sum_{n=1}^{\infty}\theta_n\left(x,t\right)\gamma^n \tag{2. 6}$$

Where

$$\theta_n\left(x,t\right) = \frac{1}{n!}\left.\frac{\partial^n\psi\left(x,t;\gamma\right)}{\partial\gamma^n}\right|_{\gamma=0} \tag{2. 7}$$

We choose properly $\Re$, $g(x)$ and γsuch that the series eq. (2. 6) is convergent i.e

$$\psi\left(x,t;1\right) = \psi\left(x,t;0\right) + \sum_{n=1}^{\infty}\theta_n\left(t,x\right) \tag{2. 8}$$

Differentiation of eq.(2. 5) with respect to γ, n^{th} times , and we get n^{th} order deformation equation

$$\Re\left[\psi\left(x,t;\gamma\right) - \lambda_n\psi\left(x,0;\gamma\right)\right] = \hbar\, R_n\left[\vec{\eta}_{n-1}\left(x,t\right)\right] \tag{2. 9}$$

Expansion solution of the eq. (2. 2) is

$$\theta_n\left(x,t\right) = \lambda_n\theta_{n-1}\left(x,t\right) + \hbar E^{-1}\left\{R_n\left[\vec{\theta}_{n-1}\left(x,t\right)\right]\right\} \tag{2. 10}$$

Where

$$R_n\left[\vec{\theta}_{n-1}\left(x,t\right)\right] = \frac{1}{(n-1)!}\left.\frac{\partial^{n-1}D\{\psi(x,t;\gamma)\}}{\partial\gamma^{n-1}}\right|_{\gamma=0}$$

and

$$\lambda_n = \begin{cases} 1 & : n > 1 \\ 0 & : n \leq 1 \end{cases}$$

3. Application

We take initial condition with

$$\theta_0\left(t,x\right) = \theta\left(0,x\right) = \left(\frac{\alpha+\beta-1}{q}\right)^{1/2}\tanh\left(px\right)e^{ikx} = g\left(x\right)$$

$$\theta_t\left(0,x\right) = \left(\frac{\alpha+\beta-1}{q}\right)^{1/2}\left(q\,\mathrm{sech}^2\left(px\right) - \beta\tanh\left(px\right)\right)e^{ikx} = h\left(x\right)$$

Applying Elzaki transformation E to eq.(1. 1) we have

$$E\left\{\theta(t,x)\right\} - \left[\frac{v^2(1+v\beta)}{(1+v)^2}g\left(x\right) + \frac{v^3h(x)}{(1+v)^2}\right] + \frac{v^2}{(1+v)^2}E\left[\left\{\alpha\left\{|\theta|^2\theta\right\} - \frac{\partial^2\{\theta(t,x)\}}{\partial x^2}\right\}\right] = 0$$

Its Non-linear equation. Now we interpolate the HAM from eq. (2. 4),
So nonlinear equation may be written as

$$D\left[\psi\left(x,t;\gamma\right)\right]=E\left\{\psi\left(x,t;\gamma\right)\right\}-\left[\frac{v^{2}\left(1+v\beta\right)}{\left(1+v\right)^{2}}g\left(x\right)+\frac{v^{3}h(x)}{\left(1+v\right)^{2}}\right]$$
$$+\frac{v^{2}}{\left(1+v\right)^{2}}E\left[\left\{\alpha\left\{\left|\psi\left(x,t;\gamma\right)\right|^{2}\psi\left(x,t;\gamma\right)\right\}-\frac{\partial^{2}\left\{\psi\left(x,t;\gamma\right)\right\}}{\partial x^{2}}\right\}\right]=0 \quad (3.\ 11)$$

