Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(10)(2020) pp. 25-45

Modelling and Analysis of the Fractional Order Ebola Virus Model with Caputo Fabrizio Derivative

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Received: 10 January, 2020 / Accepted: 10 October, 2020 / Published online: 05 October, 2020

Abstract.: In this article, to monitor the rise in the public Ebola Virus, we will discuss the complex transmission and epidemic problems. The Caputo-Fabrizio fractional derivative operator of order $\Omega \in (0, 1]$ is used to obtain fractional differential equations structure. The stability of fractional order model is developed and unique non-negative solution is tested. The numerical simulations are performed by using an iterative technique. Sumudo transformation method is used to solve the fractional order model. The nature and stability of the ebola virus system with fractional order derivative is dealt mostly. In the form of Caputo-Fabrizio, a new approach to this form of biological model associated with derivative is applied. Some new results are being viewed with the help of Sumudo transformation.

Furthermore, the existence and uniqueness of results for equilibrium solutions have been proved by Banach theorem. However, mathematical simulations are also acknowledged to evaluate the impact of the model parameters by increasing the disease and showing the effect of the Ω fractional parameter on our solutions obtained. The impact of various parameters is graphically displayed.

AMS (MOS) Subject Classification Codes: 37C75,; 93B05; 93B07; 65L07
Key Words: Ebola virus model; Caputo-Fabrizio derivative; fractional-order Ebola virus model; Sumudu transform; Existence and Uniqueness.

1. INTRODUCTION

Ebola is now under intense investigation as a fatal human virus due to the recent West-African outbreak [1,2]. The World Health Organization (WHO) has identified Ebola virus as a health emergency of growing concern for major global economic burden. Patients usually die until the antibiotic treatment happens at the lethal stage of Ebola infection. There have been many attempts to get a vaccine for Ebola disease, particularly after the outbreak of Ebola in West Africa in 2014. The results of an initial review of the July 2015 Guinea Step III efficacy vaccine study as per the WHO [3].

Further interest has recently been paid in the analysis of fractional-order derivatives in the explanation of memory effects in dynamic systems because the area involving differential equations in fraction order has been observed to have noticeable applications in different science and technology disciplines [4]. Over recent years, due to the biological processes involved, the fractional order models were given a lot of publicity. Fractional order representations are more practical and more reliable than standard order models. Therefore, by using fractional order derivatives and integrals, it is possible to explain the inherited characteristics and properties of different materials and processes. By adapting the mathematical modeling, fractional order derivative is a global operator compared to the local classical derivative. Treating real-world problems with the use of fractional order derivatives is far more efficient and reliable, particularly when trying to deal with a prototype that plays a basic role in memory or inherited property characteristics.

There are different models for Ebola spreading, starting with the simplest models of SIR and SEIR [5] and later being considered more complicated but also more practical models [6]. In [1], the writers suggest a mathematical model using classical and beta derivatives for SIRD (Susceptible-infectious-Recovered-Dead). In this template, newborn or immigration is not regarded by the group of susceptible people. The study indicates that whole country could die out in a very short span of time if small portion of people are infected and there is no good prevention [7].

In the recent year, the researcher has taken into account interest and attention in fractional calculus in various aspects for research of the said topic [8, 9, 10] There has been notable growth of derivatives and integrals of fractional orders in last couple of years as revealed by several monographs dedicated to it [11, 12]. The multitude of research papers published in scientific journals [13] studied fractional order differential-difference equation, [14] examined a mathematical model to evaluate malaria co-infection with the human immunodeficiency virus,[15] examined the local fractional diffusion and relaxation equations [16], a new fractional derivative and its implementation for polar bear hair explained [17], Wang and Liu [18,19] showed his fractional derivative applications for non-linear fractional heat transfer equation, Hu et al.[20] Some new results related fractional and fractal are also in [21,22]. The fractional complex transformation is used to turn the fractal space-time into its continuous partner, and all known analytical methods can be applied directly to the resulting equations. This paper is a fractional calculus description in a fractal system [23,24]. It is easy to use the proposed fractal derivative for discontinuous problems, and classical calculus used equations with fractal derivatives can be easily solved [25,26]. Some new results are treated with the support of Sumudu transform and Picard's successive approximation methods. Furthermore, numerical solution of the model represents different non-integer values.

2. Preliminaries

Definition 1. Let $\Theta \in H^1(c,d)$, $d > c$, $\Omega \in (0,1)$, then the new fractional order in Caputo derivative sense is as follows [4,29,30].

$$ {}^{C}D_t^{\Omega}(\Theta(t)) = \frac{N(\Omega)}{1-\Omega}\int_c^t \Theta^{'}(y) exp\left[-\Omega\frac{t-y}{1-\Omega}\right] dy, $$

where $N(\Omega)$ represents the normalization function with $N(0) = N(1) = 1$. But, if the function does not belong to $H^1(c,d)$, then the derivative can be write as

$$ {}^{C}D_t^{\Omega}(\Theta(t)) = \frac{\Omega N(\Omega)}{1-\Omega}\int_c^t (\Theta(t)-\Theta(y)) exp\left[-\Omega\frac{t-y}{1-\Omega}\right] dy. $$

Remark: If we take $\sigma = \frac{1-\Omega}{\Omega} \in [0,\infty]$, then the new Caputo derivative having fractional order as

$$ {}^{C}D_t^{\Omega}(\Theta(t)) = \frac{N(\sigma)}{\sigma}\int_c^t \Theta^{'}(y) exp\left[-\frac{t-y}{1-\sigma}\right] dy, N(0) = N(\infty) = 1 $$

Further,

$$ lim_{\sigma\to 0}\frac{1}{\sigma}exp[-\frac{t-y}{\sigma}] = \delta(y-t). $$

Definition 2. Let $\Omega \in (0,1)$, then the fractional integral of order Ω of a function $\Theta(t)$ is given by [29,30].

$$ {}^{CF}I_t^{\Omega}(\Theta(t)) = \frac{2(1-\Omega)}{(2-\Omega)N(\Omega)} + \frac{2\Omega}{(2-\Omega)N(\Omega)}\int_0^t \Theta(s)ds, t \geq 0. $$

Remark: This definition indicates that the fractional integral of the Caputo form of function of order $0 < \Omega \leq 1$ is an average of Θ function and its integral of order one. Hence we get

$$ \frac{2(1-\Omega)}{(2-\Omega)N(\Omega)} + \frac{2\Omega}{(2-\Omega)N(\Omega)} = 1. $$

This equation gives an explicit formula for

$$N(\Omega) = \frac{2}{2-\Omega}, 0 \le \Omega \le 1.$$

Assuming this derivative, the new Caputo derivative of order $0 < \Omega < 1$ has been reformulated as

$$^{C}D_t^{\Omega}(\Theta(t)) = \frac{1}{1-\Omega}\int_c^t \Theta^{'}(y)exp\left[-\Omega\frac{t-y}{1-\Omega}\right]dy.$$

Definition 3. Let $\Theta(t)$ be a function for which the Caputo-Fabrizio exists, then the Sumudu transform of the Caputo-Fabrizio fractional derivative of $\Theta(t)$ is given as [29].

$$\mathcal{S}(_0^{CF}D_t^{\Omega})(\Theta(t)) = M(\Omega)\frac{\mathcal{S}(\Theta(t)) - \Theta(0)}{1-\Omega+\Omega\Delta}$$

where the normalization function is denoted by $M(\Omega)$ with $M(0) = M(1) = 1$.

