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An Independence System As a Knot Invariant

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Abstract. This article defines an independence system for a classical knot diagram and proves that the independence system is a knot invariant for alternating knots. We also discuss the exchange property for minimal unknotting sets. Finally, we show knot diagrams where the independence system is a matroid, and there are knot diagrams where it is not.

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1. Introduction

When we draw a knot diagram, we can define a U -independence s ystem associated with it. The U-independence system can help to find new invariants for alternating knots. A more motivating factor for the U-independence system is its usefulness in examining the relationships between knots and combinatorial objects like matroids. When a U-independence system for a knot diagram is a matroid, we can say that all maximal U-independent sets have the same cardinality. As a result, all minimal unknotting sets have the same minimal cardinality. In other words, we need only find a minimal unknotting set to determine the unknotting number of a knot diagram. This concept makes the algorithmic methods of finding the unknotting number of a knot much simpler and quicker.

This paper delves further into the definition of the U-independent set and U-independence system in Section 2. In Section 3, we provide examples of independence systems and matroids, and discuss the exchange property for minimal unknotting sets. Next, in Section 4, we discuss the properties of a U-independence system and provide proof of the existence of isomorphisms between two U-independence systems of reduced alternating diagrams of a knot. Section 4 will also highlight how invariants of the U-independence system can be used as invariants of knots. Finally, we conclude this paper with proof of the various relationships between a U-independence system of a knot diagram and matroids in different families. We further intend to extend our research to other available research areas, including bridge numbers and algebraic unknotting numbers.

2. Definitions and Examples of Basic Notations

To define what a U-independent set is, we begin with an understanding of unknotting numbers and unknotting sets.

The *unknotting number* $u(D)$ of a knot diagram D is the minimum number of crossing switches required to untangle that particular knot diagram. In contrast, the *unknotting number* $u(K)$ of a knot K is the minimum number of the crossings needed to switch to the unknot that ranges over all possible diagrams of the knot K.

An *unknotting set* for a knot diagram comprises all crossings that, when switched, transform the diagram into the unknot. We define the *minimal unknotting set* as having no proper unknotting subsets.

Definition 2.1. *Minimal unknotting sets for a knot diagram have the exchange property in whenever S and R are two minimal unknotting sets and* $r \in R$ *then there exists* $s \in S$ *so that* $S - \{s\} \cup \{r\}$ *is a minimal unknotting set.*

In simpler terms, the exchange property is true if we can remove any element from a minimal unknotting set *S* and replace it with another element from some other minimal unknotting set *R* so that the resulting set is also a minimal unknotting set. The exchange property raises certain advantages: if the exchange property holds for two sets, then every minimal unknotting set for that diagram has the same size. Then, the algorithmic methods to find the minimum size of an unknotting set are easier. For example, the exchange property for the diagram of the figure-eight knot (Fig. 1(a)) holds because all the minimal unknotting sets are of cardinality one . However, to show that the exchange property does not hold, we need to show two minimal unknotting sets with different cardinalities exist. For example, the three twist knot (Fig. 1(b)) has the following minimal unknotting sets:

$$\{v_4\},\ \{v_5\}, \{v_1, v_2\}, \{v_1, v_3\} \text{ and } \{v_2, v_3\}.$$

Since the minimal unknotting sets do not have the same cardinality, the exchange property does not hold. There are cases, however, in which the exchange property still does not hold for minimal unknotting sets with the same cardinality (see subsection 3.3 for further explanation).

A property defined in a finite set, which is also a property of its subsets, is called a *hereditary property*[15]. An *independence family* I on a finite ground set E is a non-empty collection of sets $X \subset E$, satisfying the hereditary property. An *independence system* (E, I) for the set E consists of an independence family I with subsets of E. The maximal independent sets are called *bases* of (E, I). An independence system is called a *matroid* if all of its bases have the exchange property (see [15, Definition 8.2.18]).

With these terms in mind, we can formulate our key definition.

