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A Generalization to Ordinary Derivative and its Associated Integral with some applications

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Abstract.: This paper proposes a generalization to the ordinary derivative, the deformable derivative. For this, we employ a limit approach like the ordinary derivative but use a parameter varying over the unit interval. The definition makes the deformable derivative equivalent to the ordinary derivative because one's existence implies another. Its intrinsic property of continuously deforming function to its derivative, together with the graphical illustration of linear expression of the function and its derivative, renders sufficient substances to name it deformable derivative. We derive Rolle's, Mean-value and Taylor's theorems for the deformable derivative by establishing some of its basic properties. We also define the deformable integral using the fundamental theorem of calculus and discuss associated inverse, linearity, and commutativity property. In addition, we establish a connection between deformable integral and Riemann-Liouville fractional integral. As theoretical applications, we solve some fractional differential equations.

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1. Introduction

A slight variation in looking at some mathematical concepts may sometimes shed light on hidden facts. For instance, continuity and differentiability are based on limit concepts but are defined differently. However, the latter tells more geometric facts about function than the former. Similarly, a generalization of any mathematical concept, besides being a great source of motivation on its own, not just simplifies various intricate facts about it but extends its applicability to a broader class of problems. For instance, the first proof of the prime number theorem, a very popular theorem in real numbers, goes through complex analysis techniques. Likewise, there have been several generalizations of the notion of derivative to fractional derivative since the time Leibniz first asked this question to L'Hospital in his letter [9] in 1695 about a meaningful interpretation of symbol: $\dfrac{d^{1/2}y}{dx^{1/2}}$. Various types of fractional derivatives have been introduced so far. However, only a few became points of attraction for mathematicians and became popular in fractional calculus, such as Grunwald-Letnikov, Riemann-Liouville, Hadamard, Caputo, and Rieze fractional derivatives. To gain insight into fractional calculus, the reader is advised to go through [14, 13, 15].

As described above, most definitions for generalizing derivatives are in integral forms. In [8], R. Khalil introduced a fraction derivative, calling it the conformable fraction derivative in analogy to the standard one based on the limit approach. Unfortunately, his definition neither contains zero nor negative numbers. However, in the works [4, 5, 6, 11, 18, 19], various results related to fractional derivatives are investigated. In [1], we proposed a new generalized derivative named *deformable derivative*, involving limit-based definition, stated as follows:

• For some given $\alpha \in [0, 1]$, the deformable derivative of a real-valued function $g = g(t)$, in interval (a, b), is given by

$$D^\alpha g(t) = \lim_{\epsilon \to 0} \frac{(1 + \epsilon(1 - \alpha))g(t + \epsilon\alpha) - g(t)}{\epsilon}, \tag{*}$$

provided the limit exists.

It can be seen that the definition (*) behaves well with $\alpha = 0, 1$. More precisely, if $\alpha = 0$, $D^0 g(t) = g(t)$ and if $\alpha = 1$, $Dg(to) = g'(t)$ that coincide with the classical convection of ordinary derivative. Thus it can be assumed as a generalized derivative for parameter α, which is much simpler than that of Khalil's one and overcomes not only this shortcoming but ranges over a more comprehensive class of functions.

The rest of the article is arranged in the following manner. Section 2 derives a formula connecting both α-derivative and ordinary derivative, viz. $D^\alpha g(t) = \beta g(t) + \alpha Dg(t)$ and end it up with a conclusion that for a function, α-differentiability is the same as differentiability in the sense that the existence of one implies that of other. Section 3 focuses on some basic properties of the deformable derivative. We geometrically illustrate the behavior of operator D^α on some elementary functions. It also includes examples of how a deformable derivative sits between the function and its derivative. Section 4 discusses the forms of Rolle's, Mean-Value's, and Taylor's theorems in the context of the deformable derivative. The following section opens to define the deformable integral operator as given by:

$$I_a^\alpha g(t) =: \int_a^t e^{-\frac{\beta}{\alpha}(t-x)} g(x)\, d_\alpha x, \quad \text{where} \quad d_\alpha x = \frac{1}{\alpha} dx$$

for continuous function $g = g(t)$ over the interval (a,b) with respect to the parameter $0 < \alpha \leqslant 1$. It also includes some basic properties of this integral operator I_a^α. The last section employs these operators to solve some simple fractional differential equations.