3. Fractional Order Ebola Virus Model

This Ebola virus model is adapted from [27,28] and are shown in the followings equations.

$$\begin{aligned}
\frac{dS}{dt} &= \mu N - \frac{\beta_i}{N}SI - \frac{\beta_h}{N}SH - \frac{\beta_d}{N}SD - \frac{\beta_\gamma}{N}SR - \mu S,\\
\frac{dE}{dt} &= \frac{\beta_i}{N}SI + \frac{\beta_h}{N}SH + \frac{\beta_d}{N}SD + \frac{\beta_\gamma}{N}SR - \sigma E - \mu E,\\
\frac{dI}{dt} &= \sigma E - (\gamma_1 + \epsilon + \tau + \mu)I,\\
\frac{dR}{dt} &= \gamma_1 I - \gamma_2 H - (\gamma_3 + \mu)R,\\
\frac{dD}{dt} &= \epsilon I - (\sigma_1 + \xi)D,\\
\frac{dH}{dt} &= \tau I - (\gamma_2 + \sigma_2 + \mu)H,\\
\frac{dB}{dt} &= \sigma_1 D + \sigma_2 H - \xi B,\\
\frac{dC}{dt} &= \gamma_3 R - \mu C.
\end{aligned} \qquad (3.\ 1)$$

The total N population is divided into eight classes that are mutually exclusive: susceptible population is represented by (S), exposed population is represented by (E), infected population is represented by (I), hospitalized population is represented by (H), asymptomatic but still infectious population is represented by (R), population which are dead but not buried is represented by (D), buried population is represented by (B), and fully recovered population is represented by (C). Birth and death rates are considered equal and β_i, β_d, β_h and β_r represent the interaction rate of susceptible people with infective, dead, hospitalized and easily treatable people respectively. Exposed individuals become infectious at a rate σ. Where γ_1 and τ represent the per capita rate of individuals transitionary from the infectious to the asymptomatic and hospitalizedclass respectively. Dead-class individuals progress to the buried class at a rate of σ_1. Here σ_2 and γ_2 are hospitalized patients who move to the buried class and the asymptomatic classrespectively. Here γ_3 is the number of fully recovered symptomatic patients. Infectious individuals progress to the dead class at a fatality rate ϵ. Dead and buried bodies are incinerated at a rate ξ. We assume that the total population, $N = S+E+I+R+H+D+B+C$ is constant, that is, the birth and death rates, both denoted by μ, are equal to the incineration rate ξ. The model we are studying in this analysis is a Caputo-Fabrizio derivative and fractional order Ω such as $\Omega\in(0,1]$ as seen below.

$$\begin{aligned}
{}_0^{CF}D_t^{\Omega}S(t) &= \mu N - \frac{\beta_i}{N}SI - \frac{\beta_h}{N}SH - \frac{\beta_d}{N}SD - \frac{\beta_\gamma}{N}SR - \mu S,\\
{}_0^{CF}D_t^{\Omega}E(t) &= \frac{\beta_i}{N}SI + \frac{\beta_h}{N}SH + \frac{\beta_d}{N}SD + \frac{\beta_\gamma}{N}SR - \sigma E - \mu E,\\
{}_0^{CF}D_t^{\Omega}I(t) &= \sigma E - (\gamma_1 + \epsilon + \tau + \mu)I,\\
{}_0^{CF}D_t^{\Omega}R(t) &= \gamma_1 I - \gamma_2 H - (\gamma_3 + \mu)R,\\
{}_0^{CF}D_t^{\Omega}D(t) &= \epsilon I - (\sigma_1 + \xi)D,\\
{}_0^{CF}D_t^{\Omega}H(t) &= \tau I - (\gamma_2 + \sigma_2 + \mu)H,\\
{}_0^{CF}D_t^{\Omega}B(t) &= \sigma_1 D + \sigma_2 H - \xi B,\\
{}_0^{CF}D_t^{\Omega}C(t) &= \gamma_3 R - \mu C.
\end{aligned} \tag{3. 2}$$

If we want to get the classical model then we will take $\Omega = 1$. By Sumudu transform some results are obtained. By applying fixed point theorem, existence of solution is discussed. We have also proved the uniqueness of solution.

4. Caputo-Fabrizio Derivative for Ebola Model

We will get a special system solution of system (3.2) by applying the Sumudu transform on both sides of all equations of (3.2).

$$\begin{aligned}
M(\Omega)\frac{\mathcal{S}(S(t)) - S(0)}{1-\Omega+\Omega\Delta} &= \mathcal{S}[\mu N - \frac{\beta_i}{N}SI - \frac{\beta_h}{N}SH - \frac{\beta_d}{N}SD - \frac{\beta_\gamma}{N}SR - \mu S],\\
M(\Omega)\frac{\mathcal{S}(E(t)) - E(0)}{1-\Omega+\Omega\Delta} &= \mathcal{S}[\frac{\beta_i}{N}SI + \frac{\beta_h}{N}SH + \frac{\beta_d}{N}SD + \frac{\beta_\gamma}{N}SR - \sigma E - \mu E],\\
M(\Omega)\frac{\mathcal{S}(I(t)) - I(0)}{1-\Omega+\Omega\Delta} &= \mathcal{S}[\sigma E - (\gamma_1 + \epsilon + \tau + \mu)I],\\
M(\Omega)\frac{\mathcal{S}(R(t)) - R(0)}{1-\Omega+\Omega\Delta} &= \mathcal{S}[\gamma_1 I - \gamma_2 H - (\gamma_3 + \mu)R],\\
M(\Omega)\frac{\mathcal{S}(D(t)) - D(0)}{1-\Omega+\Omega\Delta} &= \mathcal{S}[\epsilon I - (\sigma_1 + \xi)D],\\
M(\Omega)\frac{\mathcal{S}(H(t)) - H(0)}{1-\Omega+\Omega\Delta} &= \mathcal{S}[\tau I - (\gamma_2 + \sigma_2 + \mu)H],\\
M(\Omega)\frac{\mathcal{S}(B(t)) - B(0)}{1-\Omega+\Omega\Delta} &= \mathcal{S}[\sigma_1 D + \sigma_2 H - \xi B],\\
M(\Omega)\frac{\mathcal{S}(C(t)) - C(0)}{1-\Omega+\Omega\Delta} &= \mathcal{S}[\gamma_3 R - \mu C].
\end{aligned}$$

By Rearranging

$$\mathcal{S}(S(t)) = S(0) + \frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}SI - \frac{\beta_h}{N}SH - \frac{\beta_d}{N}SD - \frac{\beta_\gamma}{N}SR - \mu S],$$

$$\mathcal{S}(E(t)) = E(0) + \frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\frac{\beta_i}{N}SI + \frac{\beta_h}{N}SH + \frac{\beta_d}{N}SD + \frac{\beta_\gamma}{N}SR - \sigma E - \mu E],$$

$$\mathcal{S}(I(t)) = I(0) + \frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma E - (\gamma_1+\epsilon+\tau+\mu)I],$$

$$\mathcal{S}(R(t)) = R(0) + \frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_1 I - \gamma_2 H - (\gamma_3+\mu)R], \tag{4. 3}$$

$$\mathcal{S}(D(t)) = D(0) + \frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\epsilon I - (\sigma_1+\xi)D],$$

$$\mathcal{S}(H(t)) = H(0) + \frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\tau I - (\gamma_2+\sigma_2+\mu)H],$$

$$\mathcal{S}(B(t)) = B(0) + \frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma_1 D + \sigma_2 H - \xi B],$$

$$\mathcal{S}(B(t)) = B(0) + \frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_3 R - \mu C].$$

Now applying the inverse Sumudu transform on both sides of equation (4.3), we have

$$S(t) = S(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}SI - \frac{\beta_h}{N}SH - \frac{\beta_d}{N}SD - \frac{\beta_\gamma}{N}SR - \mu S]],$$

$$E(t) = E(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\frac{\beta_i}{N}SI + \frac{\beta_h}{N}SH + \frac{\beta_d}{N}SD + \frac{\beta_\gamma}{N}SR - \sigma E - \mu E]],$$

$$I(t) = I(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma E - (\gamma_1+\epsilon+\tau+\mu)I]],$$

$$R(t) = R(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_1 I - \gamma_2 H - (\gamma_3+\mu)R]],$$

$$D(t) = D(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\epsilon I - (\sigma_1+\xi)D]],$$

$$H(t) = H(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\tau I - (\gamma_2+\sigma_2+\mu)H]],$$

$$B(t) = B(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma_1 D + \sigma_2 H - \xi B]],$$

$$C(t) = C(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_3 R - \mu C]].$$

