Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(1)(2020) pp. 17-31

Analytical and Numerical Investigation of the Hammerstein Fractional Equations

Somayyeh Dadsetadi, Kazem Nouri
Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran.
Corresponding Author: Kazem Nouri
Email: s.dadsetadi@semnan.ac.ir; knouri@semnan.ac.ir

Received: 12 March, 2019 / Accepted: 25 October, 2019 / Published online: 01 January, 2020

beginquote **Abstract.** this paper devoted to study the existence of a unique solution of the fractional Hammerstein integro-differential equations in the Banach space via the fixed point theorems. The main purpose is based on transforming the fractional equations into the integral equations of the Volterra type by using the differential transformation method and the corresponding fractional calculus characteristics. Also, we obtain the ε-modified operational matrix for the fractional integral and use the properties of modified block pulse functions to get approximate solutions. In the our presented method, the fractional Hammerstein equations are transformed into a system of algebraic equations, where the volume of computations are reduced by using the special nodes. Finally, we give some examples to demonstrate the obtained results.

AMS (MOS) Subject Classification Codes: 34A08; 26A33; 58J20; 65D15
Key Words: Fractional Hammerstein integro-differential equations, Fixed point theorem, Modified block pulse functions, Operational matrix.

1. Introduction

Fractional calculus is studied based on the generalization of integral and differential equations to any real or even complex order. It is an extension of classical calculus and therefore preserves many of basic properties. There are many usages for fractional calculus such as control theory, dynamics, viscoelasticity and electromagnetic theory (see [3, 5, 8]). The behavior of the solution of fractional integral and differential equations has been extensively studied by many authors. By utilizing the theory of fixed point and the iterative method, Zhang et al. [17] investigated several existence and uniqueness results for a new type of nonlocal multipoint boundary value problem of the Caputo fractional integro-differential equations requiring Riemann-Liouville integral boundary conditions. Also, several results on the existence of solutions have been investigated in the numerous

research papers for different kinds of integral and differential equations of fractional orders (see [1, 2, 16] and the references therein). Moreover, several numerical methods have been presented to approximate solutions of fractional equations (see [7, 9, 10, 15] and the references therein).
We study the analytical and numerical solutions of the following equations of the Hammerstein type

$$\left({}^{C}D^{\beta}\upsilon\right)(\tau) = g(\tau) + \int_0^{\tau} k(\tau,\eta)\ E\left(\upsilon(\eta)\right) d\eta, \ \ \tau \in [0,a], \ m-1 < \beta \le m, \quad (1.\ 1)$$

with the initial condition

$$\upsilon^{(j)}(0) = \upsilon_j, \quad j = 0, 1, ..., m-1, \quad (1.\ 2)$$

where ${}^{C}D^{\beta}$ denotes the Caputo fractional derivative and E is an increasing linear transformation on the Banach space χ.
In [7], Li and Sun, study the fractional differential equations using the generalized block pulse operational matrix. The modification of block pulse functions applied to numerically solve of the first kind Volterra integral equation in [9]. Motivated by the mentioned manuscripts, in this paper we established the fractional integral operational matrix based on the ε-modified block pulse functions, afterward we used the idea of converting the fractional integro-differential equations to the fractional integral equations to analyze of the problem. Also, by using the modified block pulse functions and the obtained operational matrix, numerical solutions of the equations are studied.
The structure of this paper is as follows: In Section 2, we introduce preliminaries which are used throughout the paper. In Section 3, we investigate the existence and uniqueness of solution, by converting the fractional equations to the integral equations of the Volterra type and applying the fixed point theorems. In Section 4, we first derive the operational matrix of the modified block pulse functions for the fractional integral, then we convert the fractional integro-differential equations into fractional integral equations. Furthermore, by using the ε-modified block pulse functions and the fractional integration operational matrix, we obtain an approximate solution with high accuracy. Section 5 discusses the error analysis of the our presented method. In Section 6, some numerical results are provided to clarify the method. The conclusion is given in Section 7.

2. Preliminaries

We give some basic concepts which are used further in this paper (for more details see [3] and [12]).
Throughout this paper, we consider the complete metric space (χ, d) which

$$d(h,g) = \sup_{\tau\in[0,a]} |h(\tau) - g(\tau)|$$

for all $h, g \in \chi$.