Definition 2.2. *A U-independent set is a set W of crossings in a given knot diagram such that $W \setminus S$ is not an unknotting set for every non-empty set $S \subseteq W$. In other words, a U-independent set is the set of crossings that does not contain an unknotting set.*

The definition of a U-independent set leads to the U-independence system (E, I) for a knot diagram D, where E is the set of all crossings of D, and I is the independence family consisting of the U-independent sets for D. In other words, the independence system (E, I) is the set of all subsets of E that do not contain a proper unknotting set.

We say that a U-independent set is *maximal* if it is not contained in any other U-independent set. By the definition of U-independence, every minimal unknotting set is a maximal U-independent set. While this statement is true, its converse (every maximal U-independent set is a minimal unknotting set) may not always be true.

Definition 2.3. *A reduced knot diagram is a knot diagram where no crossing can be removed just by twisting it.*

Definition 2.4. *A minimal knot diagram is a knot diagram which needs the minimum crossings to draw the knot.*

A minimal knot diagram in a Rolfsen table [13] is denoted by m_t, where m is the number of crossings in the diagram and t is the number of different knot diagrams with m crossings. An *alternating knot* is a knot that has a knot diagram in which crossings alternate under and over each other. For alternating knots, the notions of minimal and reduced diagrams coincide. Consequently, all the reduced alternating diagrams of a knot have the same number of crossings [6]. The U-independence system for a reduced alternating diagram of a knot is declared as a knot invariant by the following theorem (see Section 4.2 for its proof):

Theorem 2.5. *Let (E_1, I_1) and (E_2, I_2) be the U-independence systems of reduced alternating diagrams D_1 and D_2 of a knot K respectively. There exists an isomorphism φ between (E_1, I_1) and (E_2, I_2).*

Knots given by $c_1, c_2, \dots, c_j$ in Conway notation are denoted by $(c_1, c_2, \dots c_j)$, see [1, 5]. The following proposition describes whether the U-independence systems of families $(2n+1, 1, 2n)$, $(2n+1)$, $(2n, 2)$ of knot diagrams are matroids or not.

Proposition 2.6. *The U-independence system of each knot diagram:*
a) is not a matroid in the family $(2n+1, 1, 2n)$ for $n \geq 2$ (Fig.7);
b) is a matroid in the family $(2n+1)$ for $n \geq 1$ (Fig. 8); and
c) is not a matroid in the family $(2n, 2)$ for $n \geq 2$ (Fig.9).

We define the U-independence system for a knot diagram to examine the interplay between knots and combinatorial objects like a matroid. Another motivation is to find new invariants for alternating knots. There is a considerable advantage when a U-independence system for a knot diagram is a matroid. All maximal U-independent sets have the same cardinality if it is a matroid. As a result, all minimal unknotting sets have the same minimal cardinality. In other words, one needs only to find a minimal unknotting set to determine the knot diagram's unknotting number.

2.7. **Independence System.** An independence system (E, I) is also called *abstract simplicial complex* and a *hereditary system*[15]. The set X is called an *independent set* if $X \subset I$ and called a *dependent set* otherwise. The empty set ϕ is independent and, the set E is dependent by definition. Based on the definition of independence in different contexts, there are a variety of independence systems. For example, in linear algebra, the independence is the usual linear independence[16]. Similarly, for a simple undirected graph the property is edge-independent, i.e., a set of edges is independent if its induced graph is acyclic[15]. The reader can see [4, 17] for other graph invariants. The independent sets of each independence system (E, I) form different partitions of the ground set E. The partition of E into the smallest number of independent sets is called a *minimum partition.* The number of independent sets in a minimum partition of E is called the *I-chromatic number* denoted by $\chi(E, I)$ of (E, I) (see [18] for details).

2.8. **Matroid.** A matroid is a generalization of the linear independence in linear algebra. The formal definition of a matroid is given here which will be used later in our discussion.