The following recursive formula is given

$$S_{n+1}(t) = S_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}SI - \frac{\beta_h}{N}SH - \frac{\beta_d}{N}SD - \frac{\beta_\gamma}{N}SR - \mu S]],$$

$$E_{n+1}(t) = E_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\frac{\beta_i}{N}SI + \frac{\beta_h}{N}SH + \frac{\beta_d}{N}SD + \frac{\beta_\gamma}{N}SR - \sigma E - \mu E]],$$

$$I_{n+1}(t) = I_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma E - (\gamma_1+\epsilon+\tau+\mu)I]],$$
$$R_{n+1}(t) = R_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_1 I - \gamma_2 H - (\gamma_3+\mu)R]], \quad (4.\ 4)$$
$$D_{n+1}(t) = D_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\epsilon I - (\sigma_1+\xi)D]],$$
$$H_{n+1}(t) = H_n(0) + S^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\tau I - (\gamma_2+\sigma_2+\mu)H]],$$
$$B_{n+1}(t) = B_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma_1 D + \sigma_2 H - \xi B]],$$
$$C_{n+1}(t) = C_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_3 R - \mu C]].$$

and the solution of (4.4) is given by

$$S(t) = lim_{n\to\infty} S_n(t), E(t) = lim_{n\to\infty} E_n(t),$$
$$I(t) = lim_{n\to\infty} I_n(t), R(t) = lim_{n\to\infty} R_n(t),$$
$$D(t) = lim_{n\to\infty} D_n(t), H(t) = lim_{n\to\infty} H_n(t),$$
$$B(t) = lim_{n\to\infty} B_n(t), C(t) = lim_{n\to\infty} C_n(t).$$

4.1. **Fixed point theorem for stability analysis.**

Theorem 4.2. *Let $(Z_1, \|.\|)$ be a Banach space and P be a self-map of Z_1 satisfying $\|P_z - P_y\| \leq C\|z - P_z\| + c\|z - y\|$ for all z, y $\in Z_1$ where $0 \leq C$, $0 \leq c < 1$. Suppose that P is picard P-stable. Let the following recursive formula from (4) connected to (2.2).*

$$S_{n+1}(t) = S_n(0)+\mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}SI - \frac{\beta_h}{N}SH - \frac{\beta_d}{N}SD - \frac{\beta_\gamma}{N}SR - \mu S]],$$
$$E_{n+1}(t) = E_n(0)+\mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\frac{\beta_i}{N}SI + \frac{\beta_h}{N}SH + \frac{\beta_d}{N}SD + \frac{\beta_\gamma}{N}SR - \sigma E - \mu E]],$$
$$I_{n+1}(t) = I_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma E - (\gamma_1+\epsilon+\tau+\mu)I]],$$
$$R_{n+1}(t) = R_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_1 I - \gamma_2 H - (\gamma_3+\mu)R]],$$
$$D_{n+1}(t) = D_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\epsilon I - (\sigma_1+\xi)D]],$$
$$H_{n+1}(t) = H_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\tau I - (\gamma_2+\sigma_2+\mu)H]],$$
$$B_{n+1}(t) = B_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma_1 D + \sigma_2 H - \xi B]],$$
$$C_{n+1}(t) = C_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_3 R - \mu C]].$$

where $\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}$ is the fractional Langrange multiplier.

Theorem 4.3. *Let us define a self-map P as*

$$P(S_n(t)) = S_{n+1}(t) = S_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}SI - \frac{\beta_h}{N}SH - \frac{\beta_d}{N}SD - \frac{\beta_\gamma}{N}SR - \mu S]],$$

$$P(E_n(t)) = E_{n+1}(t) = E_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\frac{\beta_i}{N}SI + \frac{\beta_h}{N}SH + \frac{\beta_d}{N}SD + \frac{\beta_\gamma}{N}SR - \sigma E - \mu E]],$$

$$P(I_n(t)) = I_{n+1}(t) = I_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma E - (\gamma_1 + \epsilon + \tau + \mu)I]],$$

$$P(R_n(t)) = R_{n+1}(t) = R_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_1 I - \gamma_2 H - (\gamma_3 + \mu)R]],$$

$$P(D_n(t)) = D_{n+1}(t) = D_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\epsilon I - (\sigma_1 + \xi)D]],$$

$$P(H_n(t)) = H_{n+1}(t) = H_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\tau I - (\gamma_2 + \sigma_2 + \mu)H]],$$

$$P(B_n(t)) = B_{n+1}(t) = B_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma_1 D + \sigma_2 H - \xi B]],$$

$$P(C_n(t)) = C_{n+1}(t) = C_n(0) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\gamma_3 R - \mu C]].$$

The system is P-stable in $L^1(a, b)$ if

$$\begin{aligned}
&(1 - (M_1 + M_2)\frac{\beta_i}{N}f(\gamma) - (M_2 + M_3)\frac{\beta_h}{N}g(\gamma) - (M_2 + M_4)\frac{\beta_d}{N}h(\gamma) \\
&\qquad - (M_2 + M_5)\frac{\beta_\gamma}{N}I(\gamma) - \mu j(\gamma)) < 1, \\
&(1 + (M_1 + M_2)\frac{\beta_i}{N}f(\gamma) + (M_2 + M_3)\frac{\beta_h}{N}g(\gamma) + (M_2 + M_4)\frac{\beta_d}{N}h(\gamma) \\
&\qquad + (M_2 + M_5)\frac{\beta_\gamma}{N}I(\gamma) - \sigma j_1(\gamma) - \mu j(\gamma)) < 1, \\
&\{1 + \sigma f_1(\gamma) - \nu_1 g_1(\gamma) - \epsilon h_1(\gamma) - \tau I_1(\gamma) - \mu j(\gamma)\} < 1, \\
&\{1 + \nu_1 g_1(\gamma) + \nu_2 g_2(\gamma) - \nu_3 g_3(\gamma) - \mu j(\gamma)\} < 1, \\
&\{1 + \epsilon h_1(\gamma) - \sigma_1 f_2(\gamma) - \xi k(\gamma)\} < 1, \\
&\{1 + \tau I_1(\gamma) - \nu_2 g_2(\gamma) - \mu j_2(\gamma)\} < 1, \\
&\{1 + \sigma_1 f_2(\gamma) + \sigma_2 f_3(\gamma) - \xi k(\gamma)\} < 1, \\
&\{1 + \nu_3 g_3(\gamma)\} < 1.
\end{aligned} \tag{4. 5}$$

Proof: Firstly we have to show that P has a fixed point. For this, we evaluate the following. $\forall(m,n) \in N \times N$

$$P(S_n(t)) - P(S_m(t)) = S_n(t) - S_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}S_nI_n - \frac{\beta_h}{N}S_nH_n - \frac{\beta_d}{N}S_nD_n - \frac{\beta_\gamma}{N}S_nR_n - \mu S_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}S_mI_m - \frac{\beta_h}{N}S_mH_m - \frac{\beta_d}{N}S_mD_m - \frac{\beta_\gamma}{N}S_mR_m - \mu S_m]],$$

$$P(E_n(t)) - P(E_m(t)) = E_n(t) - E_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\frac{\beta_i}{N}S_nI_n + \frac{\beta_h}{N}S_nH_n + \frac{\beta_d}{N}S_nD_n + \frac{\beta_\gamma}{N}S_nR_n - \sigma E_n - \mu E_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\frac{\beta_i}{N}S_mI_m + \frac{\beta_h}{N}S_mH_m + \frac{\beta_d}{N}S_mD_m + \frac{\beta_\gamma}{N}S_mR_m - \sigma E_m - \mu E_m]],$$

$$P(I_n(t)) - P(I_m(t)) = I_n(t) - I_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma E_n - \nu_1 I_n - \in I_n - \tau I_n - \mu I_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma E_m - \nu_1 I_m - \in I_m - \tau I_m - \mu I_m]],$$

$$P(R_n(t)) - P(R_m(t)) = R_n(t) - R_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\nu_1 I_n + \nu_2 H_n - \nu_3 R_n - \mu R_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\nu_1 I_m + \nu_2 H_m - \nu_3 R_m - \mu R_m]],$$

$$P(D_n(t)) - P(D_m(t)) = D_n(t) - D_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\in I_n - \sigma_1 D_n - \xi D_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\in I_m - \sigma_1 D_m - \xi D_m]], \tag{4. 6}$$

$$P(H_n(t)) - P(H_m(t)) = H_n(t) - H_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\tau I_n - \nu_2 H_n - \sigma_2 H_n - \mu H_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\tau I_m - \nu_2 H_m - \sigma_2 H_m - \mu H_m]],$$

$$P(B_n(t)) - P(B_m(t)) = B_n(t) - B_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma_1 D_n + \sigma_2 H_n - \xi B_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\sigma_1 D_m + \sigma_2 H_m - \xi B_m]],$$

$$P(C_n(t)) - P(C_m(t)) = C_n(t) - C_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\nu_3 R_n - \mu C_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\nu_3 R_m - \mu C_m]].$$