Definition 2.1. *The Riemann-Liouville fractional integral of order* $\beta > 0$ *of a function* $\upsilon(\tau)$*, is defined as*

$$(I^{\beta}\upsilon)(\tau) = \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau-\mu)^{\beta-1}\, \upsilon(\mu)\, d\mu, \quad \tau > 0, \tag{2. 3}$$

where Γ *is the Gamma function.*

We consider the definition of the Caputo derivative which is more useful in real-life usages since it can be better able to model phenomena and be consistent with the initial conditions of the problems.

Definition 2.2. *The Caputo derivative of order* $\beta \geq 0$ *for a function* $\upsilon(\tau)$ *is defined by*

$$\left({}^{C}D^{\beta}\upsilon\right)(\tau) = \frac{1}{\Gamma(m-\beta)} \int_0^{\tau} (\tau-\mu)^{m-\beta-1}\, \upsilon^{(m)}(\mu)\, d\mu,$$

where $m = [\beta]+1$ *and* $[\beta]$ *denotes integer part of the real number* β*.*

If $\beta = m \in \mathbb{N}_0$ and the derivative $\upsilon^{(m)}(\tau)$ of order m exists, then $({}^{C}D^{m}\upsilon)(\tau)$ coincides with $\upsilon^{(m)}(\tau)$. Also, this definition implies that ${}^{C}D^{\beta}\upsilon^{(n)}(\tau) = {}^{C}D^{\beta+n}\, \upsilon(\tau)$ and ${}^{C}D^{\beta}z = 0$ (z is a constant).

Proposition 2.3. *Let* $\beta > 0$ *and* $m = [\beta]+1$*. If* $\upsilon(\tau) \in C^{m}[0,a]$*, then*

(i) $\left(I^{\beta}\, {}^{C}D^{\beta}\upsilon\right)(\tau) = \upsilon(\tau) - \sum_{j=0}^{m-1} \dfrac{\upsilon^{(j)}(0)}{j!}\tau^{j}$,

(ii) $\left({}^{C}D^{\beta}\, I^{\beta}\upsilon\right)(\tau) = \upsilon(\tau)$.

Lemma 2.4. *Problem* (1. 1) - (1. 2) *is equivalent to the Volterra integral equation*

$$\upsilon(\tau) = h(\tau) + \int_0^{\tau} P(\tau,\eta)\, E(\upsilon(\eta))\, d\eta, \tag{2. 4}$$

where

$$P(\tau,\eta) = \frac{1}{\Gamma(\beta)} \int_{\eta}^{\tau} (\tau-\mu)^{\beta-1} k(\mu,\eta)\, d\mu, \qquad h(\tau) = \sum_{j=0}^{m-1} \frac{\upsilon_j}{j!}\tau^{j} + \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau-\eta)^{\beta-1} g(\eta)\, d\eta. \tag{2. 5}$$

Proof. First, we take the integral of fractional order β from both sides of Eq. (1. 1), then by using Proposition 2.3 and with the change of integral order, we get

$$\begin{aligned}\upsilon(\tau) &= \sum_{j=0}^{m-1} \frac{\upsilon_j}{j!}\tau^{j} + \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau-\eta)^{\beta-1} g(\eta)\, d\eta + \int_0^{\tau} \left[\frac{1}{\Gamma(\beta)} \int_{\eta}^{\tau} (\tau-\mu)^{\beta-1} k(\mu,\eta)\, d\mu\right] E(\upsilon(\eta))\, d\eta \\ &= h(\tau) + \int_0^{\tau} P(\tau,\eta)\, E(\upsilon(\eta))\, d\eta,\end{aligned}$$

and we get the result. □

Definition 2.5. *Let* ω *denotes the class of those functions* $\gamma : [0,\infty) \to [0,1)$ *which satisfies the following condition*

$$\gamma(\eta_n) \to 1 \quad \textit{implies} \quad \eta_n \to 0. \tag{2. 6}$$

In the following, we introduce the epsilon modified block pulse functions and some of their properties [9]. We first have a brief reminder of the block pulse functions [4].