Definition 2.9 ([15])**.** *The independence system (E, I) consisting of a family I with subsets of a finite set E is a matroid if it satisfies the exchange property, as previously defined. To reiterate, the exchange property states that for any two maximal independent sets M_1 and M_2 and for every $x \in M_1$, there exists a $y \in M_2$ such that $(M_1 \setminus \{x\}) \cup \{y\}$ is also a maximal independent set.*

The independence systems described in Subsection 3.1 form matroids. The first independence system of linearly independent sets in a vector space is the *linear matroid*. The second whose independent sets are acyclic sets of edges for a simple undirected graph, is known as the *graphic matroid*[12]. A set of vertices in a simple graph is called a vertex independent set if no two vertices in the set are adjacent to each other. In general, the vertex independence system of a simple graph is not a matroid.

2.10. **The Exchange Property for Minimal Unknotting Sets.** If the exchange property holds for all maximal independent sets (bases) of an independence system, then those bases have the same cardinality[15]. Since minimal unknotting sets are maximal U-independent sets for a knot diagram and if the converse is also true, the following remark is worth mentioning.

Remark 2.11. *If the exchange property for minimal unknotting sets of a knot diagram holds, then all minimal unknotting sets have the same size and the unknotting number of the diagram can be determined by just finding a minimal unknotting set.*

The exchange property for the diagram of the figure eight knot (Fig. 1(a)) holds trivially because all the minimal unknotting sets are of cardinality one. To show that the exchange property does not hold, we show that there exist two minimal unknotting sets of different cardinalities. For example, the three twist knot (Fig. 1(b)) has the following minimal unknotting sets:

$$\{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_1, v_3\} \text{ and } \{v_2, v_3\}.$$

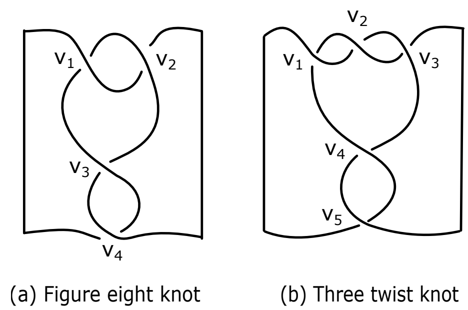


FIGURE 1

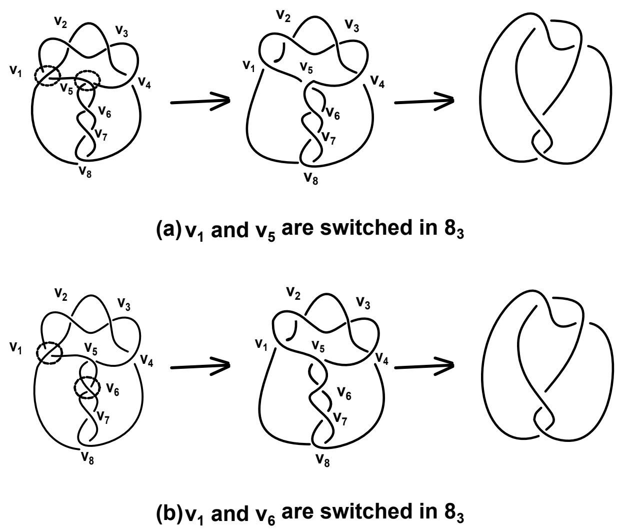


FIGURE 2

These minimal unknotting sets do not have the same cardinality, so the exchange property does not hold. However, the exchange property may still not hold for the minimal unknotting sets that have the same cardinality. For example, the minimal diagram of 8_3 knot (Fig. 2) has two minimal unknotting sets $\{v_1, v_2\}$ and $\{v_5, v_6\}$ of the same cardinality. All possible sets obtained by exchanging elements of these sets are $\{v_1, v_5\}$, $\{v_1, v_6\}$, $\{v_2, v_5\}$, and $\{v_2, v_6\}$, which are not unknotting sets. For example, when the crossings v_1 and v_5 are switched (Fig. 2(a)), the knot 8_3 is not transformed to the unknot. The set $\{v_1, v_6\}$ is also not an unknotting set (Fig. 2(b)). Similarly, $\{v_2, v_5\}$ and $\{v_2, v_6\}$ are not unknotting sets. The Table 1 lists some minimal knot diagrams up to 8 crossings that depict their exchange property for minimal unknotting sets.

