Punjab University Journal of Mathematics (2023),55(1),1-11
https://doi.org/10.52280/pujm.2023.550101

**Metric Based Fractional Dimension of Toeplitz Networks**

Hassan Zafar,[1] Muhammad Javaid[2*]
Department of Mathematics,
School of Science, University of Management and Technology,
Lahore, 54770, Pakistan.
Email: hassanzafarmath@gmail.com [1], javaidmath@gmail.com [2*]
$*$ Corresponding author javaidmath@gmail.com

Received: 14 June, 2022 / Accepted: 23 December, 2022 / Published online: 28 January, 2023

**Abstract.**: Metric dimension is one of the distance based graph - theoretic parameters which is widely used in the various disciplines of sciences such as computer science, chemistry, and engineering. The local fractional metric dimension is latest derived form of metric dimension and it is used to find the solutions of integer programming problems. In this paper, we have computed local fractional metric dimension of different families of Toeplitz networks. It is also proved that the local fractional metric dimension of these Toeplitz networks remain bounded when the order of the networks approaches to infinity.

**AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09**
**Key Words:** ————————————————-.

## 1. Introduction

The concept of locating set is introduced independently by Slater [14]. Melter and Harary [11] formally defined this concept by the name of metric dimension (MD) for connected networks. Moreover, they studied MD of different families of networks such complete, cycle and wheel networks . Later on, Chartrand et al. [7] established the bounds of metric dimension (MD) of unicyclic networks and they also proved that $M(D)$ of connected network (ℵ) is 1 iff ℵ is a path network .

Applications of MD exist in various areas of our daily life such as, coin weighting [27], drug discovery, integer programming [7], robot navigation [21], network discovery and verification [6], master mind games [8]. The concept of MD is also used in chemistry in representation of chemical structures ' [18], [19]' and to solve problems in pattern recognition and image processing [25].

Nadeem et al. ('[23],[24]') proved that some classes of Toeplitz networks have constant

MD. For the study of MD of certain families of generalized Petersen and Harary networks, see '[26], [10]'. The concept of edge MD is defined by Kelenc et al. [20] they also made a comparison among edge and standard metric dimension and for the study of constant MD of different connected networks, we refer '[1], [2]'.

Currie and Oellermann [4] defined the concept of fractional metric dimension (FMD) to find the solution of the specific integer programming problem IPP and by using FMD Feher et al. [9] purposed the optimal an improved solution of linear relaxation of the IPP . Arguman and Matthew [22] formally defined the concept of FMD and computed the exact values and FMD of many connected networks FMD of trees, unicycles and hererical product of networks computed in [5].

Asiyah et al. [3] defined the latest invariant of FMD known as local fractional metric dimension (LFMD) and they also computed the exact values of LFMD of corona product of networks . Javaid et al. [16] established the sharp bounds of LFMD and they also constructed a computational criteria to compute exact values of LFMD of the different connected networks. Recently, Javaid et al. [13] improved the lower bound of LFMD from unity and they also established a computational cetraria to compute exact values of LFMD of the connected networks under certain conditions . Upper and lower bounds of LFMD of generalized gear, convex polytopes and sunlet networks, we refer '[29, 17], [15]'. In this paper, we have computed LFMD of Toeplitz networks $\mathcal{T}_m\langle 1,2\rangle$ and $\mathcal{T}_m\langle 1,4\rangle$ in the form sharp bounds.

The remaining part of the paper is organised so that, Section II contains preliminaries, Section III deals with main results and Section IV consists of conclusion among the main results.

## 2. Notations and Preliminaries

Let $\aleph = (V(\aleph), E(\aleph))$ be a network with vertex set $V(\aleph)$ and edge set $E(\aleph) \subseteq V(\aleph) \times V(\aleph)$. A walk is a sequence $v_o, e_1, v_1, ..., v_{m-1}, e_m, v_m$ of vertices and edges such that the edge $e_i$ has end points $v_{i-1}$ and $v_i$ for $1 \leq i \leq m$. A path is a walk with no repetition of vertices. A network is connected is there exist a path joining each pair of vertices. The number of edges in the shortest path between two vertices $s$ and $t$ is called distance between them denoted by $d(s,t)$. For more about fundamental concepts, we refer to [29]. A vertex $t \in V(\aleph)$ is distinguished or resolve a pair $u, v \in V(G)$ if $d(t,v) \neq d(t,u)$. Let $R = \{x_1, x_2, x_3, ..., x_m\} \subseteq V(\aleph)$ is called as resolving set if any pair of vertices say $(x,y)$ of $\aleph$ is distinguished by some vertices of $R$ and the minimum cardinality of a resolving set is called MD of $\aleph$.