Definition 2.6. *A set of block pulse functions $\Theta(\tau)$ in the interval $[0,\ R)$ is given by*

$$\Theta(\tau) = [\theta_1(\tau)\ \ \theta_2(\tau)\ \ ...\ \ \theta_n(\tau)]^T,$$

where the jth component $\theta_j(\tau),\ (j = 1,\ 2,\ ...,\ n)$ of the BPFs vector $\Theta(\tau)$ is defined as

$$\theta_j(\tau) = \begin{cases} 1, & \tau \in [\frac{(j-1)R}{n},\ \frac{jR}{n}), \\ 0, & \text{o.w.} \end{cases}$$

Definition 2.7. *A set of ε-modified block pulse functions $\vartheta_j(\tau),\ (j = 0, 1, ..., n)$ is usually defined in the interval $[0,\ R)$ as follows*

$$\vartheta_0(\tau) = \begin{cases} 1, & \tau \in [0,\ \frac{R}{n} - \varepsilon) = J_0, \\ 0, & \text{o.w.} \end{cases}$$

$$\vartheta_n(\tau) = \begin{cases} 1, & \tau \in [R - \varepsilon,\ R) = J_n, \\ 0, & \text{o.w.} \end{cases}$$

$$\vartheta_j(\tau) = \begin{cases} 1, & \tau \in [\frac{jR}{n} - \varepsilon,\ \frac{(j+1)R}{n} - \varepsilon) = J_j, \quad (0 < j < n), \\ 0, & \text{o.w.} \end{cases}$$

Notice that if $\varepsilon = 0$ then the dimension of matrix decreases to n and we only have n block pulse functions. The advantages of ε-modified block pulse functions can be cited by easy operation and their satisfactory approximations that these advantages are due to the distinct properties of the block pulse functions. Without loss of generality and assuming $R = 1$, some of their preliminary properties are:

1. disjointness:
$$\vartheta_j(\tau)\ \vartheta_\Upsilon(\tau) = \begin{cases} \vartheta_j(\tau), & j = \Upsilon, \\ 0, & j \neq \Upsilon, \end{cases}$$
2. orthogonality:
$$\int_0^1 \vartheta_j(\tau)\ \vartheta_\Upsilon(\tau)\ d\tau = \ell\ \delta_{j\Upsilon},$$
3. completeness:
$$\int_0^1 f^2(\tau)\ d\tau = \sum_{j=0}^{\infty} f_j^2\ \ \|\ \vartheta_j\ \|^2,$$
where
$$f_j = \frac{1}{\Delta(J_j)} \int_0^1 f(\tau)\ \vartheta_j(\tau)\ d\tau = \frac{1}{\Delta(J_j)} \int_{J_j} f(\tau)\ d\tau, \tag{2. 7}$$

and $\Delta(J_j)$ is the length of the interval J_j.
If we put $\ell = \frac{R}{n}$ then the operational matrix of the ε-modified block pulse functions is

defined as follows:

$$P_{(n+1)\times(n+1)} = \begin{bmatrix} \frac{\ell-\varepsilon}{2} & \ell-\varepsilon & \ell-\varepsilon & \dots & \ell-\varepsilon & \ell-\varepsilon \\ 0 & \frac{\ell}{2} & \ell & \dots & \ell & \ell \\ 0 & 0 & \frac{\ell}{2} & \dots & \ell & \ell \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\ell}{2} & \ell \\ 0 & 0 & 0 & \dots & 0 & \frac{\varepsilon}{2} \end{bmatrix} \tag{2. 8}$$

that the above matrix has the same characteristics and applications of the matrix defined for the operational matrix of block pulse functions.

Definition 2.8. *The expansion of the continuous function $h(\tau)$ is written in terms of the ε-modified block pulse functions as follows:*

$$h(\tau) \approx \hat{h}_{n+1} = \sum_{j=0}^{n} h_j \vartheta_j(\tau) = H^T \Phi(\tau) = \Phi^T(\tau)\ H, \tag{2. 9}$$

where h_j is defined in (2. 7) *and $H = [h_0 \ \ h_1 \ \ ... \ \ h_n]^T$.*

Proposition 2.9. *With defining $\Phi(\tau) = [\vartheta_0(\tau) \ \ \vartheta_1(\tau) \ \ ... \ \ \vartheta_n(\tau)]^T$, we have*