Where $\psi\left(x,t;\gamma\right)$ is unknown function such that $\psi\left(x,t;0\right)=\theta_{0}\left(t,x\right)=\theta\left(0,x\right)=g(x)$ and if $\hbar\neq 0$,$\gamma=1$, we get solution $\psi\left(x,t;1\right)=\theta\left(t,x\right)$,
we can construct a Homotopy using eq. (2. 5)

$$\gamma\hbar D\left[\psi\left(x,t;\gamma\right)\right]=\left(1-\gamma\right)\Re\left[\psi\left(x,t;\gamma\right)-\theta_{0}\left(t,x\right)\right] \quad (3.\ 12)$$

Using Taylor's theorem and eq.(6)
we get

$$\psi\left(x,t;\gamma\right)=\psi\left(x,t\,;0\right)+\sum_{n=1}^{\infty}\theta_{n}\left(x,t\right)\gamma^{n} \quad (3.\ 13)$$

Where

$$\theta_{n}\left(x,t\right)=\frac{1}{n!}\left.\frac{\partial^{n}\psi\left(x,t;\gamma\right)}{\partial\gamma^{n}}\right|_{\gamma=0} \quad (3.\ 14)$$

Differentiation with respect to γ , nth times of eq.(12) and find The nth order Homotopy deformation function with solution

$$\theta_{n}\left(x,t\right)=\lambda_{n}\theta_{n-1}\left(x,t\right)+\hbar E^{-1}\left\{R_{n}\left[\vec{\theta}_{n-1}\left(x,t\right)\right]\right\} \quad (3.\ 15)$$

where

$$R_{n}\left[\vec{\theta}_{n-1}\left(x,t\right)\right]=E\left[\theta_{n-1}\left(x,t\right)\right]-\left(1-\lambda_{n}\right)\left[\frac{v^{2}\left(1+v\beta\right)}{\left(1+v\right)^{2}}g\left(x\right)+\frac{v^{3}h(x)}{\left(1+v\right)^{2}}\right]$$
$$+\frac{v^{2}}{\left(1+v\right)^{2}}\left\{E\left[\left\{\alpha\sum_{i}^{n-1}\left\{\left|\theta_{n-1-i}\right|^{2}\theta_{n-1-i}\right\}-\frac{\partial^{2}\left\{\theta_{n-1}(x,t)\right\}}{\partial x^{2}}\right\}\right]\right\} \quad (3.\ 16)$$

and

$$\lambda_{n}=\begin{cases}1 & :n>1\\ 0 & :n\leq 1\end{cases}$$

Put $n=1$ in (3. 15)

$$\theta_{1}\left(t,x\right)=\theta_{0}\left(t,x\right)+\hbar\left[E^{-1}\left(R_{1}\left[\vec{\theta}_{0}\left(t,x\right)\right]\right)\right] \quad (3.\ 17)$$

From eq.(3. 16) we get

$$R_{1}\left[\vec{\theta}_{0}\left(x,t\right)\right]=E\left[\theta_{0}\left(x,t\right)\right]-\left[\frac{v^{2}\left(1+v\beta\right)}{\left(1+v\right)^{2}}g\left(x\right)+\frac{v^{3}h(x)}{\left(1+v\right)^{2}}\right]$$
$$+\frac{v^{2}}{\left(1+v\right)^{2}}\left\{E\left[\left\{\alpha\left\{\left|\theta_{0}\right|^{2}\theta_{0}\right\}-\frac{\partial^{2}\left\{\theta_{0}(x,t)\right\}}{\partial x^{2}}\right\}\right]\right\} \quad (3.\ 18)$$

use initial condition in eq.(3. 17) and (3. 18) with $\beta = 2$, We get

$$\theta_1(x,t) = \hbar \begin{bmatrix} A\tanh(px)\left\{\left(1+k^2\right) - \left((1+7t) - k^2(1+t)\right)e^{-t}\right\} \\ +\left(1-(1+t)e^{-t}\right)\left\{\alpha A^3 \tanh^3(px) + 2Ap^2 \text{sech}^2(px)\tanh(px)\right\} \\ -2ikpA\left(1-(1+t)e^{-t}\right) sech^2(px) + 2Aqte^{-t}\text{sech}^2(px) \end{bmatrix} e^{ikx} \tag{3. 19}$$

where $A = \left(\frac{\alpha+1}{q}\right)^{1/2}$

similarly, put $n = 2$ in eq.(3. 15)

$$\theta_2(x,t) = \theta_1(x,t) + \hbar\left[E^{-1}\left(R_2\left[\vec{\theta}_1(x,t)\right]\right)\right] \tag{3. 20}$$