For an edge $st \in E(\aleph)$ the set of all vertices which resolve the edge $st$ is denoted by $R'(st)$ is called a local resolving neighbourhood set of that edge. A function $\phi : V(\aleph) \longrightarrow [0,1]$ is a local resolving function (LRF) if $\phi(R'(st)) \geq 1$ for any $st \in E(\aleph)$, where $\phi(R'(st)) = \sum_{x\in R'(st)} \phi(x)$. A local resolving function $\phi'$ is called minimal if $\phi' : V(\aleph) \longrightarrow [0,1]$ such that $\phi' \leq \phi$ and $\phi'(x) \neq \phi(x)$ for at least one $x \in V(\aleph)$ in not a local resolving function of $\aleph$. Then the local fractional metric dimension is defined as

$$dim_{lF}(\aleph) = \min\{|\phi| : \phi \text{ is minimal LRF of } \aleph\}.$$

The adjacency matrix, of a network is a $m \times m$ matrix whose $ij^{th}$ entry is 1 if $v_i$ and $v_j$ vertices are adjacent and 0 otherwise.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Toeplitz Matrix

A matrix is called Toeplitz if it has constant elements along its diagonals. A network is known as a Toeplitz network if its adjacency matrix is a Toeplitz matrix. Toeplitz networks have derived from Toeplitz matrices hence both have equal importance. Toeplitz matrices plays a key role in moment problem, stationary process in the orthogonal polynomials for further details see [12]. The Toeplitz network $\mathcal{T}_m\langle t_1, t_2, t_3, ...., t_m\rangle$ is a symmetric network with vertex set $\{v_1, v_2, v_3, ..., v_m\}$ and the edge between two vertices $v_i$ and $v_j$ $1 \leq i \leq j \leq m$ exits iff $|j-i| \in t_m$. For more details see Figure 1.

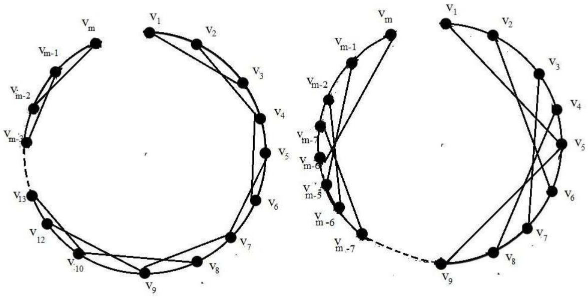

Figure 1. Toeplitz Networks $\mathcal{T}_m\langle 1, 2\rangle$ and $\mathcal{T}_m\langle 1, 4\rangle$.

## 3. Main Results

In this section our objective is to compute LRN sets and LFMD of Toeplitz networks.

**3.1 Local Resolving Neighbourhood Sets of Toeplitz Networks**

In this particular subsection, we compute LRN sets of Toeplitz networks.

**Lemma 3.1.1**

Let $\mathcal{T}_m\langle 1, 2\rangle$ be a Toeplitz network , where $m$ is odd. Then

**(a)** $|R'(v_iv_{i+1})| = \frac{m+1}{2}$ and $\bigcup\limits_{i=1}^{m} |R'(v_iv_{i+1})| = |V(\mathcal{T}_m\langle 1, 2\rangle)|$.

**(b)** $|R'(v_iv_{i+2})| < |R'(v_iv_{i+1})|$ and $|R'(v_iv_{i+2}) \cap \bigcup\limits_{i=1}^{m} |R'(v_iv_{i+1})| \geq |R'(v_iv_{i+1})|$.

**Proof**

Consider $v_iv_{i+1}, v_iv_{i+2} \in E(T_m\langle 1, 2\rangle)$ and $m+1 = 1\ (mod\ m)$, where $1 \leq i \leq m$.

**(a)** $R'(v_iv_{i+1}) = \{v_i, v_{i+1}, v_{i+3}, v_{i+4}, ..., v_m\}$ with $|R'(v_iv_{i+1})| = \frac{m+1}{2}$. Furthermore,

$|\bigcup\limits_{i=1}^{m} R'(v_iv_{i+1})| = |V(\mathcal{T}_m\langle 1,2\rangle)|$. Because of symmetry of Toeplitz networks, they possess a pattern of elements in $R'(v_nv_{n-1}), R'(v_{n-1}v_{n-2}), R'(v_{n-2}v_{n-3}), ....., R'(v_1v_2)$ as in $R'(v_iv_{i+1})$.

**(b)** $R'(v_iv_{i+2}) = V(\mathcal{T}_m\langle 1,2\rangle) - \{v_{i+1}\}$ with $|R'(v_iv_{i+1})| = m-1$. Since , $|\bigcup\limits_{i=1}^{m} R'(v_iv_{i+1})| = |V(\mathcal{T}_m\langle 1,2\rangle)|$ and therefore,$|R'(v_iv_{i+2}) \cap \bigcup\limits_{i=1}^{m} |R'(v_iv_{i+1})| \geq |R'(v_iv_{i+1})|$.

**LRN sets of Toeplitz networks** $\mathcal{T}_m\langle 1,4\rangle$.
To compute the local resolving neighbourhood sets of Toeplitz network $\mathcal{T}_m\langle 1,4\rangle$ , we have following Lemmas
**Lemma 3.1.2**
Let $\mathcal{T}_m\langle 1,4\rangle$ be a Toeplitz network, where $m \cong 1(\ mod\ 4)$. Then

**(a)** $|R'(v_iv_{i+1})| = \frac{3m+1}{4}$ and $\bigcup\limits_{i=1}^{m} |R'(v_iv_{i+1})| = |V(\mathcal{T}_m\langle 1,2\rangle)|$.

**(b)** $|R'(v_iv_{i+2})|\langle|R'(v_iv_{i+1})|$ and $|R'(v_iv_{i+2}) \cap \bigcup\limits_{i=1}^{m} |R'(v_iv_{i+1})| \geq |R'(v_iv_{i+1})|$.