i) $\Phi(\tau)\ \Phi^T(\tau) = \begin{pmatrix} \vartheta_0(\tau) & & 0 \\ & \ddots & \\ 0 & & \vartheta_n(\tau) \end{pmatrix}_{(n+1)\times(n+1)}$,

ii) $\Phi^T(\tau)\ \Phi(\tau) = 1$,

iii) $\Phi(\tau)\ \Phi^T(\tau)\ H = D_H\ \Phi(\tau)$,

iv) $\Phi^T(\tau)\ H\Phi(\tau) = \hat{H}^T\ \Phi(\tau)$,

v) $\int_0^\tau \Phi(s)\ ds \approx P\ \Phi(\tau)$,

where $D_H = diag(H)$, $\hat{H}$ is an $(n+1)$ column vector with $diag(H)$ elements, and P is the operational matrix of the ε-modified block pulse functions which is defined in (2. 8) *.*
Aaccording to (v), we can write

$$\int_0^\tau h(s)\ ds \approx \int_0^\tau H^T \Phi(s)\ ds \approx H^T P\ \Phi(\tau).$$

3. EXISTENCE AND UNIQUENESS

In this section, using the iterative method under some suitable conditions, we investigate the existence and uniqueness theorem of nonlinear equation (1. 1), which is equivalent to the nonlinear Volterra integral equation (2. 4). We define

$$P(V)(\tau) = \int_0^\tau P(\tau,\eta)\ E(\upsilon(\eta))\ d\eta, \tag{3. 10}$$

from Eqs. (2. 4) and (3. 10), we have

$$V = H + P(V), \quad H \in \chi. \tag{3. 11}$$

Now we define the operator $\mathtt{A} : \chi \to \chi$ as follows

$$\mathtt{A}V = P(V) + H, \quad V, H \in \chi, \tag{3. 12}$$

from Eqs. (3. 11) and (3. 12), we obtain

$$\mathtt{A}V = V.$$

So, we can rewrite the equation (2. 4) as follows

$$\upsilon(\tau) = h(\tau) + \int_0^\tau P(\tau,\eta)\ E(\upsilon(\eta))\ d\eta \equiv \mathtt{A}\upsilon(\tau), \tag{3. 13}$$

which means that every solution of (3. 13) is a solution of (1. 1) and vice versa.

Theorem 3.1. *Consider the nonlinear Volterra integral equation* (3. 13) *such that*

(i) $g : [0,a] \to \Re$ *and* $k : [0,a] \times [0,a] \to \Re$ *are continuous,*

(ii) $E : \chi \to \chi$ *is an increasing linear transformation and* $\gamma(\tau) = \frac{E(\tau)}{\tau} \in \omega$, $\tau \neq 0$,

(iii) $\sup_{\tau \in [0,a]} \int_0^a P^2(\tau,\eta)\ d\eta \leq \frac{1}{a}$.

Then, the integral equation (3. 13) *has a unique fixed point* $\upsilon \in \chi$.

Proof. Consider the iterative scheme

$$\upsilon_{x+1}(\tau) = h(\tau) + \int_0^\tau P(\tau,\eta)\ E(\upsilon_x(\eta))\ d\eta \equiv \mathtt{A}\upsilon_x(\tau), \quad x = 0,1,..., \tag{3. 14}$$

where $\upsilon_0 \in \chi$ is an appropriate initial guess. So,

$$\begin{aligned}
|\mathtt{A}\upsilon_x(\tau) - \mathtt{A}\upsilon_{x-1}(\tau)| &= |\int_0^\tau P(\tau,\eta)\ E(\upsilon_x(\eta))\ d\eta - \int_0^\tau P(\tau,\eta)\ E(\upsilon_{x-1}(\eta))\ d\eta| \\
&\leq \int_0^\tau |P(\tau,\eta)\ E(\upsilon_x(\eta) - \upsilon_{x-1}(\eta))| d\eta \\
&\leq \left(\int_0^a P^2(\tau,\eta)\ d\eta \right)^{\frac{1}{2}} \left(\int_0^a E^2 |\upsilon_x(\eta) - \upsilon_{x-1}(\eta)| d\eta \right)^{\frac{1}{2}}.
\end{aligned}$$