Unless specified in the article, α-derivative is sometimes referred to as the deformable derivative concerning a given α, and it is assumed that $0 \leqslant \alpha \leqslant 1$.

2. Preliminary results

This section shows a relation between the deformable derivative for the given function and its ordinary derivative, leading us to name it the deformable derivative. The relation further exposes the exciting fact that the deformable derivative's graph lies linearly between that of function and derivative.

The first result is quite natural and asserts that *differentiability implies α-differentiability*. The proof connects both operators.

Theorem 2.1. *A differentiable function g at a point $t \in (a,b)$ is always α-differentiable at that point. Moreover in this case we have*

$$D^\alpha g(t) = \beta g(t) + \alpha D g(t), \quad \textit{where } \alpha + \beta = 1. \tag{**}$$

Proof. The cases $\alpha = 0, 1$ are trivially hold. For other values of α, we have by definition

$$\begin{aligned} D^\alpha g(t) &= \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)\, g(t+\alpha\epsilon) - g(t)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \left(\frac{g(t+\alpha\epsilon) - g(t)}{\epsilon} - \beta g(t+\alpha\epsilon) \right) \\ &= \alpha \cdot D g(t) - \beta \cdot \lim_{\epsilon \to 0} g(t+\alpha\epsilon). \end{aligned}$$

Both the terms exist as g, being differentiable at t is continuous as well. Hence theorem follows. □

The second result is also natural that talks about: *Does α-differentiability imply continuity*? The answer is affirmative. However, an auxiliary result concerning the locally bounded function is required to prove it. A function is said to be *locally bounded* at a point if it is bounded in some neighborhood of that point. Formally a function g defined on (a,b) is said to be locally bounded at t if there exists positive numbers M and δ, such that

$$\left| g(t+\epsilon) \right| \leqslant M \quad \text{whenever} \quad \left| \epsilon \right| < \delta.$$

Here δ is chosen sufficiently small so that $t + \epsilon \in (a,b)$.

Lemma 2.2. *Suppose g is α-differentiable at a point $t \in (a,b)$. Then, g is locally bounded there.*

Proof. Suppose g is α-differentiable at t, there exist a number $\delta > 0$ such that

$$\begin{aligned} & \left|(1+\epsilon\beta)g(t+\epsilon\alpha) - g(t) - \epsilon \cdot D^\alpha g(t)\right| \leqslant |\epsilon|, && \text{whenever } |\epsilon| < \delta \\ \Rightarrow \quad & \left|(1+\epsilon\beta)g(t+\epsilon\alpha)\right| \leqslant |\epsilon| + \left|g(t) + \epsilon \cdot D^\alpha g(t)\right|, && \text{whenever } |\epsilon| < \delta \\ & \leqslant |\epsilon| \left(1 + \left|D^\alpha g(t)\right|\right) + \left|g(t)\right|, && \text{whenever } |\epsilon| < \delta \\ \Rightarrow \quad & \left|g(t+\epsilon\alpha)\right| \leqslant \frac{|\epsilon| \left(1 + \left|D^\alpha g(t)\right|\right) + \left|g(t)\right|}{\left|1+\beta\epsilon\right|} && \text{whenever } |\epsilon| < \delta \end{aligned}$$

This yields that g is locally bounded at t. □

The next theorem asserts that α-differentiability implies continuity.

Theorem 2.3. *Let g be α-differentiable at a point $t \in (a, b)$ for some $\alpha \in (0, 1]$. Then g is continuous there.*

Proof. For continuity, it suffices to prove the following:

$$\lim_{\epsilon \to 0} \left(g(t+\epsilon\alpha) - g(t)\right) = 0.$$

The left hand side can also be written as:

$$\begin{aligned} & \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)\, g(t+\epsilon\alpha) - g(t) - \epsilon\beta g(t+\epsilon\alpha)}{\epsilon} \epsilon \\ & = \lim_{\epsilon \to 0} \left(\frac{(1+\epsilon\beta)\, g(t+\epsilon\alpha) - g(t)}{\epsilon} \cdot \epsilon - \beta\epsilon \cdot g(t+\epsilon\alpha) \right) \\ & = D^\alpha g(t) \cdot 0 - \beta \lim_{\epsilon \to 0} \epsilon g(t+\epsilon\alpha) = 0, \quad \text{(by hypothesis)} \\ & = -\beta \lim_{\epsilon \to 0} \epsilon g(t+\epsilon\alpha) = 0. \quad \text{(by using lemma 2.2)} \end{aligned}$$