**Proof**
Consider $(v_iv_{i+1}), (v_iv_{i+2}) \in E(\mathcal{T}_m\langle 1,2\rangle)$ and $m+1 = 1\ (mod\ m)$, where $1 \leq i \leq m$.
**(a)** $R'(v_iv_{i+1}) = \{v_{i+3}, v_{i+7}, v_{i+11}, ..., v_{4m-1}\}$ with $|R'(v_iv_{i+1})| = \frac{3m+1}{4}$. Furthermore, $|\bigcup\limits_{i=1}^{m} R'(v_iv_{i+1})| = |V(\mathcal{T}_m\langle 1,2\rangle)|$. Because of symmetry of Toeplitz networks, they posses pattern of elements in $R'(v_mv_{m-1}), R'(v_{m-1}v_{m-2}), R'(v_{m-2}v_{m-3}), ....., R'(v_1v_2)$ as in $R'(v_iv_{i+1})$.

**(b)** $R'(v_iv_{i+2}) = V(\mathcal{T}_m\langle 1,2\rangle) - \{v_{i+1}\}$ with $|R'(v_iv_{i+1})| = m-1$. Since , $|\bigcup\limits_{i=1}^{m} R'(v_iv_{i+1})| = |V(\mathcal{T}_m\langle 1,2\rangle)|$ therefore,$|R'(v_iv_{i+2}) \cap \bigcup\limits_{i=1}^{m} |R'(v_iv_{i+1})| \geq |R'(v_iv_{i+1})|$.

**Lemma 3.1.3**
Let $\mathcal{T}_m\langle 1,4\rangle$ be a Toeplitz network, where $m \cong 3(\ mod\ 4)$. Then

**(a)** $|R'(v_3v_4)| = |R'(v_4v_5)| = |R'(v_{m-3}v_{m-4})| = |R'(v_{m-4}v_{m-5})| = \frac{3m-1}{4}$, $|R'(v_iv_{i+1})| = m-1$ and $\cup|R'(v_3v_4)| = |V(T_m\langle 1,4\rangle)|$.
**(b)** $|R'(v_iv_{i+4})|\langle|R'(v_3v_4)|$ and $|R'(v_iv_{i+4}) \cap \bigcup\limits_{i=1}^{m} |R'(v_iv_{i+1})| \geq |R'(v_iv_{i+1}|$.

**Proof**
Consider $(v_3v_4), (v_4v_5), (v_{m-3}v_{m-4}), (v_{m-4}v_{m-5}) \in E(T_m < 1,4\rangle)$ and $m+1 = 1\ (mod\ m)$, where $1 \leq i \leq m$.
**(a)** $R'(v_3v_4) = V(T_m\langle 1,4\rangle) - \{v_1, v_6, v_{10}, ...., v_{4m+2}\}$, $R'(v_4v_5) = V(\mathcal{T}_m\langle 1,4\rangle) - \{v_2, v_7, v_{11}, ...., v_{4m+3}\}$, $R'(v_{m-3}v_{m-4}) = V(\mathcal{T}_m < 1,4\rangle) - \{v_m, v_{m-6}, v_{m-10}, ...., v_2\}$,

$R'(v_{m-4}v_{m-5}) = V(\mathcal{T}_m\langle 1,4\rangle) - \{v_{m-1}, v_{m-6}, v_{m-10}, ...., v_1\}$ with $|R'(v_3v_4)| = \frac{3m-1}{4}$. Furthermore, $|\cup R'(v_3v_4)| = |V(\mathcal{T}_m\langle 1,4\rangle)|$.

**(b)** $R'(v_iv_{i+4}) = V(\mathcal{T}_m\langle 1,4\rangle) - \{v_{i+2}\}$ with $|R'(v_iv_{i+4})| = m-1$. Since , $|\bigcup_{i=1}^{m} R'(v_iv_{i+4})| = |V(T_4\langle 1,2\rangle)|$ therefore,$|R'(v_iv_{i+4}) \cap \cup |R'(v_3v_4)| \geq |R'(v_3v_4)|$.

**Lemma 3.1.4**
Let $\mathcal{T}_m\langle 1,4\rangle$ be a Toeplitz network, where $m \;\cong\; 0(\; mod\; 4)$ . Then

**(a)** $|R'(v_iv_{i+1})| = \frac{3m}{4}$ and $\bigcup_{i=1}^{m} |R'(v_iv_{i+1})| = |V(T_m\langle 1,2\rangle)|$, where $i \neq 2, m-1$.

**(b)** $|R'(v_iv_{i+4})|, |R'(v_2v_3)| \langle |R'(v_iv_{i+1})|$ and $|R'(v_iv_{i+4}) \cap \bigcup_{i=1}^{m} |R'(v_iv_{i+1})| \geq |R'(v_iv_{i+1}|$.