As the function E is increasing then

$$E\left(|\upsilon_x(\tau) - \upsilon_{x-1}(\tau)|\right) \leq E\left(d(\upsilon_x, \upsilon_{x-1})\right),$$

so according to (iii), we obtain

$$d^2(\upsilon_{x+1}, \upsilon_x) \leq \left(\sup_{\tau \in [0,a]} \int_0^a P^2(\tau,\eta) d\eta \right) E^2\left(d(\upsilon_x, \upsilon_{x-1})\right) a \leq E^2\left(d(\upsilon_x, \upsilon_{x-1})\right).$$

Therefore

$$\begin{aligned}
d(\upsilon_{x+1}, \upsilon_x) \leq E\left(d(\upsilon_x, \upsilon_{x-1})\right) &= \frac{E\left(d(\upsilon_x, \upsilon_{x-1})\right)}{d(\upsilon_x, \upsilon_{x-1})}\ .\ d(\upsilon_x, \upsilon_{x-1}) \\
&= \gamma\left(d(\upsilon_x, \upsilon_{x-1})\right)\ .\ d(\upsilon_x, \upsilon_{x-1}),
\end{aligned} \tag{3. 15}$$

and so the sequence $\{d(\upsilon_{x+1}, \upsilon_x)\}$ is descending and bounded. Therefore there exists $\zeta \geq 0$ such that $\lim_{x\to\infty} d(\upsilon_{x+1}, \upsilon_x) = \zeta$. Suppose $\zeta > 0$. Then by (3. 15), we have

$$\frac{d(\upsilon_{x+1}, \upsilon_x)}{d(\upsilon_x, \upsilon_{x-1})} \leq \gamma\left(d(\upsilon_x, \upsilon_{x-1})\right), \quad x = 1, 2, \ldots .$$

According to the above inequality, we conclude $\gamma \notin \omega$ because $\lim_{x\to\infty} \gamma\left(d(\upsilon_x, \upsilon_{x-1})\right) = 1$ where $\lim_{x\to\infty} \left(d(\upsilon_x, \upsilon_{x-1})\right) = \zeta > 0$ and this is a contradiction. So $\zeta = 0$ and therefore $\lim_{x\to\infty} d(\upsilon_{x+1}, \upsilon_x) = 0$. Now we show that $\{\upsilon_x\}$ is a Cauchy sequence. Contrariwise, suppose that

$$\lim_{\lambda, x\to\infty} \sup d(\upsilon_x, \upsilon_\lambda) > 0. \tag{3. 16}$$

By the triangle inequality and Eq. (3. 15), we have

$$\begin{aligned} d(\upsilon_x, \upsilon_\lambda) &\leq d(\upsilon_x, \upsilon_{x+1}) + d(\upsilon_{x+1}, \upsilon_{\lambda+1}) + d(\upsilon_{\lambda+1}, \upsilon_\lambda) \\ &\leq d(\upsilon_x, \upsilon_{x+1}) + \left(\gamma\left(d(\upsilon_x, \upsilon_\lambda)\right) d(\upsilon_x, \upsilon_\lambda)\right) + d(\upsilon_{\lambda+1}, \upsilon_\lambda), \end{aligned}$$

hence

$$d(\upsilon_x, \upsilon_\lambda)\left[1 - \gamma\left(d(\upsilon_x, \upsilon_\lambda)\right)\right] \leq d(\upsilon_x, \upsilon_{x+1}) + d(\upsilon_{\lambda+1}, \upsilon_\lambda).$$

Thus, we have

$$d(\upsilon_x, \upsilon_\lambda) \leq \left(1 - \gamma\left(d(\upsilon_x, \upsilon_\lambda)\right)\right)^{-1}\left[d(\upsilon_x, \upsilon_{x+1}) + d(\upsilon_{\lambda+1}, \upsilon_\lambda)\right].$$