By taking norm on both sides of the first equation of (4.6),we get

$$\|P(S_n(t)) - P(S_m(t))\| = \|S_n(t) - S_m(t) + \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}S_n I_n - \frac{\beta_h}{N}S_n H_n - \frac{\beta_d}{N}S_n D_n - \frac{\beta_\gamma}{N}S_n R_n - \mu S_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}S_m I_m - \frac{\beta_h}{N}S_m H_m - \frac{\beta_d}{N}S_m D_m - \frac{\beta_\gamma}{N}S_m R_m - \mu S_m]]\| \quad (4.\ 7)$$

By triangular inequality, this equation becomes

$$\|P(S_n(t)) - P(S_m(t))\| \leq \|S_n(t) - S_m(t)\| + \|\mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}S_n I_n - \frac{\beta_h}{N}S_n H_n - \frac{\beta_d}{N}S_n D_n - \frac{\beta_\gamma}{N}S_n R_n - \mu S_n]] - \mathcal{S}^{-1}[\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}\mathcal{S}[\mu N - \frac{\beta_i}{N}S_m I_m - \frac{\beta_h}{N}S_m H_m - \frac{\beta_d}{N}S_m D_m - \frac{\beta_\gamma}{N}S_m R_m - \mu S_m]]\| \quad (4.\ 8)$$

By more simplification we get

$$\|P(S_n(t)) - P(S_m(t))\| \leq \|S_n(t) - S_m(t)\| + \mathcal{S}^{-1}[\mathcal{S}\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}[\|\frac{-\beta_i}{N}I_n(S_n - S_m)\| + \|\frac{-\beta_i}{N}S_m(I_n - I_m)\| + \|\frac{-\beta_h}{N}H_n(S_n - S_m)\| + \|\frac{-\beta_h}{N}S_m(H_n - H_m)\| + \|\frac{-\beta_d}{N}D_n(S_n - S_m)\| + \|\frac{-\beta_d}{N}S_m(D_n - D_m)\| + \|\frac{-\beta_\gamma}{N}R_n(S_n - S_m)\| + \|\frac{-\beta_\gamma}{N}S_m(R_n - R_m)\| + \|\mu(S_n - S_m)\|]]. \quad (4.\ 9)$$

Because both solutions have the same role, we can assume that

$$\|S_n(t) - S_m(t)\| \cong \|I_n(t) - I_m(t)\|,$$
$$\|S_n(t) - S_m(t)\| \cong \|H_n(t) - H_m(t)\|,$$
$$\|S_n(t) - S_m(t)\| \cong \|D_n(t) - D_m(t)\|,$$

$$\|S_n(t) - S_m(t)\| \cong \|R_n(t) - R_m(t)\|.$$

By putting this in equation (4.9),we have

$$\begin{aligned} \|P(S_n(t)) - P(S_m(t))\| &\le \|S_n(t) - S_m(t) \\ +\mathcal{S}^{-1}[\mathcal{S}\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}[\|\frac{-\beta_i}{N} I_n(S_n - S_m)\| + \|\frac{-\beta_i}{N} S_m(S_n - S_m)\| \\ +\|\frac{-\beta_h}{N} H_n(S_n - S_m)\| + \|\frac{-\beta_h}{N} S_m(S_n - S_m)\| + \|\frac{-\beta_d}{N} D_n(S_n - S_m)\| \\ +\|\frac{-\beta_d}{N} S_m(S_n - S_m)\| + \|\frac{-\beta_\gamma}{N} R_n(S_n - S_m)\| \\ +\|\frac{-\beta_\gamma}{N} S_m(S_n - S_m)\| + \|\mu(S_n - S_m)\|]]. \end{aligned} \tag{4. 10}$$

Since I_n, S_m, H_n, D_n, R_n are bounded, therefore we have five positive constants M_1, M_2, M_3, M_4 and M_5 $\forall$ t such that $\|I_n\| < M_1$, $\|S_m\| < M_2$, $\|H_n\| < M_3$, $\|D_n\| < M_4$ and $\|R_n\| < M_5$, $\forall(m,n) \in N \times N$. Substitute in equation (10), we get

$$\begin{aligned} \|P(S_n(t)) - P(S_m(t))\| \le (1 - (M_1 + M_2)\frac{\beta_i}{N} f(\gamma) - (M_2 + M_3)\frac{\beta_h}{N} g(\gamma) \\ -(M_2 + M_4)\frac{\beta_d}{N} h(\gamma) - (M_2 + M_5)\frac{\beta_\gamma}{N} I(\gamma) - \mu j(\gamma))\|S_n - S_m\|, \end{aligned} \tag{4. 11}$$

Here f,g,h,I,j are functions from $\mathcal{S}^{-1}[\mathcal{S}\frac{(1-\Omega+\Omega\Delta)}{M(\Omega)}]$.

Similarly we can get

$$\begin{aligned} \|P(E_n(t)) - P(E_m(t))\| \le (1 + (M_1 + M_2)\frac{\beta_i}{N} f(\gamma) + (M_2 + M_3)\frac{\beta_h}{N} g(\gamma) \\ +(M_2 + M_4)\frac{\beta_d}{N} h(\gamma) + (M_2 + M_5)\frac{\beta_\gamma}{N} I(\gamma) - \sigma j_1(\gamma) - \mu J(\gamma))\|S_n - S_m\|, \\ \|P(I_n(t)) - P(I_m(t)\| \le \{1 + \sigma f_1(\gamma) - \nu_1 g_1(\gamma) - \epsilon h_1(\gamma) - \tau I_1(\gamma) - \mu j(\gamma)\} \|S_n - S_m\|, \\ \|P(R_n(t)) - P(R_m(t)\| \le \{1 + \nu_1 g_1(\gamma) - \nu_2 g_2(\gamma) - \nu_3 g_3(\gamma) - \mu j(\gamma)\} \|S_n - S_m\|, \\ \|P(D_n(t)) - P(D_m(t)\| \le \{1 + \epsilon h_1(\gamma) - \sigma_1 f_2(\gamma) - \xi k(\gamma)\} \|S_n - S_m\|, \\ \|P(Hn(t)) - P(H_m(t)\| \le \{1 + \tau I_1(\gamma) - \nu g_2(\gamma) - \mu j(\gamma)\} \|S_n - S_m\|, \\ \|P(B_n(t)) - P(B_m(t)\| \le \{1 + \sigma_1 f_2(\gamma) + \sigma_2 f_3(\gamma) - \xi k(\gamma)\} \|S_n - S_m\|, \\ \|P(C_n(t)) - P(C_m(t)\| \le \{1 + \nu_3 g_3(\gamma) - \mu j(\gamma)\} \|S_n - S_m\|. \end{aligned} \tag{4. 12}$$