**Proof**
Consider $(v_iv_{i+1}), (v_iv_{i+2}) \in E(\mathcal{T}_m\langle 1,2\rangle)$ and $m+1 \;=\; 1\;(mod\; m)$, where $1 \;\leq\; i \;\leq\; m$.
**(a)** $R'(v_iv_{i+1}) = \{v_{i+3}, v_{i+7}, v_{i+11}, ..., v_{4m-1}\}$ with $|R'(v_iv_{i+1})| = \frac{3m+1}{4}$. Furthermore, $|\bigcup_{i=1}^{m} R'(v_iv_{i+1})| = |V(T_m\langle 1,2\rangle)|$. Because of symmetry of Toeplitz networks, they posses pattern of elements in $R'(v_mv_{m-1}), R'(v_{m-2}v_{m-3}), ....., R'(v_1v_2)$ as in $R'(v_iv_{i+1})$.

**(b)** $R'(v_iv_{i+4}) = V(T_m\langle 1,2\rangle) - \{v_{i+2}\}$,$R'(v_2v_3) = V(T_m\langle 1,2\rangle) - \{v_{i+3}, v_{i+7}, v_{i+11}, v_{4m-1}\}$ with $|R'(v_iv_{i+1})| = m-1$ . Hence $|R'(v_iv_{i+4})|, |R'(v_2v_3)| \langle |R'(v_iv_{i+1})|$.

3.1. **LFMD of Toeplitz Networks.** In this subsection our aim is to compute LFMD of Toeplitz networks.

**Theorem 3.2.1 :** If $\mathcal{T}_3 < 1,2\rangle$ be a Toeplitz network. Then, $dim_{lF}(\mathcal{T}_3 < 1,2\rangle) = \frac{3}{2}$.
**Proof :**
The possible local RN sets of $\mathcal{T}_3\langle 1,2\rangle$ are :

$R'(v_1v_2) = \{v_1, v_2\}$, $R'(v_2v_3) = \{v_2, v_3\}$ and $R'(v_1v_3) = \{v_1, v_3\}$. Each of local RN sets has order 2 therefore, we define a constant mapping $f : V(\mathcal{T}_3\langle 1,2\rangle) \to [0,1]$ hence, $dim_{lF}(\mathcal{T}_3\langle 1,2\rangle) = \frac{3}{2}$.

**Theorem 3.2.2 :** Let $\mathcal{T}_m\langle 1,2\rangle$ be a Toeplitz network, where $m$ is odd. Then

$$\frac{m}{m-1} \leq dim_{lF}(\mathcal{T}_m\langle 1,2\rangle) \leq \frac{2m}{m+1}.$$

**Proof**
For different vertices of $\mathcal{T}_m\langle 1,2\rangle$, we have two cases

**Case 1**
From the local RN sets, $R'(v_1v_2)$, $R'(v_2v_3)$, $R'(v_3v_4)$ and $R'(v_4v_5)$ have minimum cardinality and their union is $V(\mathcal{T}_5\langle 1,2\rangle)$. Hence a real valued function $f : V(\mathcal{T}_5\langle 1,2\rangle) \to [0,1]$

TABLE 1. Local RN sets of Toeplitz networks $\mathcal{T}_5\langle 1,2\rangle$ .

| RN Sets | $R'(v_1v_2)$ | $R'(v_2v_3)$ | $R'(v_3v_4)$ | $R'(v_1v_3)$ | $R'(v_2v_4)$ | $R'(v_3v_5)$ |
|---|---|---|---|---|---|---|
| Deleted Vertices | $v_3, v_5$ | $v_1, v_4$ | $v_1, v_3$ | $v_2$ | $v_3$ | $v_4$ |

defined by $f(v) = \frac{1}{3}, \forall\ v \ \in\ V(T_5\langle 1,2\rangle)$ is a LRF. Therefore, $dim_{lF}(\mathcal{T}_5\langle 1,2\rangle) \leq \frac{5}{3}$.

From, above local RN sets, $R'(v_1v_3), R'(v_2v_4), R'(v_3v_5)$ have maximum cardinality and their union is $V(T_5\langle 1,2\rangle)$. Hence a real valued function $f : V(\mathcal{T}_5\langle 1,2\rangle) \to [0,1]$ defined by $f(v) = \frac{1}{4}, \forall v \in V(\mathcal{T}_5\langle 1,2\rangle))$ is an upper LRF. Therefore, $dim_{lF}(\mathcal{T}_5\langle 1,2\rangle) \geq \sum\limits_{i=1}^{5} \frac{1}{4} = \frac{5}{4}$. Consequently,

$$\frac{5}{4} \leq dim_{lF}(\mathcal{T}_5\langle 1,2\rangle) \leq \frac{5}{3}.$$

**Case 2**

For $m \geq 7$, where $m$ is odd by Lemma 3.1.1 $|R'(y)| \leq |R'(x)|$ and $|\bigcup\limits_{i=1}^{n} R'(y)| = |V(\mathcal{T}_5\langle 1,2\rangle)|$, where $R'(x)$ are the other local RN sets. Therefore, we define a real valued function $f : V(\mathcal{T}_m\langle 1,2\rangle) \to [0,1]$ which is a minimal LRF and it is defined as $f(v) = \frac{2}{m+1} \forall\ v \ \in\ V(\mathcal{T}_m\langle 1,2\rangle)$. Consequently, $dim_{lF}(\mathcal{T}_m\langle 1,2\rangle) \leq \frac{2m}{m+1}$.