From eq.(3. 16) we get

$$R_2\left[\vec{\theta}_1(x,t)\right] = E\left[\theta_1(x,t)\right] + \frac{v^2}{(1+v)^2}E\left[\left\{\alpha\left(|\theta_1|^2\theta_1 + |\theta_0|^2\theta_0\right) - \frac{\partial^2\{\theta_1(x,t)\}}{\partial x^2}\right\}\right] \tag{3. 21}$$

if Put the value of $R_2\left[\vec{\theta}_1(x,t)\right]$ in eq.(3. 20)
use the value of $M = (1+k^2) - \left((1+7t) - k^2(1+t)\right)e^{-t}$ and $R = \{1-(1+t)e^{-t}\}$
then we get $\theta_2(x,t)$ as

$$\theta_2(x,t) = E^{-1}\left[\frac{v^2}{(1+v)^2}E\left[\alpha\left\{\hbar^3\{[4A^3Mq^2t^2e^{-2t}sech^4(px)\tanh(px)\right.\right.\right.$$
$$+A^3M^3\tanh^3(px) + 4A^3p^4MR^2\text{sech}^4(px)\tanh^3(px) + 4\alpha A^5p^2MR^2\text{sech}^2(px)\tanh^5 px$$
$$+4A^3M^2qte^{-t}\tanh^2(px)\text{sech}^2(px) + 3\alpha A^5M^2R\tanh^5(px) + \alpha^3A^9R^3\tanh^9(px)$$
$$+4\alpha A^5MRqte^{-t}\tanh^4(px)\text{sech}^2(px) + 8A^3p^2qte^{-t}MR\tanh^2(px)\text{sech}^4(px)$$
$$+2A^3M^2qte^{-t}\tanh^2(px)\text{sech}^2(px) + 2\alpha^2A^7R^2qte^{-t}\tanh^6(px)\text{sech}^2(px)$$
$$+8A^3p^4R^2qte^{-t}\tanh^2(px)\text{sech}^6(px) + 8\alpha A^5R^2p^2qte^{-t}\tanh^4(px)\text{sech}^4(px)$$
$$+8A^3Mq^2t^2e^{-2t}\tanh(px)\text{sech}^4(px) + 8\alpha A^5R^2p^2qte^{-t}\tanh^4(px)\text{sech}^4(px)$$
$$+8A^3p^2qte^{-t}MR\tanh^2(px)\text{sech}^4(px) + 8k^2p^2A^3R^2qte^{-t}\text{sech}^6(px)$$
$$+8A^3q^3t^3e^{-3t}\text{sech}^6(px) + 4\alpha k^2p^2A^5R^3\tanh^3 px\text{sech}^4(px)$$
$$+4\alpha A^5Rq^2t^2e^{-2t}\tanh^3(px)\text{sech}^4(px) + 4\alpha A^5p^4R^3\tanh^5(px)\text{sech}^4(px)$$
$$+4\alpha^2A^7R^3p^2\text{sech}^2(px)\tanh^7(px) + 4\alpha k^2p^2A^5R^3\tanh^3(px)\text{sech}^4(px)$$
$$+4\alpha A^5MRqte^{-t}\tanh^4(px)\text{sech}^2(px) + 4\alpha^2A^7R^2qte^{-t}\tanh^6(px)\text{sech}^2(px)+$$
$$+2\alpha^2A^7MR^2\tanh^7 px + 4\alpha A^5p^2MR^2\text{sech}^2(px)\tanh^5(px)$$
$$+2A^3p^2M^2R\tanh^3(px)\text{sech}^2(px) + 8A^3Rp^2q^2t^2e^{-2t}\tanh(px)\text{sech}^6(px)$$
$$+2\alpha^2p^2A^7MR^3\tanh^7(px)\text{sech}^2(px) + 4\alpha p^2A^5MR^2\tanh^5(px)\text{sech}^2(px)$$
$$+2A^3p^2M^2R\tanh^3(px)\text{sech}^2(px) + 8A^3p^2q^2t^2Re^{-2t}\tanh(px)\text{sech}^6(px)$$