$Knot$	$ExchangeProp.\ holds$	$Knot$	$ExchangeProp.\ holds$
3_1	yes	7_4	no
4_1	yes	7_5	no
5_1	yes	7_6	no
5_2	no	7_7	no
6_1	no	8_1	no
6_2	no	8_2	no
6_3	no	8_3	no
7_1	yes	8_4	no
7_2	no	8_5	no
7_3	no	8_6	no

TABLE 1

3. U-INDEPENDENCE SYSTEM OF A KNOT DIAGRAM

3.1. **Basic Properties.**

1.Minimality and maximality

The idea of converting a minimality in one sense to a maximality in another sense was first introduced by Boutin (see [3] where *det-independent* and *res-independent* sets were defined for determining and resolving sets respectively in simple graphs). Also, see [19] for some corrections in [3]. In this paper, the definition of a U-independent set (see Definition 2.2) is slightly different than the one given in [3]. This definition is modified to suit our purpose.

2.U-independent set may be unknotting as well as non-unknotting.

The knot diagram of 7_3 (Fig. 3(a)) given in the Rolfsen knot table [13] has the unknotting number two. Let $E = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be the set of all crossings in the minimal diagram (Fig. 3(a)). Some of the unknotting sets are $W_1 = \{v_1, v_2\}, W_2 = \{v_1, v_3\}, W_3 = \{v_1, v_4\}, W_4 = \{v_2, v_3\}, W_5 = \{v_2, v_4\}$ and $W_6 = \{v_3, v_4\}$. All these unknotting sets are U-independent. For example, $W_1 \setminus \{v_1\}$ and $W_1 \setminus \{v_2\}$ are not unknotting sets. There may be other U-independent sets not necessarily unknotting sets, e.g., $\{v_1, v_5\}$ is not an unknotting set but U-independent because $\{v_1, v_5\} \setminus \{v_1\}$ and $\{v_1, v_5\} \setminus \{v_5\}$ are not unknotting sets.

3.Minimum and minimal unknotting sets

A *minimum unknotting set* has the smallest cardinality among all the minimal unknotting sets of a knot diagram. This smallest cardinality is actually $u(D)$ of the knot diagram. Every minimum unknotting set is minimal, but the converse may not always be true. For example, for the knot diagram 7_3 (Fig. 3(a)), all minimal unknotting sets are:

$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}$ $\{v_3, v_4\}, \{v_1, v_5, v_6\}, \{v_1, v_5, v_7\}, \{v_1, v_6, v_7\}, \{v_2, v_5, v_6\}, \{v_2, v_5, v_7\}, \{v_2, v_6, v_7\}, \{v_3, v_5, v_6\}, \{v_3, v_5, v_7\}, \{v_3, v_6, v_7\}, \{v_4, v_5, v_6\}, \{v_4, v_5, v_7\}$ and $\{v_4, v_6, v_7\}$. Only W_1, W_2, W_3, W_4, W_5 and W_6 are minimum unknotting sets.

4.Unknotting number and minimal diagram

We know that while a knot K has infinite many knot diagrams, it is not necessarily true that $u(K)$ is always obtained from a minimal diagram of K. In addition, there might be a knot diagram of K, not necessarily a minimal one that has the same unknotting number as $u(K)$. For many knots listed in the Rolfsen Table of knots in [13], $u(K)$ is the same for the minimal and other diagrams of K. However, for the knot 10_8 ($(5,1,4)$) in Conway notation [5]), the minimal diagram (Fig. 3(b)) is unknotted by switching at least three crossings with the minimum unknotting set $\{v_2, v_4, v_6\}$. There is another diagram (Fig. 3(c)) of 10_8 which turns to the unknot by switching only 2 crossings with a minimum unknotting set $\{v_6, v_9'\}$. The unknotting number of this diagram is actually the unknotting number of the knot 10_8 (see [2, 11]). An unknotting number, called $u_{min}(K)$, can be defined for each minimal diagram of a knot K, see [14]. Note that for a knot K, the following inequality holds:

$$u(K) \leq u_{min}(K).$$

3.2. **U-independence System as knot invariant.**

Definition 3.3 ([9]). *Let (E_1, I_1) and (E_2, I_2) be two independence systems. Let there exist a bijection $\varphi : E_1 \to E_2$ such that $\varphi(X) \in I_2$ if and only if $X \in I_1$. Then, (E_1, I_1) and (E_2, I_2) are said to be isomorphic.*

In order to prove that the U-independence system is a knot invariant for an alternating knot, the following well-known conjecture of Tait (proved in [10] by Menasco and Thistlethwaite) is needed.