This completes theorem. □

A strong version of Theorem 2.3 is now given as an easy consequence in the following corollary:

Corollary 2.4. *An α-differentiable function g in (a, b) is differentiable as well.*

Proof. For the existence of derivative we use its definition

$$\begin{aligned} Dg(t) &= \frac{1}{\alpha} \cdot \lim_{\epsilon \to 0} \frac{g(t+\alpha\epsilon) - g(t)}{\epsilon} \\ &= \frac{1}{\alpha} \cdot \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)\, g(t+\alpha\epsilon) - g(t) - \epsilon\beta g(t+\alpha\epsilon)}{\epsilon} \\ \Rightarrow \quad Dg(t) &= \frac{1}{\alpha} \left(\lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)\, g(t+\alpha\epsilon) - g(t)}{\epsilon} - \beta \cdot \lim_{\epsilon \to 0} g(t+\alpha\epsilon) \right) \end{aligned}$$

By using hypothesis and theorem 2.3, we get the result done. □

We summarise all these by saying that two concepts α-differentiability and classical differentiability of a function defined in (a, b) are equivalent in the sense that the existence of one implies other. We write it as a separate theorem.

Theorem 2.5. *Let g be defined in (a, b). For any α, g is α-differentiable if and only if it is differentiable.*

Remark 2.6. *We make a remark here that over the interval (a, b), the existence of α-derivative with respect to one particular value of $\alpha > 0$ is sufficient for the existence of α-derivative with respect to all other values of α.*

Though the most important case for α is when $\alpha \in [0, 1]$ but what happens if $\alpha \in (n, n+1]$ for any natural number n and what should be definition? The extension is made in a very natural way.

Definition 2.7. *Suppose g is n-times differentiable at $t \in (a, b)$. For given $\alpha \in (n, n+1]$, we extend deformable derivative in a very natural way and define it by the following limit:*

$$D^{\alpha} g(t) =: \lim_{\epsilon \to 0} \frac{(1 + \epsilon\{\beta\}) D^n g\,(t + \epsilon\{\alpha\}) - D^n g(t)}{\epsilon}$$

where $\{\alpha\}$ is the fractional part of α and $\{\alpha\} + \{\beta\} = 1$.

As the consequence of the above definition, if $g^{(n+1)}$ exists, we have

$$D^{\alpha} g(t) = \{\beta\} D^n g(t) + \{\alpha\} D^{n+1} g(t).$$

3. Basic properties of deformable derivative

Apart from discussing some basic properties of the deformable derivative, such as linearity and commutativity, this section deals with fundamental theorems: Rolle's, Mean-Value, and Taylor's theorems. The geometric illustration of D^{α}, concerning some of the elementary functions, is also included.

Theorem 3.1. *The operator D^{α} possesses the following properties:*

(a) *Linearity: $D^{\alpha}(ag + bh) = aD^{\alpha} g + bD^{\alpha} h$.*
(b) *Commutativity: $D^{\alpha_1} \cdot D^{\alpha_2} = D^{\alpha_2} \cdot D^{\alpha_1}$. In general, we have $D^{\alpha} \cdot D^n = D^n \cdot D^{\alpha}$, where n is a positive integer.*
(c) *$D^{\alpha}(\lambda) = \beta\lambda$, where λ is a constant function.*
(d) *$D^{\alpha}(g \cdot h) = (D^{\alpha} g) \cdot h + \alpha g \cdot Dh$, i.e., the Leibniz rule does not holds for D^{α}.*

Proof. Linearity is evident from definition. Commutativity follows readily by noticing the symmetries in the expression below:

$$D^{\alpha_1}(D^{\alpha_2} g) = \beta_1\beta_2 g + (\alpha_1\beta_2 + \alpha_2\beta_1) Dg + \alpha_1\alpha_2 D^2 g,$$

where $\alpha_i + \beta_i = 1$ for $i = 1, 2$. Using the relation: $D^{\alpha} = \beta I + \alpha D$, the third and fourth can be easily established. Violation of Leibniz rule in part (d) motivates to regard D^{α} as fractional derivative. Readers are advised to see [17]. □

Most of the familiar functions behave well with respect to differentiation, so their deformable derivatives can be obtained from expression (**) in theorem 2.1. We present the deformable derivatives of some of the elementary functions in the following proposition.