$$+16A^3p^4R^2qte^{-t}\tanh^2(px)\text{sech}^4(px)+4A^3MRkpqte^{-t}\tanh(px)\text{sech}^4(px)$$
$$+2\alpha^2A^7p^2R^3\tanh^7(px)\text{sech}^2(px)+8A^3p^6R^3\tanh^3(px)\text{sech}^6(px)$$
$$+8A^3p^2qtMRe^{-t}\tanh^2(px)\text{sech}^4(px)+8\alpha A^5R^2p^2qte^{-t}\tanh^4(px)\text{sech}^4(px)$$
$$+4\alpha A^5MR^2p^2\tanh^5(px)\text{sech}^2(px)+8A^3MR^2p^4\text{sech}^4(px)\tanh^3(px)$$
$$+8\alpha A^5p^4R^3\tanh^5(px)\text{sech}^4(px)+8k^2p^4A^3R^3\text{sech}^6(px)\tanh(px)]$$
$$-2i[A^3M^2kpR\tanh^2(px)\text{sech}^2(px)+4A^3Rpq^2t^2ke^{-2t}\text{sech}^6(px)$$
$$+4kp^5A^3R^3\tanh^2(px)\text{sech}^6(px)+4\alpha kp^3A^5R^3\text{sech}^4(px)\tanh^4(px)$$
$$+4\alpha A^5R^2kpqte^{-t}\text{sech}^4(px)\tanh^3(px)+8kp^3qtA^3R^2e^{-t}\tanh(px)\text{sech}^6(px)$$
$$+2\alpha A^5MR^2kp\tanh^4(px)\text{sech}^2(px)+\alpha^2kpA^7R^3\tanh^6(px)\text{sech}^2(px)$$
$$+4A^3MR^2p^3k\tanh^2(px)\text{sech}^4(px)+4k^3p^3A^3R^3\text{sech}^6(px)]\}e^{ikx}$$
$$+\{(\frac{\alpha+\beta-1}{q})^{\frac{3}{2}}\tanh^3(px)\}e^{ikx}\Big\}$$
$$-\Big\{i\hbar k[AMp\text{sech}^2(px)+4Apqte^{-t}\tanh px\text{sech}^2(px)+R\{3\alpha A^3p\tanh^2(px)\text{sech}^2(px)$$
$$+4Ap^3\tanh^2 px\text{sech}^2(px)+2Ap^3\text{sech}^4(px)\}-4ikpAR\text{sech}^2(px)\tanh(px)]e^{ikx}$$
$$+\hbar[2AMp^2\text{sech}^2(px)\tanh px$$
$$+4Ap^2qte^{-t}\text{sech}^4(px)+8Ap^2qte^{-t}\text{sech}^2(px)\tanh^2(px)+R\{6\alpha A^3p^2\tanh(px)\text{sech}^4(px)$$
$$+6\alpha A^3p^2\text{sech}^2(px)\tanh^3(px)+8Ap^4\tanh(px)\text{sech}^4(px)+8Ap^4\tanh^3(px)\text{sech}^2(px)$$
$$+8Ap^4\tanh(px)\text{sech}^4(px)\}$$
$$-8ikp^2AR\text{sech}^2(px)\tanh^2(px)-4ikp^2AR\text{sech}^4(px)]e^{ikx}$$
$$+i\hbar k[AMp\text{sech}^2(px)+4Apqte^{-t}\tanh(px)\text{sech}^2(px)$$
$$+R\{3\alpha A^3p\tanh^2(px)\text{sech}^2(px)+4Ap^3\tanh^2(px)\text{sech}^2(px)+2Ap^3\text{sech}^4(px)\}$$
$$-4ikpAR\text{sech}^2(px)\tanh(px)]e^{ikx}-\hbar k^2[AM\tanh(px)+2Aqte^{-t}\text{sech}^2(px)$$
$$+R\{\alpha A^3\tanh^3(px)+2Ap^2\text{sech}^2(px)\tanh(px)\}-2ikpAR\text{sech}^2(px)]e^{ikx}\Big\}\Big]\Big] \tag{3. 22}$$