From, Lemma 3.1.1 both $|R'(v_iv_{i+2})|, |R'(v_jv_{j+2})|\rangle|R'(x)|$, where $R'(x)$ are other local RN sets. Hence the real valued function $f' : V(T_m\langle 1,2\rangle)) \to [0,1]$ and it is defined as $f'(v) = \frac{1}{m-1}\ \ \forall\ v \ \in\ V(\mathcal{T}_m\langle 1,2\rangle))$ is a maximal lower LRF so $dim_{lF}(\mathcal{T}_m\langle 1,2\rangle) \geq \frac{m}{m-1}$. Consequently,

$$\frac{m}{m-1} \leq dim_{lF}(\mathcal{T}_m\langle 1,2\rangle) \leq \frac{2m}{m+1}.$$

**Theorem 3.2.3 :** If $\mathcal{T}_5\langle 1,4\rangle$ be a Toeplitz network, then

$$dim_{lF}(\mathcal{T}_5\langle 1,4\rangle) = \frac{5}{4}.$$

**Proof**

All the local RN sets $|R'(v_1v_2)| = |R'(v_2v_3)| = |R'(v_3v_4)| = |R'(v_4v_5)| = |R'(v_1v_5)| = 4$ therefore, we define a constant mapping $f : V(\mathcal{T}_5\langle 1,4\rangle) \to [0,1]$ as $f(v) = \frac{1}{4}, \forall\ v \ \in\ V(\mathcal{T}_5\langle 1,4\rangle)$. The possible local RN sets of $\mathcal{T}_5\langle 1,4\rangle$ are

$$dim_{Lf}(\mathcal{T}_5\langle 1,4\rangle) = \frac{5}{4}.$$

TABLE 2. Local RN sets of Toeplitz networks $\mathcal{T}_5\langle 1,4\rangle$

| RN Sets | $R'(v_1v_2)$ | $R'(v_2v_3)$ | $R'(v_3v_4)$ | $R'(v_4v_5)$ | $R'(v_1v_5)$ |
|---|---|---|---|---|---|
| Deleted Vertices | $v_4$ | $v_5$ | $v_1$ | $v_2$ | $v_3$ |

**Theorem 3.2.4 :** Let $\mathcal{T}_m\langle 1,4\rangle$, be a Toeplitz network, where $m \cong 1(\ mod\ 4)$. Then

$$\frac{m}{m-1} \leq dim_{lF}(T_m\langle 1,4\rangle) \leq \frac{4m}{3m+1}.$$

**Proof.** For different vertices of $\mathcal{T}_m\langle 1,4\rangle$, we have following cases

**Case 1**
For $m = 9$ the possible local RN sets are,

TABLE 3. Local RN sets of Toeplitz Network $\mathcal{T}_9\langle 1,4\rangle$.

| RN Sets | $R'(v_1v_2)$ | $R'(v_2v_3)$ | $R'(v_3v_4)$ | $R'(v_5v_6)$ | $R'(v_6v_7)$ | $R'(v_7v_8)$ | $R'(v_8v_9)$ | $R'(v_1v_5)$ |
|---|---|---|---|---|---|---|---|---|
| Deleted Vertices | $v_4, v_8$ | $v_5, v_9$ | $v_1, v_6$ | $v_3, v_8$ | $v_4, v_8$ | $v_5, v_1$ | $v_6, v_2$ | $v_3$ |

TABLE 4. Local RN sets of Toeplitz Network $\mathcal{T}_9\langle 1,4\rangle$.

| RN Sets | $R'(v_2v_6)$ | $R'(v_3v_7)$ | $R'(v_4v_8)$ | $R'(v_5v_8)$ |
|---|---|---|---|---|
| Deleted Vertices | $v_4$ | $v_5$ | $v_6$ | $v_7$ |

The local RN sets have $R'(v_1v_2)$, $R'(v_2v_3)$, $R'(v_3v_4)$, $R'(v_4v_5)$, $R'(v_5v_6)$, $R'(v_6v_7)$, $R'(v_7v_8)$ and $R'(v_8v_9)$, have minimum cardinality and their union is $V(\mathcal{T}_9\langle 1,4\rangle)$. Therefore, we define a minimal LLRF $f : V(T_9\langle 1,4\rangle) \rightarrow [0,1]$, as $f(v) = \frac{1}{7}, \forall\ v \in V(\mathcal{T}_9\langle 1,4\rangle$ hence $dim_{lF}(\mathcal{T}_9\langle 1,4\rangle) \leq \frac{9}{7}$.

From, above the local RN sets, $R'(v_1v_5)$, $R'(v_2v_6)$, $R'(v_5v_9)$ and $R'(v_3v_7)$ have cardinality 8. Furthermore their union is $V(\mathcal{T}_9\langle 1,4\rangle)$. Therefore, we define a maximal LLRF $f' : V(\mathcal{T}_9\langle 1,4\rangle) \rightarrow [0,1]$ as $f'(v) = \frac{1}{8}$ hence, $dim_{lF}(T_9\langle 1,4\rangle) \geq \frac{9}{8}$. Consequently,

$$\frac{9}{8} \leq dim_{lF}(T_9\langle 1,4\rangle) \leq \frac{9}{7}.$$

**Case 2**
For $n \geq 13$, where $m \cong (1\ mod\ 4)$ by Lemma 3.1.2 $|R'(v_iv_{i+1})| \leq R'(x)$ and $|\bigcup_{i=1}^{n} R'(v_iv_{i+1})| = |V(\mathcal{T}_m\langle 1,4\rangle|$, where $R'(x)$ are other local RN sets. Therefore, we define a minimal ULRF $f : V(T_m\langle 1,4\rangle) \rightarrow [0,1]$ as $f(v) = \frac{4}{n+1}, \forall\ v \in V(\mathcal{T}_m\langle 1,4\rangle$ hence , $dim_{lf}(T_m\langle 1,4\rangle) \leq \frac{4m}{m+1}$.