Proposition 3.2.

(1) $D^{\alpha}(t^r) = \beta t^r + r\alpha t^{r-1}, \quad r \in \mathbb{R}$.
(2) $D^{\alpha}(e^t) = e^t$.

(3) $D^{\alpha}(\sin t) = \beta \sin t + \alpha \cos t$.
(4) $D^{\alpha}(\log t) = \beta \log t + \dfrac{\alpha}{t}$, $t > 0$.

It is intuitively clear that the operator D^{α} is continuous with respect to parameter α. However, we leave it for the reader to prove. Instead, we focus on the geometric realization of the ideal with some examples. The following figures depict not only continuity phenomenon but also explain its nature of deforming function to its derivative.

4. Some important theorems on deformable derivative

This section focuses on Rolle's theorem, Mean Value theorem, and Taylor's theorems for the deformable derivative.

Theorem 4.1. (Rolle's theorem for deformable derivative) *Let $g : [a, b] \longrightarrow \mathbb{R}$ be a function satisfying:*

(i) *g is continuous on $[a, b]$*
(ii) *g is α-differentiable in (a, b)*
(iii) *$g(a) = g(b)$.*

Then there exists a point $p \in (a, b)$ such that $D^{\alpha}g(p) = \beta g(p)$.

Proof. By applying corollary 2.5, g is differentiable in (a, b). Hence g holds all conditions of classical 4.1. Then there $\exists$ a point $p \in (a, b)$ such that $Dg(p) = 0$. Hence using equation (**) of theorem 2.1, we have $D^{\alpha}g(p) = \beta g(p)$. □

Mean Value theorem is a consequence of 4.1 so is the case with deformable derivative.

Theorem 4.2. (Mean Value theorem for deformable derivative) *Let $g : [a, b] \longrightarrow R$ be a function satisfying:*

(i) *g is continuous on $[a, b]$*
(ii) *g is α- differentiable in (a, b).*

Then, there $\exists$ a point $p \in (a, b)$ such that

$$D^{\alpha}g(p) = \beta g(p) + \alpha \frac{g(b) - g(a)}{b - a}.$$

Proof. We consider a function h defined by:

$$h(t) =: g(t) - g(a) - \frac{g(b) - g(a)}{b - a} t$$

Notice that $h(x)$ holds all the conditions of 4.1, there exists $p \in (a, b)$ such that $D^{\alpha}h(p) = \beta h(p)$. This yields the desired expression in the theorem. □

Theorem 4.3. (Taylor's theorem for deformable derivative) *Suppose g is n-times α-differentiable such that all α-derivatives are continuous on $[a, a + h]$. Then*

$$g(a + h) = \sum_{k=0}^{n-1} \frac{h^k}{k!\alpha^k} \left(D_k^{\alpha} g(a) - \beta \frac{(1-\theta)^{k-n+1} h}{\alpha n} D_k^{\alpha} g(a + \theta h) \right) + \frac{h^n}{n!\alpha} D_n^{\alpha} g(a + \theta h),$$

where $D_k^{\alpha} = D^{\alpha} D^{\alpha} \cdots D^{\alpha}$ (k-times), $0 < \theta < 1$.

Proof. Consider a function ϕ defined by:

$$\phi(t) = \sum_{k=0}^{n-1} \frac{(a+h-t)^k}{k!\alpha^k} D_k^\alpha(t) + \frac{A}{n!\alpha^n}(a+h-t)^n, \tag{4. 1}$$

where A is a constant to be chosen A such that $\phi(a+h) = \phi(a)$. This yields

$$\frac{A}{n!\alpha^n} h^n = g(a+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!\alpha^k} D_k^\alpha g(a). \tag{4. 2}$$

Now by hypothesis, ϕ is α-differentiable in $(a, a+h)$. Using part (d) of theorem 3.1, the α-derivative $D^\alpha \phi$ is given by

$$D^\alpha \phi(t) = \frac{(a+h-t)^{n-1}}{\alpha^{n-1}(n-1)!} D_n^\alpha g(t) + \frac{A}{\alpha^n n!}\left(\beta(a+h-t)^n - \alpha n(a+h-t)^{n-1}\right). \tag{4. 3}$$