$\theta_2\,(x,t)$ has the Elzaki transform and inverse Elzaki transform find as follow

First term of (3. 22)

$$E^{-1}\left[\frac{v^2}{(1+v)^2}E\left\{4\alpha A^3\hbar^3q^2Mt^2e^{-2t}\text{sech}^4(px)\tanh(px)\right\}\right]$$
$$=\left\{4\alpha A^3\hbar^3q^2\text{sech}^4(px)\tanh(px)\right\}E^{-1}\left[\frac{v^2}{(1+v)^2}E\left\{Mt^2e^{-2t}\right\}\right]$$

$$= \left\{4\alpha A^3\hbar^3 q^2 \mathrm{sech}^4(px)\tanh(px)\right\}$$
$$E^{-1}\left[(1+k^2)\frac{v^6}{(1+2v)^3(1+v)^2} + (k^2-1)\frac{v^6}{(1+3v)^3(1+v)^2}\right.$$
$$\left. + 2(k^2-7)\frac{v^7}{(1+3v)^4(1+v)^2}\right] \quad (3.\ 23)$$

Inverse of Elzaki Transform E^{-1} may be finded by theorem (3.1) of [12] as follow let T(v)=E{f(t)} then

$$E^{-1}\left[T(v)\right] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{vt} vT\left(\frac{1}{v}\right)dv = \sum Residue\ of\ \left\{e^{vt}vT\left(\frac{1}{v}\right)\right\} \quad (3.\ 24)$$

then eq.(3. 23) convert as

$$= \left\{4\alpha A^3\hbar^3 q^2 \mathrm{sech}^4(px)\tanh(px)\right\}\left[(1+k^2)\sum Residue\ of\ \left\{\frac{e^{vt}}{(v+2)^3(1+v)^2}\right\}\right.$$
$$+ (k^2-1)\sum Residue\ of\ \left\{\frac{e^{vt}}{(v+3)^3(1+v)^2}\right\}$$
$$\left. + 2(k^2-7)\sum Residue\ of\ \left\{\frac{e^{vt}}{(v+3)^4(1+v)^2}\right\}\right. \quad (3.\ 25)$$

$$= 4\alpha A^3\hbar^3 q^2 \mathrm{sech}^4(px)\tanh(px)\left[(1+k^2)\left\{(t-3)e^{-t} + (t^2+4t+6)e^{-2t}\right\}\right.$$
$$+ (k^2-1)\left\{\frac{(2t-3)\,e^{-t}}{2^4} + \frac{\left(2t^2+4t+3\right)e^{-3t}}{2^3}\right\}$$
$$\left. + \frac{(k^2-7)}{2^3}\left\{(t-3)e^{-t} - (2t^3+6t^2+9t+6)e^{-3t}\right\}\right] \quad (3.\ 26)$$