Theorem 3.4. *([The Tait flyping conjecture]) Given reduced alternating diagrams D_1, D_2 of a knot (or link), it is then possible to transform D_1 to D_2 by a sequence of* flypes *(Fig. 4).*

Proof of Theorem 2.5. Let v_i be a crossing in the diagram D_1. Apply the flype (Fig. 4) to D_1 to remove the crossing v_i and create a new crossing with the same label v_i. More precisely, the tangle (the shaded disc in Fig. 4) is turned upside-down to map the crossing (one to its left) to the crossing (one to its right). During the application of the flype all the unknotting/not unknotting sets of the diagram D_1 are preserved. Consequently, all the U-independent sets are preserved in the process. By Theorem 3.4, the diagram D_1 can be converted to D_2, through a sequence of the flypes, preserving the U-independent sets. As a result, an isomorphism φ between (E_1, I_1) and (E_2, I_2) is established. □

Theorem 2.5 further states that the U-independence system (defined for a reduced alternating diagram D of a knot K) itself and all its invariants are knot invariants. The number $u_{min}(K)$ can also be defined as the cardinality of a U-independent set which is also a minimum unknotting set. The number is not a complete invariant, i.e., there are non-isotopic

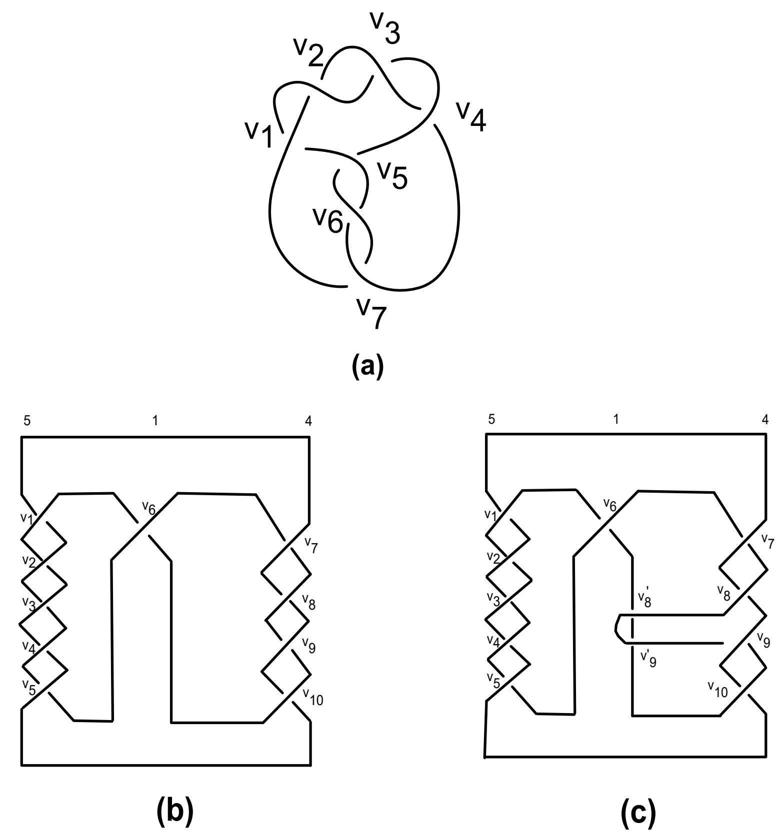


FIGURE 3. (a) Minimal diagram of 7_3 knot (b) Minimal diagram of 10_8 knot (c) Non-minimal diagram of 10_8 knot.