where

$$\begin{aligned} (1 - (M_1 + M_2)\frac{\beta_i}{N} f(\gamma) - (M_2 + M_3)\frac{\beta_h}{N} g(\gamma) - (M_2 + M_4)\frac{\beta_d}{N} h(\gamma) \\ -(M_2 + M_5)\frac{\beta_\gamma}{N} I(\gamma) - \mu j(\gamma)) < 1, \\ (1 + (M_1 + M_2)\frac{\beta_i}{N} f(\gamma) + (M_2 + M_3)\frac{\beta_h}{N} g(\gamma) + (M_2 + M_4)\frac{\beta_d}{N} h(\gamma) \\ +(M_2 + M_5)\frac{\beta_\gamma}{N} I(\gamma) - \sigma j_1(\gamma) - \mu j(\gamma)) < 1, \\ \{1 + \sigma f_1(\gamma) - \nu_1 g_1(\gamma) - \epsilon h_1(\gamma) - \tau I_1(\gamma) - \mu j(\gamma)\} < 1, \\ \{1 + \nu_1 g_1(\gamma) + \nu_2 g_2(\gamma) - \nu_3 g_3(\gamma) - \mu j(\gamma)\} < 1, \end{aligned} \tag{4. 13}$$

$$\{1 + \epsilon h_1(\gamma) - \sigma_1 f_2(\gamma) - \xi k(\gamma)\} < 1,$$
$$\{1 + \tau I_1(\gamma) - \nu_2 g_2(\gamma) - \mu j_2(\gamma)\} < 1,$$
$$\{1 + \sigma_1 f_2(\gamma) + \sigma_2 f_3(\gamma) - \xi k(\gamma)\} < 1,$$
$$\{1 + \nu_3 g_3(\gamma)\} < 1.$$

Thus the non linear P-self mapping has a fixed point. Now we will show that P satisfies the condition in Theorem 4.2.
Let (4.11) and (4.12) hold and therefore using

$$C = (0, 0, 0, 0, 0, 0, 0, 0)$$

$$C = \begin{cases} (1 - (M_1 + M_2)\frac{\beta_i}{N} f(\gamma) - (M_2 + M_3)\frac{\beta_h}{N} g(\gamma) - (M_2 + M_4)\frac{\beta_d}{N} h(\gamma) \\ -(M_2 + M_5)\frac{\beta_\gamma}{N} I(\gamma) - \mu j(\gamma)), \\ (1 + (M_1 + M_2)\frac{\beta_i}{N} f(\gamma) + (M_2 + M_3)\frac{\beta_h}{N} g(\gamma) + (M_2 + M_4)\frac{\beta_d}{N} h(\gamma) \\ +(M_2 + M_5)\frac{\beta_\gamma}{N} I(\gamma) - \sigma j_1(\gamma) - \mu j(\gamma)), \\ \{1 + \sigma f_1(\gamma) - \nu_1 g_1(\gamma) - \epsilon h_1(\gamma) - \tau I_1(\gamma) - \mu j(\gamma)\}, \\ \{1 + \nu_1 g_1(\gamma) + \nu_2 g_2(\gamma) - \nu_3 g_3(\gamma) - \mu j(\gamma)\}, \\ \{1 + \epsilon h_1(\gamma) - \sigma_1 f_2(\gamma) - \xi k(\gamma)\}, \\ \{1 + \tau I_1(\gamma) - \nu_2 g_2(\gamma) - \mu j_2(\gamma)\}, \\ \{1 + \sigma_1 f_2(\gamma) + \sigma_2 f_3(\gamma) - \xi k(\gamma)\} \\ \{1 + \nu_3 g_3(\gamma)\}. \end{cases} \tag{4. 14}$$

Then the above shows that condition of Theorem 4.2 exist for the nonlinear mapping P. Hence all the conditions in Theorem 4.3 are satisfied for the defined non-linear mapping P. Hence P is Picard P-stable.

4.4. **Uniqueness of the Special Solution.** Here, by using iteration method we will show the uniqueness of special solution of equation (3.2). We suppose that equation (3.2) has an exact solution by which, the special solution converges for a large number m. Hilbert space $H = L(a, b) \times (0, T)$ is defined by $y : (a, b) \times (0, T) \longrightarrow$ R such that $\int\int uydudy < \infty$. We have the following operator

$$P(S, E, I, R, D, H, B, C) = \begin{cases} \mu N - \frac{\beta_i}{N} SI - \frac{\beta_h}{N} SH - \frac{\beta_d}{N} SD - \frac{\beta_\gamma}{N} SR - \mu S, \\ \frac{\beta_i}{N} SI + \frac{\beta_h}{N} SH + \frac{\beta_d}{N} SD + \frac{\beta_\gamma}{N} SR - \sigma E - \mu E, \\ \sigma E - (\gamma_1 + \epsilon + \tau + \mu) I, \\ \gamma_1 I = \gamma_2 H - (\gamma_3 + \mu) R, \\ \epsilon I - (\sigma_1 + \xi) D, \\ \tau I - (\gamma_2 + \sigma_2 + \mu) H, \\ \sigma_1 D + \sigma_2 H - \xi B, \\ \gamma_3 R - \mu C. \end{cases}$$

The purpose of this part is to prove that the inner product of
$P\Big((Z_{11} - Z_{12}, Z_{21} - Z_{22}, Z_{31} - Z_{32}, Z_{41} - Z_{42}, Z_{51} - Z_{52},$

$Z_{61}-Z_{62},Z_{71}-Z_{72},Z_{81}-Z_{82}),(W_1,W_2,W_3,...,W_8)\Big)$,
where $(Z_{11}-Z_{12}),(Z_{21}-Z_{22}),...,(Z_{81}-Z_{82})$ are special solution of system. However

$$P\Big((Z_{11}-Z_{12},Z_{21}-Z_{22},Z_{31}-Z_{32},...,Z_{81}-Z_{82}),(W_1,W_2,W_3,...,W_8)\Big)$$
$$=\begin{cases}(\frac{-\beta_i}{N}(Z_{11}-Z_{12})(Z_{31}-Z_{32})-\frac{\beta_h}{N}(Z_{11}-Z_{12})(Z_{61}-Z_{62})\\ -\frac{\beta_d}{N}(Z_{11}-Z_{12})(Z_{51}-Z_{52}))-\frac{\beta_\gamma}{N}(Z_{11}-Z_{12})(Z_{41}-Z_{42})-\mu(X_{11}-Z_{12}),W_1),\\ (\frac{\beta_i}{N}(Z_{11}-Z_{12})(Z_{31}-Z_{32})+\frac{\beta_h}{N}(Z_{11}-Z_{12})(Z_{61}-Z_{62})\\ +\frac{\beta_d}{N}(Z_{11}-Z_{12})(Z_{51}-Z_{52})+\frac{\beta_\gamma}{N}(Z_{11}-Z_{12})(Z_{41}-Z_{42})\\ -(\sigma+\mu)(Z_{21}-Z_{22}),W_2),\\ (\sigma(Z_{21}-Z_{22})-(\nu_1+\epsilon+\tau+\mu)(Z_{31}-Z_{32}),W_3),\\ (\nu_1(Z_{31}-Z_{32})+\nu_2(Z_{61}-Z_{62})-(\nu_3+\mu)(Z_{41}-Z_{42}),W_4),\\ (\epsilon(Z_{31}-Z_{32})-(\sigma_1+\xi)(Z_{51}-Z_{52}),W_5),\\ (\tau(Z_{31}-Z_{32})-(\nu_2+\sigma_2+\mu)(Z_{61}-Z_{62}),W_6),\\ (\sigma_1(Z_{51}-Z_{52})+\sigma_2(Z_{61}-Z_{62})-\xi(Z_{71}-Z_{72}),W_7),\\ (\nu_3(Z_{41}-Z_{42})-\mu(Z_{81}-Z_{82}),W_8).\end{cases} \quad (4.\ 15)$$