Hence ϕ satisfies all the conditions of 4.1. So there is some $\theta \in (0,1)$ such that

$$D^\alpha \phi(a+\theta h) = \beta \phi(a+\theta h).$$

Using equations (1), (2) and (3), we have

$$g(a+h) = \sum_{k=0}^{n-1} \frac{h^k}{k!\alpha^k}\left(D_k^\alpha g(a) - \beta \frac{(1-\theta)^{k-n+1}h}{\alpha n} D_k^\alpha g(a+\theta h)\right) + \frac{h^n}{n!\alpha} D_n^\alpha g(a+\theta h).$$

This completes the theorem. □

5. Deformable Integral

An integral, being an inverse operator to a derivative, plays an equally important role as the derivative. The section defines a deformable integral as an inverse operator for the deformable derivative, and we discuss some basic properties of this deformable integral. All functions considered in this section are assumed to be continuous.

Definition 5.1. *Let g be a continuous function defined on $[a,b]$. For $\alpha \in (0,1]$, we define deformable integral of order α, denoted by $I_a^\alpha g$, by the integral:*

$$I_a^\alpha g(t) := \int_a^t e^{-\frac{\beta}{\alpha}(t-x)} g(x)\, d_\alpha x \tag{5. 4}$$

where $d_\alpha x = \dfrac{1}{\alpha} dx$ and $\alpha + \beta = 1,\ \alpha \in (0,1]$.

The reason for calling it deformable integral is because of being an inverse operator to the deformable derivative. This is done in the following theorem, a version of the fundamental theorem of calculus. That is, it roughly says that deformable integral I_a^α is the right inverse operation of α-differentiation D^α.

Theorem 5.2 (Inverse Property). *Let g be a continuous function defined on $[a,b]$. Then, $I_a^\alpha g$ is α-differentiable in (a,b). In fact, we have $D^\alpha\left(I_a^\alpha g(x)\right) = g(x)$. Conversely, suppose h is a continuous anti-α-derivative of g over (a,b), that is $h = D^\alpha g$. Then, we have*

$$I_a^\alpha\left(D^\alpha g(t)\right) = I_a^\alpha\left(h(t)\right) = g(t) - e^{\frac{\beta}{\alpha}(a-t)} g(a).$$

Proof. Since g is given to be continuous so in view of theorem 2.5, $I_a^\alpha g$ is α-differentiable. If we set $h = I_a^\alpha g$ then we have

$$D^\alpha \left(I_a^\alpha g(t)\right) = D^\alpha h(t) = \alpha Dh(t) + \beta h(t).$$

We know that a particular solution of the differential equation: $\alpha Dh + \beta h = g$ is given as

$$h(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_a^t e^{\frac{\beta}{\alpha} x} g(x) dx.$$

Thus the first part of the theorem is complete. For the second part, we have

$$\begin{aligned} h(t) &= D^\alpha g(t) = \alpha Dg(t) + \beta g(t) \\ \Rightarrow \quad I_a^\alpha h(t) &= \alpha I_a^\alpha \left(Dg(t)\right) + \beta I_a^\alpha g(t) \\ &= e^{\frac{-\beta}{\alpha} t} \int_a^t e^{\frac{\beta}{\alpha} x} g'(x) dx + \beta I_a^\alpha g(t) \\ &= e^{\frac{-\beta}{\alpha} t} \left(\left[e^{\frac{\beta}{\alpha} x} g(x) \right]_a^t - \frac{\beta}{\alpha} \int_a^t e^{\frac{\beta}{\alpha} x} g(x) dx \right) + \beta I_a^\alpha g(t) \\ \Rightarrow \quad I_a^\alpha h(t) &= g(t) - e^{\frac{\beta}{\alpha}(a-t)} g(a). \end{aligned}$$

This completes the second part. □

Some basic properties of deformable fractional integral are contained in following theorem:

Theorem 5.3. *The operator I_a^α possesses the following properties:*

(a) *Linearity:* $I_a^\alpha (bg + ch) = bI_a^\alpha g + cI_a^\alpha h$.
(b) *Commutativity:* $I_a^{\alpha_1} \circ I_a^{\alpha_2} = I_a^{\alpha_2} \circ I_a^{\alpha_1}$, *where* $\alpha_i + \beta_i = 1$, $i = 1, 2$.