Since $\limsup_{\lambda, x\to\infty} d(\upsilon_x, \upsilon_\lambda) > 0$ and $\lim_{x\to\infty} d(\upsilon_{x+1}, \upsilon_x) = 0$, then

$$\limsup_{\lambda, x\to\infty} \left(1 - \gamma\left(d(\upsilon_x, \upsilon_\lambda)\right)\right)^{-1} = +\infty,$$

from the above equation, we conclude $\limsup_{\lambda, x\to\infty} \gamma\left(d(\upsilon_x, \upsilon_\lambda)\right) = 1$ and since $\gamma \in \omega$, we obtain

$$\lim_{\lambda, x\to\infty} \sup d(\upsilon_x, \upsilon_\lambda) = 0.$$

This contradicts with (3. 16), shows $\{\upsilon_x\}$ is a Cauchy sequence in χ, and $\{\upsilon_x\}$ is a convergent sequence in χ, that is

$$\exists \upsilon \in \chi, \quad \lim_{x\to\infty} \upsilon_x = \upsilon.$$

Now by taking the limit of both sides of (3. 14), we have

$$\begin{aligned} \upsilon(\tau) = \lim_{x\to\infty} \upsilon_{x+1}(\tau) &= \lim_{x\to\infty} \left(h(\tau) + \int_0^\tau P(\tau, \eta)\, E(\upsilon_x(\eta))\, d\eta\right) \\ &= h(\tau) + \int_0^\tau P(\tau, \eta)\, E(\lim_{x\to\infty} \upsilon_x(\eta))\, d\eta \\ &= h(\tau) + \int_0^\tau P(\tau, \eta)\, E(\upsilon(\eta))\, d\eta \equiv \mathtt{A}\upsilon(\tau). \end{aligned}$$

Thus, there exists a solution $\upsilon \in \chi$ such that $\mathtt{A}\upsilon = \upsilon$. It is clear that the fixed point of $\mathtt{A}$ is unique. $\square$

4. IMPLEMENTATION OF THE FRACTIONAL HAMMERSTEIN EQUATIONS

In this section, we first describe how to obtain the operational matrix of modified block pulse functions for the fractional integral, and then explain how to implement the method for the numerical approximation of the solution of the fractional integro- differential equations.
For $\vartheta_j(\tau)$, applying definitions of the Riemann-Liouville fractional integral of order $\beta > 0$ and convolution product, we have

$$I^{\beta}\vartheta_j(\tau) = \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau-\mu)^{\beta-1}\vartheta_j(\mu)\, d\mu = \frac{1}{\Gamma(\beta)}[\tau^{\beta-1} * \vartheta_j(\tau)],$$

due to the definition of the ε-modified block pulse functions, we have

$$I^{\beta}\vartheta_0(\tau) = \begin{cases} \frac{\tau^{\beta}}{\Gamma(\beta+1)} & ; \quad \tau \in [0,\ \ell-\varepsilon), \\ \frac{\tau^{\beta}-(\tau-\ell+\varepsilon)^{\beta}}{\Gamma(\beta+1)}; & \quad \tau \in [\ell-\varepsilon,\ R), \end{cases} \tag{4. 17}$$

$$I^{\beta}\vartheta_j(\tau) = \begin{cases} 0 & ; \quad \tau \in [0,\ j\ell-\varepsilon), \\ \frac{(\tau-j\ell+\varepsilon)^{\beta}}{\Gamma(\beta+1)} & ; \quad \tau \in [j\ell-\varepsilon,\ (j+1)\ell-\varepsilon), \\ \frac{(\tau-j\ell+\varepsilon)^{\beta}-(\tau-(j+1)\ell+\varepsilon)^{\beta}}{\Gamma(\beta+1)}; & \quad \tau \in [\ (j+1)\ell-\varepsilon,\ R), \end{cases} \tag{4. 18}$$

where $j = 1, ..., n-1$, and

$$I^{\beta}\vartheta_n(\tau) = \begin{cases} 0 & ; \quad \tau \in [0,\ R-\varepsilon), \\ \frac{(\tau-R+\varepsilon)^{\beta}}{\Gamma(\beta+1)}; & \quad \tau \in [R-\varepsilon,\ R). \end{cases} \tag{4. 19}$$