By evaluating first equation of (4.14)

$$\begin{aligned}(\frac{-\beta_i}{N}(Z_{11}-Z_{12})(Z_{31}-Z_{32})-\frac{\beta_h}{N}(Z_{11}-Z_{12})(Z_{61}-Z_{62})-\frac{\beta_d}{N}(Z_{11}-Z_{12})(Z_{51}-Z_{52})\\ -\frac{\beta_\gamma}{N}(Z_{11}-Z_{12})(Z_{41}-Z_{42})-\mu(Z_{11}-Z_{12}),W_1)\\ \cong(\frac{-\beta_i}{N}(Z_{11}-Z_{12})(Z_{31}-Z_{32}),W_1)+(-\frac{\beta_h}{N}(Z_{11}-Z_{12})(X_{61}-X_{62}),W_1)\\ +(-\frac{\beta_d}{N}(Z_{11}-Z_{12})(Z_{51}-Z_{52}),W_1)+(-\frac{\beta_\gamma}{N}(Z_{11}-Z_{12})(Z_{41}-Z_{42}),W_1)\\ +(-\mu(Z_{11}-Z_{12}),W_1),W_1)\end{aligned} \quad (4.\ 16)$$

Since both the solution play same role, we can suppose that
$(Z_{11}-Z_{12})\cong(Z_{21}-Z_{22})\cong(Z_{31}-Z_{32})\cong,...,\cong(Z_{81}-Z_{82})$.
Then the equation (4.15) becomes

$(\frac{-\beta_i}{N}(Z_{11}-Z_{12})^2-\frac{\beta_h}{N}(Z_{11}-Z_{12})^2-\frac{\beta_d}{N}(Z_{11}-Z_{12})^2-\frac{\beta_\gamma}{N}(Z_{11}-Z_{12})^2$
$-\mu(Z_{11}-Z_{12}),W_1)$.

By relationship between norm and inner product , we will get

$$
\begin{aligned}
&(\frac{-\beta_i}{N}(Z_{11}-Z_{12})^2-\frac{\beta_h}{N}(Z_{11}-Z_{12})^2-\frac{\beta_d}{N}(Z_{11}-Z_{12})^2-\frac{\beta_\gamma}{N}(Z_{11}-Z_{12})^2\\
&\qquad -\mu(Z_{11}-Z_{12}),W_1)\\
&\cong(\frac{-\beta_i}{N}(Z_{11}-Z_{12})^2,W_1)+(-\frac{\beta_h}{N}(Z_{11}-Z_{12})^2,W_1)+(-\frac{\beta_d}{N}(Z_{11}-Z_{12})^2,W_1)\\
&\qquad +(-\frac{\beta_\gamma}{N}(Z_{11}-Z_{12})^2,W_1)+(-\mu(Z_{11}-Z_{12}),W_1)\\
&\leq\frac{\beta_i}{N}\|(Z_{11}-Z_{12})^2\|\|W_1\|+\frac{\beta_h}{N}\|(Z_{11}-Z_{12})^2)\|\|W_1\|+\frac{\beta_d}{N}\|(Z_{11}-Z_{12})^2)\|\|W_1\|\\
&\qquad +\frac{\beta_\gamma}{N}\|(Z_{11}-Z_{12})^2)\|\|W_1\|+\mu\|(Z_{11}-Z_{12}\|\|W_1\|\\
&=(\frac{\beta_i}{N}\varpi_1+\frac{\beta_h}{N}\varpi_1+\frac{\beta_d}{N}+\frac{\beta_\gamma}{N}\varpi_1+\mu)\|(Z_{11}-Z_{12})\|\|W_1\|.
\end{aligned}
\tag{4. 17}
$$

Repeating the same pattern from second to eighth equation of (4.14), we find

$$
\begin{aligned}
&(\frac{\beta_i}{N}(Z_{11}-Z_{12})^2+\frac{\beta_h}{N}(Z_{11}-Z_{12})^2+\frac{\beta_d}{N}(Z_{11}-Z_{12})^2+\frac{\beta_\gamma}{N}(Z_{11}-Z_{12})^2\\
&\qquad -(\sigma+\mu)(Z_{21}-Z_{22}),W_2)\\
&\leq(\frac{\beta_i}{N}\varpi_2+\frac{\beta_h}{N}\varpi_2+\frac{\beta_d}{N}\varpi_2+\frac{\beta_\gamma}{N}\varpi_2+(\mu+\sigma))\|(Z_{21}-Z_{22})\|\|W_2\|,\\
&(\sigma(Z_{21}-Z_{22})-(\nu_1+\epsilon+\tau+\mu)(Z_{31}-Z_{32}),W_3)\\
&\qquad \leq(\sigma+\nu_1+\epsilon+\tau+\mu)\|(Z_{31}-Z_{32})\|\|W_3\|,\\
&(\nu_1(Z_{31}-Z_{32})+\nu_2(Z_{61}-Z_{62})-(\nu_3+\mu)(Z_{41}-Z_{42}),W_4)\\
&\qquad \leq(\nu_1+\nu_2+\nu_3+\mu)\|(Z_{41})-Z_{42}\|\|W_4\|,\\
&(\epsilon(Z_{31}-Z_{32})-(\sigma_1+\xi)(Z_{51}-Z_{52}),W_5)\\
&\qquad \leq(\epsilon+\sigma_1+\xi)\|(Z_{51}-Z_{52})\|\|W_5\|,\\
&(\tau(Z_{31}-Z_{32})-(\nu_2+\sigma_2+\mu)(Z_{61}-Z_{62}),W_6)\\
&\qquad \leq(\tau+\nu_2+\sigma_2+\mu)\|(Z_{61}-Z_{62})\|\|W_6\|,\\
&(\sigma_1(Z_{51}-Z_{52})+\sigma_2(Z_{61}-Z_{62})-\xi(Z_{71}-Z_{72}),W_7)\\
&\qquad \leq(\sigma_1+\sigma_2+\xi)(\|(Z_{71}-Z_{72})\|\|W_7\|),\\
&(\nu_3(Z_{41}-Z_{42})-\mu(Z_{81}-Z_{82}),W_8)\\
&\qquad \leq(\nu_3+\mu)\|(Z_{81}-Z_{82})\|\|W_8\|.
\end{aligned}
\tag{4. 18}
$$

Putting equation (4.16) and (4.17) in (4.14), we get

$$P\Big((Z_{11}-Z_{12},Z_{21}-Z_{22},Z_{31}-Z_{32},...,Z_{81}-Z_{82}),(W_1,W_2,W_3,...,W_8)\Big)$$

$$\leq \begin{cases} (\frac{\beta_i}{N}\varpi_1+\frac{\beta_h}{N}\varpi_1+\frac{\beta_d}{N}+\frac{\beta_\gamma}{N}\varpi_1+\mu)\|(Z_{11}-Z_{12})\|\|W_1\|, \\ (\frac{\beta_i}{N}\varpi_2+\frac{\beta_h}{N}\varpi_2+\frac{\beta_d}{N}\varpi_2+\frac{\beta_\gamma}{N}\varpi_2+(\mu+\sigma))\|(Z_{21}-Z_{22})\|\|W_2\|, \\ (\sigma+\nu_1+\epsilon+\tau+\mu)\|(Z_{31}-Z_{32})\|\|W_3\|, \\ (\nu_1+\nu_2+\nu_3+\mu)\|(Z_{41})-Z_{42}\|\|W_4\|, \\ (\epsilon+\sigma_1+\xi)\|(Z_{51}-Z_{52})\|\|W_5\|, \\ (\tau+\nu_2+\sigma_2+\mu)\|(Z_{61}-Z_{62})\|\|W_6\|, \\ (\sigma_1+\sigma_2+\xi)(\|(Z_{71}-Z_{72})\|\|W_7\|), \\ (\nu_3+\mu)\|(Z_{81}-Z_{82})\|\|W_8\|. \end{cases}$$