Proof. Linearity readily follows from definition (5. 4). For commutativity, we consider

$$\begin{aligned} I_a^{\alpha_1} \circ I_a^{\alpha_2} g(t) &= I_a^{\alpha_1} \left(\int_a^t e^{-\frac{\beta_2}{\alpha_2}(t-\theta)} g(\theta)\, d_{\alpha_2}\theta \right) \\ &= \int_a^t e^{-\frac{\beta_1}{\alpha_1}(t-x)} \left(\int_a^x e^{-\frac{\beta_2}{\alpha_2}(x-\theta)} g(\theta)\, d_{\alpha_2}\theta \right) d_{\alpha_1} x \\ &= \frac{1}{\alpha_1\alpha_2} e^{-\frac{\beta_1}{\alpha_1} t} \int_a^t \int_a^x e^{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)x} e^{\frac{\beta_2}{\alpha_2}\theta} g(\theta)\, d\theta dx. \end{aligned}$$

Now reversing the order of integration, we get

$$\begin{aligned}
I_a^{\alpha_1} \circ I_a^{\alpha_2} g(t) &= \frac{1}{\alpha_1 \alpha_2} e^{-\frac{\beta_1}{\alpha_1} t} \int\limits_a^t \int\limits_\theta^t e^{(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2})x} e^{\frac{\beta_2}{\alpha_2}\theta} g(\theta)\, dx d\theta \\
&= \frac{1}{\alpha_1 \alpha_2} e^{-\frac{\beta_1}{\alpha_1} t} \int\limits_a^t e^{\frac{\beta_2}{\alpha_2}\theta} g(\theta) \left(\int\limits_\theta^t e^{(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2})x} dx \right) d\theta \\
&= \frac{1}{\beta_1 \alpha_2 - \beta_2 \alpha_1} \left(e^{-\frac{\beta_2}{\alpha_2} t} \int\limits_a^t e^{\frac{\beta_2}{\alpha_2}\theta} g(\theta) d\theta - e^{-\frac{\beta_1}{\alpha_1} t} \int\limits_a^t e^{\frac{\beta_1}{\alpha_1}\theta} g(\theta) d\theta \right) \\
&= \frac{1}{\beta_1 \alpha_2 - \beta_2 \alpha_1} \left(\alpha_2 I_a^{\alpha_2} - \alpha_1 I_a^{\alpha_1} \right) g(t).
\end{aligned}$$

Interchanging the role of α_1 and α_2, it follows that

$$I_a^{\alpha_2} I_a^{\alpha_1} g(t) = \frac{1}{\beta_2 \alpha_1 - \beta_1 \alpha_2} \left(\alpha_1 I_a^{\alpha_1} - \alpha_2 I_a^{\alpha_2} \right) g(t) = I_a^{\alpha_1} I_a^{\alpha_2} g(t).$$

This completes the proof. □

We end up the section with a list of the deformable integrals of some elementary functions in the following proposition and leave their verification for the reader.

Proposition 5.4.

(1) $I_a^\alpha \sin t = \dfrac{1}{\alpha^2 + \beta^2} \Big(\beta \sin t - \alpha \cos t + e^{\frac{\beta}{\alpha}(a-t)} \left(\alpha \cos a - \beta \sin a \right) \Big).$

(2) $I_a^\alpha e^t = \Big(e^t - e^{\frac{(a - \beta t)}{\alpha}} \Big).$

(3) $I_a^\alpha \lambda = \dfrac{\lambda}{\beta} \left(1 - e^{\frac{\beta}{\alpha}(a-t)} \right)$, *where* λ *is a constant.*

(4) $I_0^\alpha t^n = \dfrac{1}{\beta} \left(\displaystyle\sum_{k=0}^n \frac{(-1)^k\, n!}{(n-k)!} \left(\frac{\alpha}{\beta} \right)^k t^{n-k} + (-1)^{n+1}\, n! \left(\frac{\alpha}{\beta} \right)^n e^{-\frac{\beta}{\alpha} t} \right).$

6. Connection to Riemann-Liouville Fractional Integral

This section explores the deformable integral operator and discovers its connection to Riemann-Liouville fractional integral operator. Throughout the section, we assume that all functions considered are defined in the interval $[0, \infty)$.