Set $(I^{\beta}\Phi)(\tau) \ \simeq\ Q\ \Phi(\tau)$. So, we have

$$(I^{\beta}\vartheta_0)(\tau) \simeq\ q_{00}\vartheta_0(\tau) + \sum_{j=1}^{n-1} q_{0j}\vartheta_j(\tau) + q_{0n}\vartheta_n(\tau),$$

so

$$\int_0^R I^{\beta}\vartheta_0(\tau)\, d\tau \simeq\ (\ell-\varepsilon)q_{00} + \ell\sum_{j=1}^{n-1} q_{0j} + \varepsilon q_{0n},$$

according to (4. 17) can be written

$$q_{00} = \frac{(\ell-\varepsilon)^{\beta}}{\Gamma(\beta+2)},$$

$$q_{0j} = \frac{((j+1)\ell-\varepsilon)^{\beta+1}-(j\ell)^{\beta+1}-(j\ell-\varepsilon)^{\beta+1}+((j-1)\ell)^{\beta+1}}{\ell.\ \Gamma(\beta+2)}, \quad j = 1,\ ...,\ n-1,$$

$$q_{0n} = \frac{R^{\beta+1}-(R-\ell+\varepsilon)^{\beta+1}-(R-\varepsilon)^{\beta+1}+(R-\ell)^{\beta+1}}{\varepsilon.\ \Gamma(\beta+2)},$$

now for $\vartheta_i(\tau),\ (i = 1,\ ...,\ n-1)$, according to (4. 18) can be written

$q_{ij} = \frac{\ell^\beta}{\Gamma(\beta+2)}[(j-i+1)^{\beta+1} - 2(j-i)^{\beta+1} + (j-i-1)^{\beta+1}], \quad j = i+1,\ ...,\ n-1,$

$q_{in} = \frac{1}{\varepsilon.\ \Gamma(\beta+2)}[(R-i\ell+\varepsilon)^{\beta+1} - (R-(i+1)\ell+\varepsilon)^{\beta+1} - (R-i\ell)^{\beta+1} + (R-(i+1)\ell)^{\beta+1}],$

eventually for $\vartheta_n(\tau)$, according to (4. 19), we have

$q_{n0} = 0,$

$q_{nj} = 0, \quad j = 1,\ ...,\ n-1,$

$q_{nn} = \frac{\varepsilon^\beta}{\Gamma(\beta+2)}.$

Finally, we obtain

$$Q_{(n+1)\times(n+1)} = \frac{1}{\Gamma(\beta+2)} \begin{pmatrix} (\ell-\varepsilon)^\beta & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{n-1} & \xi_n \\ 0 & \ell^\beta & \ell^\beta\eta_1 & \ell^\beta\eta_2 & \cdots & \ell^\beta\eta_{n-2} & \zeta_1 \\ 0 & 0 & \ell^\beta & \ell^\beta\eta_1 & \cdots & \ell^\beta\eta_{n-3} & \zeta_2 \\ 0 & 0 & 0 & \ell^\beta & \cdots & \ell^\beta\eta_{n-4} & \zeta_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \ell^\beta & \zeta_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \varepsilon^\beta \end{pmatrix} \tag{4. 20}$$

where

$\xi_k = \frac{1}{\ell}\ [((k+1)\ell-\varepsilon)^{\beta+1} - (k\ell)^{\beta+1} - (k\ell-\varepsilon)^{\beta+1} + ((k-1)\ell)^{\beta+1}], \quad k = 1,\ ...,\ n-1,$

$\xi_n = \frac{1}{\varepsilon}\ [R^{\beta+1} - (R-\ell+\varepsilon)^{\beta+1} - (R-\varepsilon)^{\beta+1} + (R-\ell)^{\beta+1}],$

$\eta_k = [(k+1)^{\beta+1} - 2k^{\beta+1} + (k-1)^{\beta+1}], \quad k = 1,\ ...,\ n-1-i,$

$\zeta_k = \frac{1}{\varepsilon}\ [(R-k\ell+\varepsilon)^{\beta+1} - (R-(k+1)\ell+\varepsilon)^{\beta+1} - (R-k\ell)^{\beta+1} + (R-(k+1)\ell)^{\beta+1}], \quad k = 1,\ ...,\ n-1.$