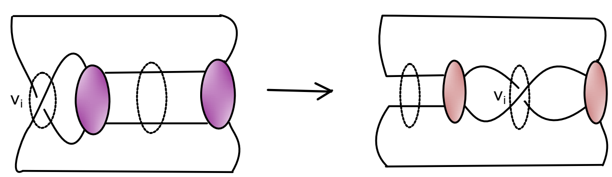
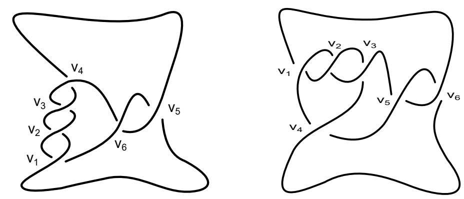


FIGURE 4. flype

knots having the same $u_{min}(K)$. However, other invariants of the U-independence systems of non-isotopic alternating knots may distinguish them where $u_{min}(K)$ fails to do so. Here are two such examples.

The Number of U-independent Sets of a Fixed Cardinality

Example 3.5. *Consider the reduced diagram (Fig. 5(a)) of knot 6_1 and the reduced diagram (Fig. 5(b)) of knot 6_2. For the knot 6_1, the set of all crossings $E = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is divided into two disjoint sets A and B: the set $A = \{v_1, v_2, v_3, v_4\}$ and $B = \{v_5, v_6\}$.*



(a) Reduced diagram of 6_1 (b) Reduced diagram of 6_2

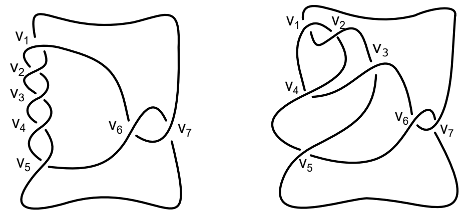
FIGURE 5

In A, no single crossing switch turns the knot into the unknot. In contrast, when any crossing in B is switched, the knot is unknotted. All possible subsets of $\{v_1, v_2, v_3, v_4\}$ of cardinality two are minimal unknotting sets. Furthermore, every subset of cardinality three, four, or five contains an unknotting set. Thus, all the U-independent sets are $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_1, v_2\}$, $\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}$. *There are six U-independent sets of size* 2. *For the knot 6_2, the set $E = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ (Fig. $5(b)$) is divided into three disjoint sets A, B, and C: in the set $A = \{v_1, v_2, v_3\}$, there is no crossing in A which turns the knot to the unknot; the set $B = \{v_4\}$ is an unknotting set; and the set $C = \{v_5, v_6\}$ contains no unknotting set. When any two crossings from $A \cup B$ are switched, the knot is unknotted. However, there is no unknotting set of cardinality 2 in $B \cup C$. Furthermore, every subset of cardinality* 3, 4, *or* 5 *contains an unknotting set. Thus, the U-independent sets are* $\{v_1\}$, $\{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_3\}, \{v_2, v_5\}$, $\{v_2, v_6\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_5, v_6\}$. *There are* 10 *U-independent sets of size* 2. *The knots* 6_1 *and* 6_2 *are distinguished by the number of U-independent sets of cardinality* 2.

The *I-chromatic Number*

The set of crossings of a reduced knot diagram D can be partitioned into U-independent sets and the minimum number of such U-independent sets gives a minimum partition of E. The number of U-independent sets in a minimum partition of E gives the *I-chromatic number* $\chi(E, I)$. The number $\chi(E, I)$ for D can be used as a knot invariant in combination with $u_{min}(K)$. In other words, two alternating knots can be distinguished by $\chi(E, I)$ if the knots have the same $u_{min}(K)$.