The deformable integral defined in (5. 4) can also be written as follows:

$$I_0^\alpha \left(g(t) \right) = \frac{1}{\alpha} \int_0^t e^{-\frac{\beta}{\alpha}(t-x)} g(x) dx = \frac{1}{\alpha} \int_0^t g(t-x) e^{-\frac{\beta}{\alpha} x} dx.$$

We know that Riemann-Liouville fractional integral for parameter $\gamma \in \mathbb{C}$, $\mathrm{Re}(\gamma) > 0$ is defined by the integral:

$${}_R I_+^\gamma \left(g(t) \right) =: \frac{1}{\Gamma(\gamma)} \int_0^t (t-x)^{\gamma - 1} g(x) dx.$$

From these two above equations, we can get

$$I_0^\alpha\left(x^{\gamma-1}\right) = \frac{1}{\alpha}\int_0^t (t-x)^{\gamma-1} e^{-\frac{\beta}{\alpha}x}dx = \frac{\Gamma(\gamma)}{\alpha}{}_R I_+^\gamma\left(e^{-\frac{\beta}{\alpha}t}\right).$$

This yields that

$$\alpha \cdot I_0^\alpha\left(x^{\gamma-1}\right) = \Gamma(\gamma)\cdot {}_R I_+^\gamma\left(e^{-\frac{\beta}{\alpha}t}\right). \tag{6. 5}$$

A Special case when $\gamma = n$ a natural number, we get

$$\alpha \cdot I_0^\alpha\left(x^{n}\right) = n! \cdot {}_R I_+^{n+1}\left(e^{-\frac{\beta}{\alpha}t}\right).$$

7. Applications to deformable Differential Equations

We solve some simple linear deformable differential equations using deformable derivative operator D^α. In first example we discuss method of solving homogeneous linear, while in second non-homogeneous linear deformable differential equations.

Example 7.1. *Consider the deformable differential equation:*

$$D^\alpha y(t) + Q(t)y(t) = 0,$$

where $Q(t)$ is continuous. Using the expression given in (**), the equation gets transformed to

$$\begin{aligned} \alpha Dy + \beta y + Q(t)y &= 0 \\ \Rightarrow \quad Dy + \frac{(\beta + Q(t))}{\alpha}y &= 0. \end{aligned}$$

We get simple first order linear ordinary differential equation whose general solution is given by

$$y = Ce^{\frac{-\left(\beta t + \int Q(t)dt\right)}{\alpha}},$$

where C is arbitrary constant.

Example 7.2. *We now consider a non-homogeneous linear deformable equation:*

$$D^{1/2}y + y = te^{-t}.$$

This can be written as

$$\frac{1}{2}y + \frac{1}{2}Dy + y = te^{-t} \quad \Rightarrow \quad Dy + 3y = 2te^{-t}.$$

whose general solution is given by

$$y(t) = Ce^{-3t} + \left(t - \frac{1}{2}\right)e^{-t},$$

where C is a constant.

Example 7.3. *The deformable differential equation :*

$$D^{\alpha_2}[D^{\alpha_1}y(t)] = 0$$

is equivalent to the following second order homogeneous differential equation:

$$\alpha_1\alpha_2 D^2y + (\alpha_1\beta_2 + \alpha_2\beta_1)Dy + \beta_1\beta_2 y = 0.$$

The roots its auxiliary equation are:

$$\frac{-\beta_1}{\alpha_1} \text{ and } \frac{-\beta_2}{\alpha_2}.$$

Hence in case of distinct roots the general solution of the deformable differential equation is

$$y = C_1 e^{-\frac{\beta_1}{\alpha_1}t} + C_2 e^{-\frac{\beta_2}{\alpha_2}t},$$

and in case of repeated roots, we have

$$y = (C_1 + C_2 t)e^{-\frac{\beta}{\alpha}t}.$$

Example 7.4. *Consider another problem*

$$D_2^{1/4}y(t) - \frac{3}{2}D^{1/4}y(t) + y(t) = 0$$

with boundary conditions $y = 0$, $D^{1/4}y = 1$ when $t = 0$.
which is equivalent to the following second order homogeneous differential equation:

$$D^2y + 7y = 0$$

whose general solution is given by

$$y(t) = A\cos\sqrt{7}t + B\sin\sqrt{7}t,$$

where A and B are arbitrary constants.
using boundary conditions, we have, $A = 0$ and $B = \frac{4\sqrt{7}}{7}$.