From, Lemma 3.1.2, $|R'(v_iv_{i+4})|\rangle|R'(x)|$, where $R'(x)$ are other local RN sets. Hence, we define minimal LLRF $f' : V(T_m\langle 1,4\rangle)) \rightarrow [0,1]$ as $f'(v) = \frac{1}{m-1} \ \forall\ v \ \in \ V(T_m\langle 1,4\rangle)$, therefore $dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \geq \frac{m}{m-1}$. Consequently,

$$\frac{m}{m-1} \leq dim_{lF}(T_m\langle 1,4\rangle) \leq \frac{4m}{3m+1}.$$

**Theorem 3.2.5 :** Let $\mathcal{T}_m\langle 1,4\rangle$, be a Toeplitz network , where $m \ \cong \ 3 \ (\ mod\ 4)$. Then

$$\frac{m}{m-1} \leq dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \leq \frac{(4m)}{3m-1}.$$

**Proof :** For different vertices of $\mathcal{T}_m\langle 1,4\rangle$, we have following cases
**Case 1** For $m = 7$ the possible local RN sets are,

TABLE 5. Local RN sets of Toeplitz network $\mathcal{T}_7\langle 1,4\rangle$.

| RN Sets | $R'(v_1v_2)$ | $R'(v_2v_3)$ | $R'(v_3v_4)$ | $R'(v_4v_5)$ | $R'(v_5v_6)$ | $R'(v_6v_7)$ | $R'(v_1v_5)$ |
|---|---|---|---|---|---|---|---|
| Deleted Vertices | $v_4, v_5$ | $v_1, v_6$ | $v_1, v_6$ | $v_2, v_7$ | $v_3$ | $v_4$ | $v_3$ |

TABLE 6. Local RN sets of Toeplitz network $\mathcal{T}_7\langle 1,4\rangle$.

| RN Sets | $R'(v_2v_6)$ | $R'(v_3v_7)$ |
|---|---|---|
| Deleted Vertices | $v_4$ | $v_5$ |

The local RN sets $R'(v_3v_4)$, $R'(v_4v_5)$, have minimum cardinality and their union is $V(T_7\langle 1,4\rangle)$. Therefore, we define a minimal ULRF $f : V(\mathcal{T}_7\langle 1,4\rangle) \rightarrow [0,1]$, as $f(v) = \frac{1}{5}, \forall\ v \ \in V(\mathcal{T}_7\langle 1,4\rangle$ hence $dim_{lF}(\mathcal{T}_7\langle 1,4\rangle) \leq \frac{7}{5}$.

From, above the local RN sets, $R'(v_1v_5)$, $R'(v_2v_6)$ have cardinality 6 and their union is $V(T_7\langle 1,4\rangle)$. Therefore, we define a maximal ULRF $f' : V(\mathcal{T}_7\langle 1,4\rangle) \rightarrow [0,1]$ as $f'(v) = \frac{1}{6}$ hence $dim_{lF}(T_7\langle 1,4\rangle) \geq \frac{7}{6}$. Consequently,

$$\frac{7}{6} \leq dim_{lF}(\mathcal{T}_7\langle 1,4\rangle) \leq \frac{7}{5}.$$

**Case 2**
For $m \geq 11$, where $m \ \cong \ 3(\ mod\ 4)$ by Lemma 3.1.3 $|R'(v_{n-3}v_{n-4})| \leq R'(x)$ and $|\bigcup_{i=1}^{m} R'(v_{m-3}v_{m-4})| = |V(\mathcal{T}_m\langle 1,4\rangle|$, where $R'(x)$ are other local RN sets. Therefore, we define a minimal LLRF $f : V(\mathcal{T}_m\langle 1,2\rangle) \rightarrow [0,1]$ as $f(v) = \frac{4}{3n+1}, \forall\ v \ \in \ V(T_m\langle 1,4\rangle$. Consequently, $dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \leq \frac{4m}{3m-1}$. Simlarly, $|R'(v_iv_{i+4})|\rangle|R'(x)|$, where $R'(x)$

are other local RN sets. Hence we define a maximal LLRF$f' : V(T_m\langle 1,4\rangle)) \to [0,1]$ as $f'(v) = \frac{1}{m-1} \ \forall\ v \in V(T_m\langle 1,4\rangle)$ . Hence $dim_{lF}(T_m\langle 1,4\rangle) \geq \frac{m}{m-1}$. Consequently,

$$\frac{m}{m-1} \leq dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \leq \frac{4m}{3m+1}.$$

**Theorem 3.2.6**
Let $\mathcal{T}_m\langle 1,4\rangle$, be a Toeplitz network, where $m \cong 0\ (\ mod\ 4)$. Then

$$\frac{m}{m-1} \leq dim_{lF}(T_m\langle 1,4\rangle) \leq \frac{4}{3}.$$

**Proof**
For different vertices of $\mathcal{T}_m\langle 1,4\rangle$, we have following cases
**Case 1**
For $m = 8$ the possible local RN sets are,