Similarly we may Calculate the other terms of eq.(3. 22). we achieve the solution $\hbar = -1$ in the same manner

$$\theta\left(t,x\right) = \left(\tfrac{\alpha+1}{q}\right)^{1/2}\tanh\left(px+qt\right)e^{i(kx+\omega t)}$$

With condition

$$1-\omega^2+k^2+2i\omega = 0, \quad \begin{vmatrix} p & 2 \\ q & 2 \end{vmatrix} = \alpha$$

i.e. approximation solution will be

$$\theta\left(t,x\right) = \left(\frac{\alpha+1}{q}\right)^{1/2}\tanh\left\{\frac{1}{2}\left(\left(\alpha+2q\right)x+2qt\right)\right\}e^{i(kx+(i\pm k)t)} \quad (3.\ 27)$$

[b]0.49

FIGURE 1. $|\theta_1|$
[b]0.49

FIGURE 2. $|\theta|$

FIGURE 3. Fig. (A) and Fig. (B) are the soliton solutions of eq.(1. 1) with $\alpha = 1$, $k = -0.5$, $q = 0.5$, $\hbar = -1$

4. ESSENCE

A new expansion HAET method is productively functional to solve the non-linear wave system. The used method is much simpler in comparing to other methods because this method is straightforward and its calculation procedure is very concise. If $\alpha \neq 0$ then the solution is a new solitary wave solution for nonlinear telegraph Equation. It is evident that when compute more terms for the Homotopy Analysis series the numerical results are getting much closer to the corresponding analytical solutions given in fig-b. If $\alpha = 0$ then the solution of given equation are approximate to the linear telegraph equation [15]. Numerical representation (Fig.1) is show the significant character to this method. Result $\theta_1(x,t)$ is put in the given equation and satisfied its. Therefore, the applied method is quite efficient and practically well suited and could be more effectively used to solve various NLEEs which regularly arise in science, engineering and other technical arenas.

REFERENCES

[1] M.J. Ablowitz , P.A. Clarkson *Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform.*, Cam-bridge Univ. Press, Cambridge (1990).

[2] G. Adomian, *A review of the decomposition method and some recent results for nonlinear equations*,Computers and Mathematics with Applications **21**(1991) 101-127.

[3] G. Adomian, *Nonlinear Stochastic Differential Equations.*,ournal of Mathematical Analysis and Applications **55**(1976) 441-452.

[4] S. S. Ali, MS. Chaudhary, *On a new integral transform and solution of some integral equations*,International Journal of Pure and Applied Mathematics.**73(3)**, (2011) 299-308.

[5] A. Biswas, M.Fessak, S. Johnson, S. Beatrice, D. Milovic,Z. Jovanoski, R. Kohl and F. Majid *Optical soliton perturbation in non-Kerrlaw media*, Traveling wave solution. Optics and Laser Tech., **44** (2012) 263-268.

[6] L. Debnath. *Nonlinear partial differential equations for Scientists and Engineers*, Birkhauser, Boston (1997).

[7] A. Devi, et.al. *Solution of Partial Differential Equations by Elzaki Transform.*,International Journal of Mathematical Archive , **9,** No.1(2018) EISSN 2229-5046.

[8] T.M. Elzaki and M. S. Elzaki, *On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients*,Advances in Theoretical and Applied Mathematics. **2(4)** (2012), 104-111. ISSN 2224-5804

[9] T. M. Elzaki and M. S. Elzaki, *On a new integral transform and solution of some integral equations*,Advances in Theoretical and App. Math. **1**,(2011) 115-123.

[10] T.M. Elzaki. *On a new integral transform and solution of some integral equations*,Global Journal of Pure and Applied Mathematics. **1** (2011), 57-64. ISSN 0973-1768.

[11] T. M. Elzaki, *On the New Integral Transform ?ELzaki Transform? Fundamental Properties Investigations and Applications*,Global Journal of Mathematical Sciences: Theory and Practical, **4,** No. 1(2012) 1-13.