Example 3.6. *Consider the reduced diagram (Fig. 6(a)) of knot 7_2 and the reduced diagram (Fig. 6(b)) of knot 7_7. For the knots 7_2, $E = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ (Fig. 6(a)) is divided into two disjoint subsets A and B: $A = \{v_1, v_2, v_3, v_4, v_5\}$ and $B = \{v_6, v_7\}$. The set A contains no unknotting set of cardinality one and two. Each subset of A with cardinality three is a minimal unknotting set. Every crossing in B unknots the knot, but B itself is not an unknotting set. Every subset of E containing $\{v_6\}$ or $\{v_7\}$ is not a minimal unknotting set. Furthermore, every set of cardinality four, five and six contains an unknotting set. Thus, the U-independent sets are:*
$\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_7\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}$, $\{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}$, $\{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}$. *A minimum partition is $\{\{v_1, v_2, v_3\}, \{v_4, v_5\}, \{v_6\}, \{v_7\}\}$ and $\chi(E, I) = 4$. For the knot*



(a) Reduced diagram of 7_2 knot (b) Reduced diagram of 7_7 knot

FIGURE 6

7_7, the set $E = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ (Fig. $6(b)$) is divided into three disjoint subsets A, B, and C. The set $A = \{v_1, v_2, v_3\}$, $B = \{v_4, v_5\}$, and $C = \{v_6, v_7\}$. In the set A, no unknotting set of cardinality one exists; for cardinality two, all of the sets are unknotting sets except for $\{v_1, v_2\}$. In B, $\{v_4\}$ and $\{v_5\}$ are unknotting sets but B itself is not an unknotting set. In C, neither a set of cardinality one nor C itself is an unknotting set. Every set of cardinality three, four, five and contains an unknotting set. Thus, the U-independent sets are: $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_7\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_6\}, \{v_1, v_7\}, \{v_2, v_3\}, \{v_2, v_6\}, \{v_2, v_7\}, \{v_3, v_6\}, \{v_3, v_7\}, \{v_6, v_7\}$.
A minimum partition is $\{\{v_1, v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6, v_7\}\}$ and $\chi(E, I) = 5$. Hence, the knots 7_2 and 7_7 are distinguished by $\chi(E, I)$.

4. U-INDEPENDENCE AS A MATROID

4.1. **Family** $(2n + 1, 1, 2n)$**.** For a knot K in the family $(2n + 1, 1, 2n)$ with $n \geq 2$, $u(K) = n < u_{min}(K) = n + 1$, see [2]. The unknotting number of a knot in this family can be obtained from the diagram (Fig. 7).

Proof of Proposition 2.6(a). For the knot diagram D (Fig. 7) with $u(D) = n$, there are two minimal unknotting sets $\{w, u'_3, u'_5, u'_7, \dots u'_{2n-1}\}$ and $\{v_2, v_4, v_6, \dots, v_{2n}, w\}$ of cardinalities n and $n+1$ respectively. Consequently, there are two maximal U-independent sets of different cardinalities. By Remark 2.11, the U-independence system is not a matroid. □

4.2. **Family** $(2n + 1)$**.** The following result may be known to an expert in knot theory. Anyhow, it is proved here for the sake of completion.

Lemma 4.3. *A knot diagram D in the family $(2n + 1)$ has $u(D) = n$.*

Proof. Apply induction on n. For $n = 1$, $(2n+1)$ is a reduced diagram of trefoil knot with $u(D) = 1$. Suppose $u(D) = m$ for $(2m+1)$. For $n = m+1$, $(2(m+1)+1) = (2m+3)$ is a family of knot diagram with $2m + 3$ alternating crossings (Fig. 8). When the crossing v_{2m+3} is switched, the crossing v_{2m+2} is also killed and the knot diagram $(2m + 1)$ is obtained (Fig.8). By induction, $u(D) \leq m + 1$. The knot diagram $(2m + 1)$ can not be unknotted by fewer than m crossings because if $m-1$ crossings are switched, then $2(m-1)$ alternating crossings are untangled and a reduced diagram of trefoil knot is obtained. It follows that the unknotting number of $(2m + 3)$ is $m + 1$. □