TABLE 7. Local RN sets of Toeplitz networks $\mathcal{T}_8\langle 1,4\rangle$

| RN Sets | $R'(v_1v_2)$ | $R'(v_2v_3)$ | $R'(v_3v_4)$ | $R'(v_5v_6)$ | $R'(v_6v_7)$ | $R'(v_7v_8)$ | $R'(v_1v_5)$ |
|---|---|---|---|---|---|---|---|
| Deleted Vertices | $v_4, v_8$ | $v_5$ | $v_1, v_6$ | $v_2, v_7$ | $v_4$ | $v_1, v_5$ | $v_3$ |

TABLE 8. Local RN sets of Toeplitz networks $\mathcal{T}_8\langle 1,4\rangle$

| RN Sets | $R'(v_2v_6)$ | $R'(v_5v_9)$ |
|---|---|---|
| Deleted Vertices | $v_4$ | $v_7$ |

The local RN sets $R'(v_1v_2)$, $R'(v_3v_4)$, $R'(v_4v_5)$, $R'(v_5v_6)$, $R'(v_7v_8)$, have minimum cardinality and their union is $V(T_8\langle 1,4\rangle)$. Therefore, we define a minimal ULRF $f : V(\mathcal{T}_8\langle 1,4\rangle) \to [0,1]$, as $f(v) = \frac{1}{6}, \forall\ v \in V(\mathcal{T}_8\langle 1,4\rangle$. Hence $dim_{lF}(\mathcal{T}_8\langle 1,4\rangle) \leq \frac{4}{3}$.

The local RN sets $R'(v_1v_5)$, $R'(v_2v_6)$, $R'(v_5v_9)$ and $R'(v_3v_7)$, have maximum cardinality and their union is $V(\mathcal{T}_8\langle 1,4\rangle)$. Therefore, we define a maximal LLRF $f' : V(\mathcal{T}_8\langle 1,4\rangle) \to [0,1]$ as $f'(v) = \frac{1}{7}$ hence $dim_{lF}(\mathcal{T}_8\langle 1,4\rangle) \geq \frac{8}{7}$. Consequently,

$$\frac{8}{7} \leq dim_{lF}(\mathcal{T}_8\langle 1,4\rangle) \leq \frac{4}{3}.$$

**Case 2**

For $m \geq 12$, where $m \cong 0\ (\ mod\ 4)$ by Lemma 3.1.4 $|R'(v_iv_{i+1})| \leq R'(x)$ and $|\bigcup_{i=1}^{n} R'(v_iv_{i+1})| = |V(T_m\langle 1,4\rangle|$, where $R'(x)$ are other local RN sets . Therefore, we a minimal ULRF $f : V(\mathcal{T}_m\langle 1,4\rangle) \to [0,1]$ as $f(v) = \frac{4n}{3}, \forall\ v \in V(T_m\langle 1,4\rangle$. Hence,

$dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \leq \frac{4}{3}$. Likewise, $|R'(v_iv_{i+4})|\rangle|R'(x)|$, where $R'(x)$ are other local RN sets. Hence, we define a minimal LLRF $f' : V(T_m\langle 1,4\rangle)) \rightarrow [0,1]$ as $f'(v) = \frac{1}{m-1}$ $\forall\, v \in V(\mathcal{T}_m\langle 1,4\rangle)$ hence $dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \geq \frac{m}{m-1}$. Consequently,

$$\frac{m}{m-1} \leq dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \leq \frac{4}{3}.$$

## 4. CONCISION

In this manuscript, we have computed local fractional metric dimension of different families of generalized Toeplitz networks in the form of exact values and bounds. Furthermore, the upper bound of local fractional metric dimension of $\mathcal{T}_m\langle 1,4\rangle$ is constant. It has also observed that local fractional metric dimension of these Toeplitz networks remain bounded when $m \rightarrow \infty$.

• Boundedness of Toeplitz networks illustrated in Table 8.

TABLE 9. Boundedness of LFMD of Toeplitz networks.

| Network | LFMD | Lower bound | Upper bound | Comment |
|---|---|---|---|---|
| $\mathcal{T}_m\langle 1,2\rangle$ | $\frac{m}{m-1} \leq dim_{lF}\,(\mathcal{T}_m\langle 1,2\rangle) \leq \frac{2m}{m+1}$. | 1 | 2 | Bounded |
| $\mathcal{T}_m\langle 1,4\rangle$ When $m \cong 3(\ mod\ 4)$ | $\frac{m}{m-1} \leq dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \leq \frac{4m}{3m-1}$. | 1 | $\frac{4}{3}$ | Bounded |
| $\mathcal{T}_m\langle 1,4\rangle$ When $m \cong 0(\ mod\ 4)$ | $\frac{m}{m-1} \leq dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \leq \frac{4}{3}$. | 1 | $\frac{4}{3}$ | Bounded |
| $\mathcal{T}_m\langle 1,4\rangle$ When $m \cong 1(\ mod\ 4)$ | $\frac{m}{m-1} \leq dim_{lF}(\mathcal{T}_m\langle 1,4\rangle) \leq \frac{4m}{3m+1}$. | 1 | $\frac{4}{3}$ | Bounded |

## REFERENCES

[1] M. Ali, G. Ali, U. Ali and M. T. Rahim. *On cycle related graphs with constant metric dimension, Open Journal of Discrete Mathematics.* **4** (2012) 21-23.