which yields the result

$$y(t) = \frac{4\sqrt{7}}{7}\sin\sqrt{7}t.$$

8. CONCLUSION

This paper has presented a new fractional derivative, the deformable derivative, and its inverse operator. The definition is based on the limit approach, using a parameter that varies over a unit interval. The simple nature of the definition tells us how deformable and ordinary derivatives imply each other. The linear relation of deformable derivative in terms of function and its derivative is presented nicely. The deformable form of Rolle's, Mean-value and Taylor's theorem makes the whole theory more efficient by enabling shorter between simpler proofs based on the knowledge of its basic properties. A relation between the deformable integral operator and Riemann-Liouville integral operator is also established. The novelty of the proposed work reflects in solved numerical examples. The proposed operator converts fractional-order parts into a differential equation with constant coefficients. Thus, the theory developed in this paper helps to solve fractional differential equations. Future work may focus on obtaining the deformable Laplace Transform [12, 3, 2] and the deformable Euler's Theorem [10, 7, 16].

We end the paper with some critical questions yet to be answered.

(i) What are the geometric interpretation and physical significance of the deformable derivative?
(ii) Is there any similarity between the classical fractional derivative and deformable derivative?
(iii) The deformable derivative is equivalent to the ordinary derivative but not the same, so it could be used to analyze functions.

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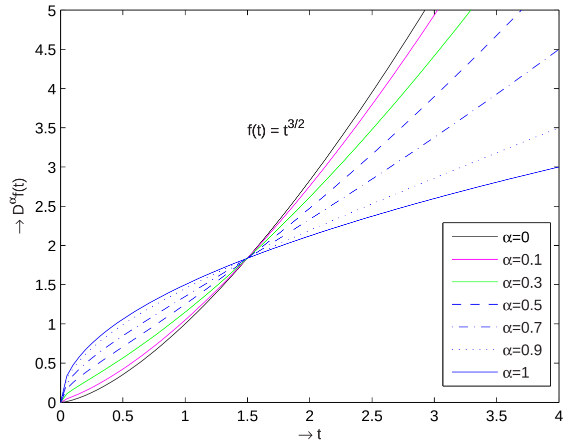
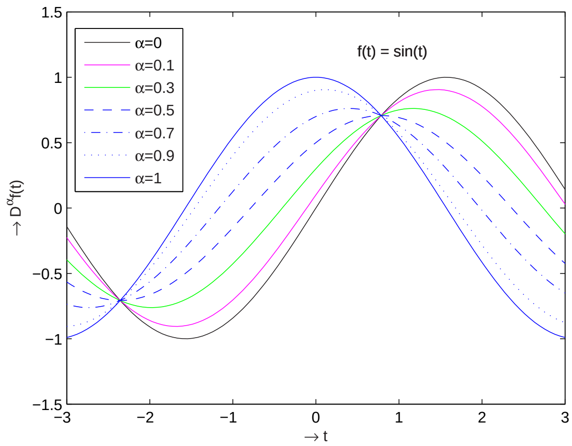


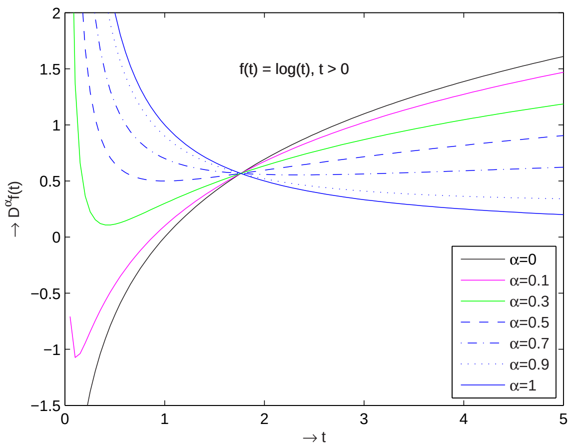
[b]0.45

Figure 2: D^{α} operating on $t^{3/2}$



[b]0.45

Figure 3: D^{α} operating on $\sin t$



[b]0.45

Figure 4: D^{α} operating on $\log t, t > 0$

Figure 5: D^{α}-derivative of different functions, $\alpha \in (0, 1)$