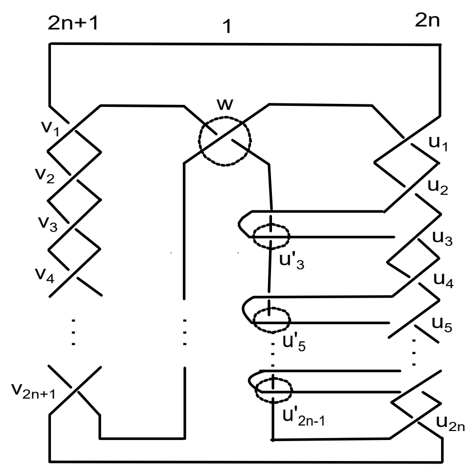


FIGURE 7. $(2n+1, 1, 2n)$

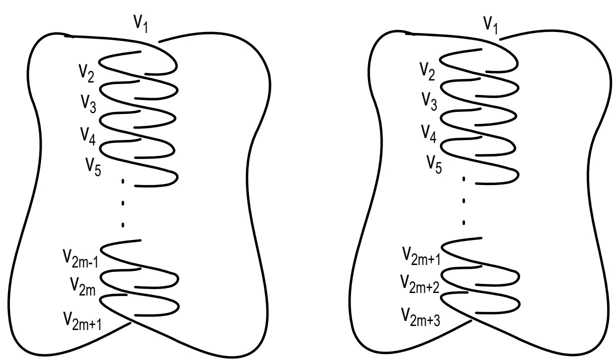


FIGURE 8. $(2m+1)$ and $(2m+3)$

Proof of Proposition 2.6(b) The diagram (Fig. 8) has the property that every subset $A \subset E = \{v_1, v_2, v_3, \ldots, v_{2n}, v_{2n+1}\}$ with $\mid A \mid = n$ is an unknotting set. By Lemma 4.3, the set A must be a minimal unknotting set (a maximal U-independent set). There is no maximal U-independent set of cardinality $< n$. Also, there is no U-independent set B of cardinality $> n$ because B contains an unknotting set of cardinality n. It follows that a U-independent set is maximal if and only if it is a minimal unknotting set and of cardinality n. The exchange property holds for all maximal U-independent sets as every subset of E of cardinality n is a maximal U-independent. Hence, the U-independence system is a matroid by Definition 2.9. $\square$

4.4. **Family** $(2n, 2)$**.** For $n \geq 1$, each diagram D (Fig. 9) has $u(D) = 1$ and its U-independence system is not a matroid except for the figure eight knot, i.e., when $n = 1$.

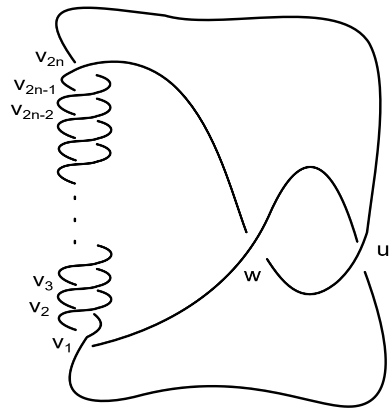


FIGURE 9. $(2n, 2)$

Proof of Proposition 2.6(c). For the diagram (Fig. 9), the sets $\{w\}$ and $\{v_1, v_2, v_3, v_4, \ldots, v_n\}$ are minimal unknotting sets of cardinality 1 and n respectively. Thus, there are two maximal U-independent sets having different cardinalities. By Remark 2.11, the U-independence system is not a matroid. □

5. CONCLUSION

This completes our introduction to U-independence system of a classical knot. On the same lines, independence systems can also be defined for a knot diagram with respect to other invariants like bridge numbers and algebraic unknotting numbers. The corresponding invariants of these independence systems may also be used as knot invariants in combination with these invariants. Every independence system (E, I) is an abstract simplicial complex, see [8]. Therefore, the homology of (E, I) can be investigated for more refined invariants of the corresponding knots. Similarly, independence systems can be associated with and studied for virtual knots [7]. There is much that implores further investigation. For example, one can show that the U-independence systems for reduced alternating diagrams of 6_1 and 6_3 are isomorphic. The knots 6_1 and 6_3 are not mirror images of each other. We can then ask the following open question:

Question. Does there exist two non-isotopic alternating knots (not the mirror image of each other) that have the same number of crossings in their reduced alternating diagram and the same $u_{min}(K) \geq 2$ with isomorphic U-independent systems?

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