But for sufficiently large values of m_i, with $i=1,2,3,...,8$ both the solution converge to exact solution, using the topological concept, there exist eight very small positive parameters $l_{m_1},l_{m_2},...l_{m_8}$ such that

$$\|S-Z_1\|,\|S-Z_2\| < \frac{l_{m_1}}{3(\frac{\beta_i}{N}\varpi_1+\frac{\beta_h}{N}\varpi_1+\frac{\beta_d}{N}+\frac{\beta_\gamma}{N}\varpi_1+\mu)\|(Z_{11}-Z_{12})\|\|W_1\|},$$

$$\|E-Z_1\|,\|E-Z_2\| < \frac{l_{m_2}}{3(\frac{\beta_i}{N}\varpi_2+\frac{\beta_h}{N}\varpi_2+\frac{\beta_d}{N}\varpi_2+\frac{\beta_\gamma}{N}\varpi_2+(\mu+\sigma))\|(Z_{21}-Z_{22})\|\|W_2\|},$$

$$\|I-Z_1\|,\|I-Z_2\| < \frac{l_{m_3}}{3(\sigma+\nu_1+\epsilon+\tau+\mu)\|(Z_{31}-Z_{32})\|\|W_3\|},$$

$$\|R-Z_1\|,\|R-Z_2\| < \frac{l_{m_4}}{3(\nu_1+\nu_2+\nu_3+\mu)\|(Z_{41})-Z_{42}\|\|W_4\|},$$

$$\|D-Z_1\|,\|D-Z_2\| < \frac{l_{m_5}}{3(\epsilon+\sigma_1+\xi)\|(Z_{51}-Z_{52})\|\|W_5\|},$$

$$\|H-Z_1\|,\|H-Z_2\| < \frac{l_{m_6}}{3(\tau+\nu_2+\sigma_2+\mu)\|(Z_{61}-Z_{62})\|\|W_6\|},$$

$$\|B-Z_1\|,\|B-Z_2\| < \frac{l_{m_7}}{3(\sigma_1+\sigma_2+\xi)(\|(Z_{71}-Z_{72})\|\|W_7\|},$$

$$\|C-Z_1\|,\|C-Z_2\| < \frac{l_{m_7}}{3(\nu_3+\mu)\|(Z_{81}-Z_{82})\|\|W_8\|}.$$

By putting the exact solution in the right side of above equation and implementing the triangular inequality by taking

$$M=max(m_1,m_2,m_3,...,m_8),\ l=max(l_{m_1},l_{m_2},...l_{m_8})$$

we get

$$\begin{bmatrix} (\frac{\beta_i}{N}\varpi_1+\frac{\beta_h}{N}\varpi_1+\frac{\beta_d}{N}+\frac{\beta_\gamma}{N}\varpi_1+\mu)\|(Z_{11}-Z_{12})\|\|W_1\| \\ (\frac{\beta_i}{N}\varpi_2+\frac{\beta_h}{N}\varpi_2+\frac{\beta_d}{N}\varpi_2+\frac{\beta_\gamma}{N}\varpi_2+(\mu+\sigma))\|(Z_{21}-Z_{22})\|\|W_2\| \\ (\sigma+\nu_1+\epsilon+\tau+\mu)\|(Z_{31}-Z_{32})\|\|W_3\| \\ (\nu_1+\nu_2+\nu_3+\mu)\|(Z_{41})-Z_{42}\|\|W_4\| \\ (\epsilon+\sigma_1+\xi)\|(Z_{51}-Z_{52})\|\|W_5\|, \\ (\tau+\nu_2+\sigma_2+\mu)\|(Z_{61}-Z_{62})\|\|W_6\| \\ (\sigma_1+\sigma_2+\xi)(\|(Z_{71}-Z_{72})\|\|W_7\| \\ (\nu_3+\mu)\|(Z_{81}-Z_{82})\|\|W_8\| \end{bmatrix} < \begin{bmatrix} l \\ l \\ l \\ l \\ l \\ l \\ l \\ l \end{bmatrix}$$

As l is a very small positive parameter, with the help of topological idea, we get

$$\begin{bmatrix} (\frac{\beta_i}{N}\varpi_1 + \frac{\beta_h}{N}\varpi_1 + \frac{\beta_d}{N} + \frac{\beta_\gamma}{N}\varpi_1 + \mu)\|(Z_{11} - Z_{12})\|\|W_1\| \\ (\frac{\beta_i}{N}\varpi_2 + \frac{\beta_h}{N}\varpi_2 + \frac{\beta_d}{N}\varpi_2 + \frac{\beta_\gamma}{N}\varpi_2 + (\mu+\sigma))\|(Z_{21} - Z_{22})\|\|W_2\| \\ (\sigma + \nu_1 + \epsilon + \tau + \mu)\|(Z_{31} - Z_{32})\|\|W_3\| \\ (\nu_1 + \nu_2 + \nu_3 + \mu)\|(Z_{41}) - Z_{42}\|\|W_4\| \\ (\epsilon + \sigma_1 + \xi)\|(Z_{51} - Z_{52})\|\|W_5\|, \\ (\tau + \nu_2 + \sigma_2 + \mu)\|(Z_{61} - Z_{62})\|\|W_6\| \\ (\sigma_1 + \sigma_2 + \xi)(\|(Z_{71} - Z_{72})\|\|W_7\| \\ (\nu_3 + \mu)\|(Z_{81} - Z_{82})\|\|W_8\| \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

But it is clear that
$(\frac{\beta_i}{N}\varpi_1 + \frac{\beta_h}{N}\varpi_1 + \frac{\beta_d}{N} + \frac{\beta_\gamma}{N}\varpi_1 + \mu) \neq 0$, $(\frac{\beta_i}{N}\varpi_2 + \frac{\beta_h}{N}\varpi_2 + \frac{\beta_d}{N}\varpi_2 + \frac{\beta_\gamma}{N}\varpi_2 + (\mu+\sigma)) \neq 0$,

$(\sigma + \nu_1 + \epsilon + \tau + \mu) \neq 0$,$(\nu_1 + \nu_2 + \nu_3 + \mu) \neq 0$,$(\epsilon + \sigma_1 + \xi) \neq 0(\tau + \nu_2 + \sigma_2 + \mu) \neq 0$,

$(\sigma_1 + \sigma_2 + \xi) \neq 0$,$(\nu_3 + \mu) \neq 0$
therefore we get
$\|Z_{11} - Z_{12}\| = 0$, $\|Z_{21} - Z_{22}\| = 0$, $\|Z_{31} - Z_{32}\| = 0$, $\|Z_{41} - Z_{42}\| = 0$,

$\|Z_{51} - Z_{52}\| = 0$,$\|Z_{61} - Z_{62}\| = 0$, $\|Z_{71} - Z_{72}\| = 0$, $\|Z_{81} - Z_{82}\| = 0$
which shows that
$Z_{11} = Z_{12}$, $Z_{21} = Z_{22}$, $Z_{31} = Z_{32}$, $Z_{41} = Z_{42}$,

$Z_{51} = Z_{52}$, $Z_{61} = Z_{62}$, $Z_{71} = Z_{72}$, $Z_{81} = Z_{82}$
In this way, we have completed the proof.

5. Results

Numerical simulation was performed to analyze the control with Caputo Fabrizio fractional technique of fractional Ebola virus model. By using this technique, numerical outcomes of susceptible population (S), exposed (E), infected (I), hospitalized (H), asymptomatic but still infectious (R), population which are dead but not buried (D), buried (B), and fully recovered population (C) for different fractional value of Ω with initial conditions $S(0) = 18000$, $E(0) = 0$, $I(0) = 15$, $R(0) = 0$, $D(0) = 0$, $H(0) = 0$, $B(0) = 0$, $C(0) = 0$ and values of parameters $\sigma = \frac{1}{11.4}$, $\mu = \frac{14}{1000}$, $\beta_i = 0.14$, $\beta_d = 0.40$, $\beta_h = 0.29$, $\beta_r = 0.185$, $\gamma_1 = \frac{1}{10}$, $\epsilon = \frac{1}{9.6}$, $\sigma_1 = \frac{1}{2}$, $\sigma_2 = \frac{1}{4.6}$, $\gamma_2 = \frac{1}{5}$, $\tau = \frac{1}{5}$, $\gamma_3 = \frac{1}{30}$, $\xi = \frac{14}{1000}$ are obtained. The solution of given model consists of non-linear system of fractional differential equation has been presented by using Caputo fabrizio derivative. To observe the effects of factors on the mechanics of the fractional order model, it can be observed at different numerical ways having the value of given parameter with finite time. These simulations reveal that a value change affects the model's dynamics. First, we examine the disease-free equilibrium point of the fractional order model and also verify the non-negative solution of S, E, I, R, D, H, B, C and its stability is shown in figures 1 to 7. In these figures we can see that susceptible population (S), exposed (E), infected (I), hospitalized (H), asymptomatic but still infectious population (R), population which are dead but not buried (D), buried (B), and fully recovered population (C) has more degree of freedom comparatively ordinary derivative. Here we can also observe that the fractional value results are more reliable as compared to classical derivative. It gives better approach to desired value to control the disease. The graphs of the approximate solutions against different fractional order Ω are provided as in Figures $1-7$. In figure 1, susceptible population starts increasing by decreasing the fractional while in figure 2, 3, 4, 5 and 6 i.e, infected, asymptomatic but still infectious population, population which are dead but not buried, hospitalized and buried population starts decreasing by decreasing the fractional value respectively. In figure 7, completely recovered population start increasing by increasing fractional values which show that the vaccination has minor effects on our population.