[2] M. Ali, M. T. Rahim, G Ali. *On two families of graphs with constant metric dimension, Journal of Prime Research in Mathematics,* **8** (2012) 95- 101.

[3] S. Aisyah, M. Utoyo and L. Susilowati. *On the local fractional metric dimension of corona product networks, IOP Conference, Earth Environ. Sci. Hungarica,* **243** (2019) 14.

[4]

[5] S. Arumugam, V. Mathew. *The fractional metric dimension of graphs,Discrete Math,* **9** (2012) 1584- 1590.

[6] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalk, L. Ram *Network discovery and verification, IEEE Access* **24** (2006) 2168-2181.

[7] G. Chartrand, L. Eroh, M. Johnson, O. Oellermann. *Resolvability in graphs and the metric dimension of a graph*, *Discrete Appl. Math.* **105**, (2000), 19- 28.

[8] V. Chvtal Mastermind *Combinatorica*, **3** (1983) 325-329.

J. Currie, O.R. Oellermann. *The metric dimension and metric independence of a graph, J. Combin.* **39** (2001) 157167.

[9] M. Fehr, S. Gosselin, and O. R. Oellermann. *The metric dimension of Cayley digraphs, Discrete Math,* **306**, (2006) 31-41.

[10] C. Grigorious, P. Manuel, M. Millera, B. Rajand, S. Stephen. *On the metric dimension of circulant and Harary graphs, Applied Mathematics and Computation*. **248** (2014) 47-54.

[11] F. Harary and R. Melter. *On the metric dimension of a graph*, *Ars Combin*, **2** (1976) 191 - 195.

[12] G. Heinig and K. Rost. *Algebraic Methods for Toeplitz-Like Matrices Operators*, *Boston*, *MA*, USA: Birkhuser, (1984).

[13] M. Javaid, H. Zafar, Q. Zhu and A. M. Alanazi. *Improved Lower Bound of LFMD with Applications of Prism-Related Networks*, *Mathematical Problems in Engineering*, **2021** (2021) 9 pages.

[14] P. J. Slater. *Leaves of trees, Congressus Numerantium*, **14** (1975) 549559.

[15] M. Javaid, H. Zafar, A. Aljaedi, A. M. Alanazi. *Boundedness of Convex Polytopes Networks via Local Fractional Metric Dimension, Mathematical Problems in Engineering,* **2021** (2021).

[16] M. Javaid, M. Raza, P. Kumam and J. B .Liu. *Sharp upper bounds of local fractional metric dimesion of connected graphs*, *IEEE ACCESS* **8**, (2020) 172329-172342.

[17] M. Javaid, H. Zafar, E. Bonyah. *Fractional Metric Dimension of Generalized Sunlet Networks, Journal of Mathematics*, **2021** (2021).

[18] M.A. Johnson, *Structure-activity maps for visualizing the graph variables arising in drug design, Journal of Biopharmaceutical Statistics,* **3** (1993) 203-236.

[19] M.A. Johnson, R. Carb-Dorca and P. Mezey. *Browsable structure-activity datasets*, *Advances in Molecular Similarity JAI Press.* (1998) 153-170.

[20] A. Kelenc, N. Tratnik, I. G Yero. *Uniquely identifying the edges of a graph The edge metric dimension,Discret. Appl. Math.* **2018** (2018).

[21] S. Khuller, B. Raghavachari and A. Rosenfeld. *Landmarks in Graphs Disc Applied Mathematics*, **70** (1996) 217 -229.

[22] D A. Krismanto and S. W. Saputro. *Fractional Metric Dimension of Tree and Unicyclic Graph, Procedia Computer Science* **74**, (2015) 47-52.

[23] J. B. Liu, M. F. Nadeem, H. M. A. Siddiqui and W. Nazir. *Computing metric dimension of Certain families of toeplitz graphs*, *IEEE Access*, **7**, (2019) 126734-126741.

[24]

[25] R.A. Melter, I. Tomescu *Metric bases in digital geometry, Computer Vision, Graphics, and Image Processing,* **25** (1984) 113-121.

M. F. Nadeem, S. Qu, A. Ahmad and M . Azeem. *Metric Dimension of Some Generalized Families of Toeplitz Graphs, Mathematical Problems in Engineering*, **2022**, (2022).

[26] Z. Shao, S. M. Sheikholeslami, Pu Wu, Jia-Biao Liu. *The Metric Dimension of Some Generalized Petersen Graphs, Discrete Dynamics in Nature and Society*,**2018** (2018) 10 pages.

[27] H. Shapiro, S. Sderberg. *A combinatory detection problem Amer, Math. Monthly,* **70**, (1963) 1066-1070.

[28] D. B. West. Introduction to Graph Theory, Edition 2,*Prentice Hall*, USA, 2011.

[29] H. Zafar, M. Javaid, E. Bonyah. *Computing LF-Metric Dimension of Generalized Gear Networks, Mathematical Problems in Engineering,* **2021** (2021).