[12] T. M. Elzaki, *Application of new transform ?Elzaki transform? to partial differential equations*, Global Journal of Pure and Applied Mathematics, **7,** No. 1 (2011) 65-70.

[13] T. M. Elzaki, *On Some Applications of New Integral Transform "Elzaki Transform*,The Global Journal of Mathematical Sciences: Theory and Practical, **4,** No. 1(2012) 15-23.

[14] F. Hussain, *Solution of 1-dimensional Wave equation by Elzaki Transform.*,International Journal of Multidisciplinary Research and Development, 4, No. 10 (2017) 64-67.

[15] D. Kaya, *A new approach to the telegraph equation; an application of the decomposition method*,Bulletin Of The Institute Of Mathematics Academia Sinica, **28,** No. 1(2000) 51-57.

[16] A. Kumar and R. D. Pankaj, *Laplace-modified Decomposition Method for The Generalized Hiroota-Satauma Coupled Kdv Equation* ,Canadian Journal of Basic and Applied Sciences (CJBAS) **03(04)** (2015) 126-133, ISSN 2292-3381

[17] A. Kumar, R. D.Pankaj and K. Sharma, *A Description of Non-linear wave equations by homotopy analysis transforms method*,International Journal of Scientific & Engineering Research, **8,** No. 6(2017) 2096-2104, ISSN-2229-5518

[18] S. Liao, *The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems*, Ph.D. Thesis, Shanghai Jiao Tong University(1992)

[19] S. Liao *Notes on the homotopy analysis method: some definitions and theories.* ,Nonlinear Science Numerical Simulation **14**(2009) 983?997.

[20] M. Ohta and G. Todorova. *Instability of standing waves for nonlinear Klein-Gordon equation and related system.*,GAKUTO International Series, Mathematical Sciences and Applications, Vol. **26** (2006) Nonlinear Dispersive Equations, pp.189-200

[21] R. D. Pankaj and A. Kumar, *Solutions of the Coupled Klein-Gordon Equation by Modified Exp-Function Method*,Indian Journal Of Applied Research Volume, **3,** No. 3(2013)276-278 ISSN - 2249-555X.

[22] R. D. Pankaj, A. Kumar and S. Chandrawati, *A Description of the Coupled Schrdinger-KDV Equation of Dusty Plasma*,International Journal of Mathematics Trends and Technology (IJMTT) **52,** No. 8(2017),537-544 ISSN: 2231-5373.

[23] R. D. Pankaj, A. Kumar and S. Chandrawati *Evolution of Modulational instability in travelling wave solution of non-linear partial differential equation.*,International Journal of Engineering Technologies and Management Research, 5, No. 1(2018) 1-7 ISSN: 2454-1907.

[24] R. D. Pankaj, *Analytical Methods to Study the Wave Wave Interaction in Ionic Media*,Lap Lambert academic publication, Germany(2016) ISBN: 978-3-659-9780-2

[25] R. D. Pankaj, *A Novel Narration for Propagation of Wave in Relaxing Medium by Expansion Scheme*,Advances and Applications in Mathematical Sciences, **20,** No. 8 (2021) 1517-1523 ISSN 0974-6803

[26] S. Qiang, *Solving the Klein-Gordon equation by means of the homotopy analysis method*,Applied Mathematics and Computation **169** (2005) 355-365.

[27] E. Topkara, D. Milovic, A. K. Sarma, E. Zerrad and A. Biswas. *On the New Integral Transform (ELzaki Transform) Fundamental Properties Investigations and Applications*,Optical solitons with non-Kerr law nonlinearity and inter-modal dispersion with time-dependent coefficients, Comm. Nonlinear Sc. Num. Sim., **15** (2010), 2320-2330.

[28] L. Shijun, *On the homotopy analysis method for nonlinear problems*,Applied Mathematics and Computation **147**,(2004) 499?513.