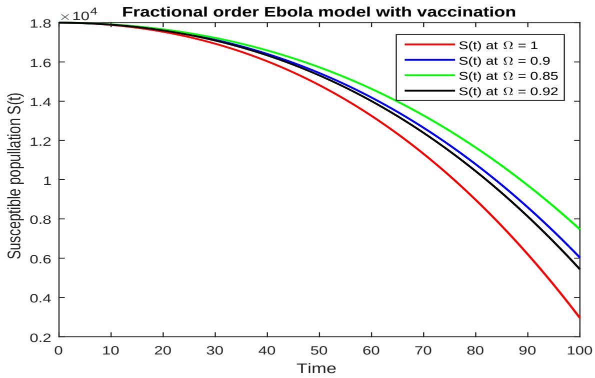

FIGURE 1. Graph for $S(t)$ in a time t (days) at four values of Ω.

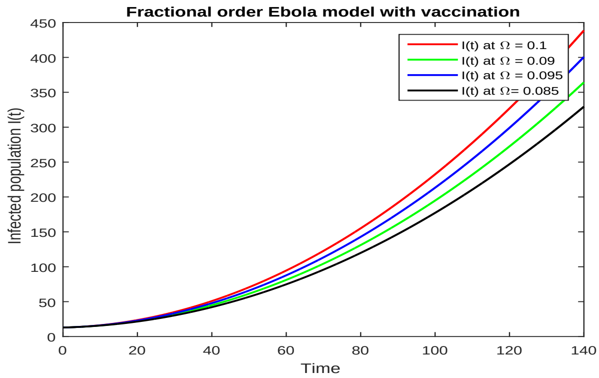

FIGURE 2. Graph for $I(t)$ in a time t (days) at four values of Ω.

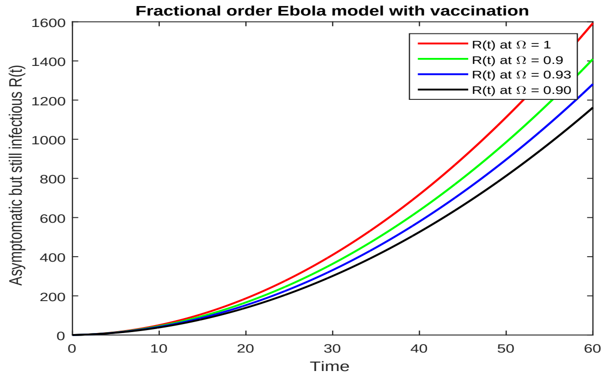

FIGURE 3. Graph for $R(t)$ in a time t (days) at four values of Ω.

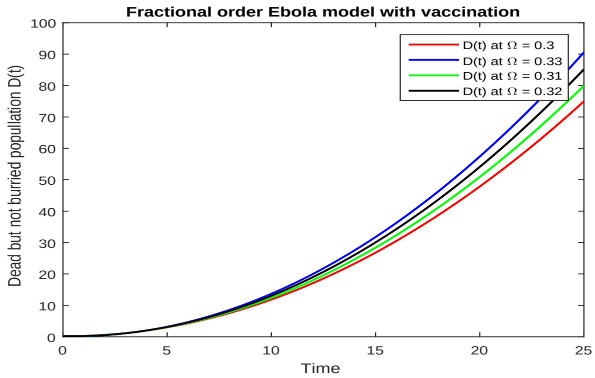

FIGURE 4. Graph for $D(t)$ in a time t (days) at four values of Ω.

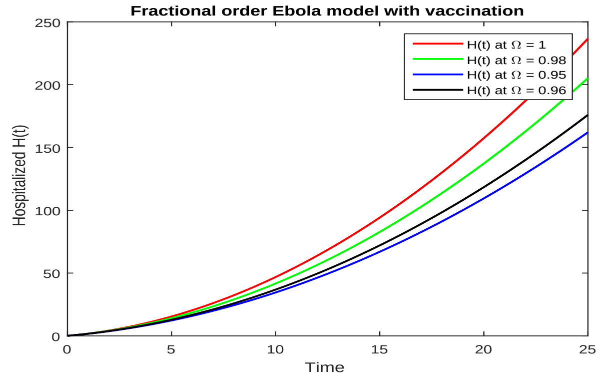

FIGURE 5. Graph for $H(t)$ in a time t (days) at four values of Ω.

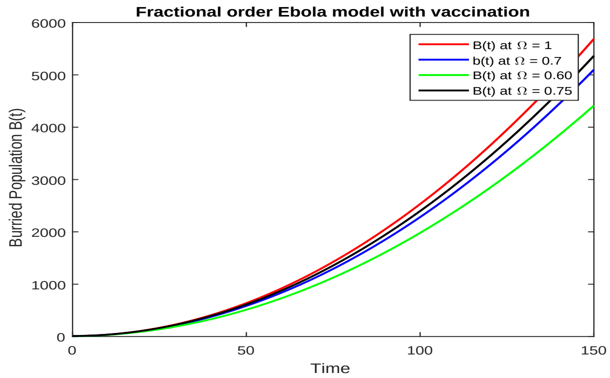

FIGURE 6. Graph for $B(t)$ in a time t (days) at four values of Ω.

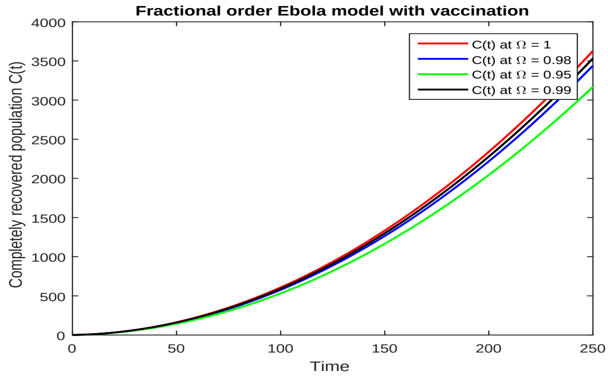

FIGURE 7. Graph for $C(t)$ in a time t (days) at four values of Ω.

6. CONCLUSION

A nonlinear fractional order model with Caputo-Fabrizio for the treatment of compartment of Ebola virus is discussed in this article. The basis of this fractional model, we made up of non-singular exponentially decreasing kernels that appears in the Caputo-Fabrizio derivation. Theoretical and numerical investigation of the bio-medical Ebola virus model are presented. It shows the stability, uniqueness and applicability of the model for the control of Ebola disease in the society. Fixed point theory and Picard Lindelof approach is used to derive solutions. The effect of the different values of fractional order is analyzed through analysis of numerical results. We have found that fractional order systems show richer dynamics than the classical version of the integer order. In the future, the above research can be applied to more complex physics models. Convergence analysis shows the efficiency and precision of the proposed method. Our results shows that fractional-order models can provide better data turnarounds than integer-order models in some situations. It is obvious that for adequate turnarounds, the models need further improvement and the implementation of these improvements will greatly improve future models.

7. ACKNOWLEDGMENT

The authors are very thankful to the anonymous referees for their valuable suggestions to improve the paper in its present form.

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