Q is called the ε-modified operational matrix for the fractional integral. In (4. 20), if $\beta = 1$ then matrix Q is equal to matrix P. Therefore, the matrix Q is a generalization of the ε-modified operational matrix for integration P.
Note that, if $g(\tau)$ is a continuous function, according to the above description, we can write

$$(I^\beta g)(\tau) \approx I^\beta G^T \Phi(\tau) \approx G^T Q\ \Phi(\tau). \tag{4. 21}$$

Here we solve the fractional Hammerstein integro-differential equations (1. 1)-(1. 2) by using εMBPFs. According to Proposition 2.3, problem (1. 1) can be rewritten as follows:

$$\upsilon(\tau) = g_0(\tau) + I^{\beta} g(\tau) + I^{\beta} \int_0^{\tau} k(\tau,\eta)\ E\left(\upsilon(\eta)\right) d\eta, \tag{4. 22}$$

where $g_0(\tau) = \sum_{j=0}^{m-1} \frac{\upsilon_j}{j!}\ \tau^j$. We first assume $E(\upsilon(\eta)) = w(\eta)$, then we have

$$w(\tau) = E(g_0(\tau) + I^{\beta} g(\tau) + I^{\beta} \int_0^{\tau} k(\tau,\eta)\ w(\eta) d\eta), \tag{4. 23}$$

now from (2. 9), we consider the following approximations

$$\begin{aligned} w(\tau) &\approx W^T \Phi(\tau) = \Phi^T(\tau)\ W, \\ g(\tau) &\approx G^T \Phi(\tau) = \Phi^T(\tau)\ G, \\ k(\tau,\eta) &\approx \Phi^T(\tau) K\ \Phi(\eta), \end{aligned} \tag{4. 24}$$

where $W,\ G$ and $\Phi(\tau)$ are $(n+1\times 1)$ column vectors and K is a matrix which is defined as follows

$$k_{ij} = \frac{1}{\Delta(J_i).\,\Delta(J_j)} \int_0^1 \int_0^1 k(\tau,\ \eta)\ \vartheta_i(\tau)\ \vartheta_j(\eta)\ d\tau\ d\eta, \qquad i,\ j = 0,\ ...\ n.$$

By putting the above approximations in the Eq. (4. 23) and using Proposition 2.9 and Eq. (4. 21), we have

$$W^T \Phi(\tau) = E(g_0(\tau) + G^T Q\ \Phi(\tau) + \hat{B}^T Q\ \Phi(\tau)),$$

where $B = K D_W P$ and $\hat{B}$ is an $(n+1\times 1)-$ matrix which elements are the same to the diagonal entries of matrix B.
Now, using the following nodes

$$\tau_r = \frac{t_r + t_{r+1}}{2}, \qquad r = 1,\ 2,\ ...,\ n+1,$$

where $t = [0,\ \frac{R}{n} - \varepsilon,\ \frac{2R}{n} - \varepsilon,\ ...,\ R-\varepsilon,\ R]$, we have

$$W^T \Phi(\tau_r) = E(g_0(\tau_r) + G^T Q\ \Phi(\tau_r) + \hat{B}^T Q\ \Phi(\tau_r)),\ \ r = 1,\ 2,\ ...,\ n+1. \tag{4. 25}$$

Now Eq. (4. 25) gives a system of algebraic equations, so we find unknown (n+1)-vector W. By substituting $E(\upsilon(\eta)) = w(\eta)$, in Eq. (4. 22) and applying the relationship (4. 21) and Proposition 2.9, we have

$$\widehat{\upsilon}_{\varepsilon}(\tau) = g_0(\tau) + G^T Q\ \Phi(\tau) + \hat{B}^T Q\ \Phi(\tau).$$

Therefore, we can obtain the solution of equation (1. 1) as follows:

$$\upsilon(\tau) \approx \overline{\upsilon}(\tau) = \frac{1}{k} \sum_{j=0}^{k-1} \widehat{\upsilon}_{\varepsilon_j}(\tau),$$

where $\varepsilon_j = \frac{j\ell}{k}$, $j = 0, 1, \ldots, k-1$ and $\widehat{\upsilon}_{\varepsilon_j}(\tau)$ are the approximate solutions of $\upsilon(\tau)$ which are respectively expanded in terms of ε_jMBPFs. To calculate the error bound of εMBPFs according to Theorem 2 in [9] can be written

$$\|\upsilon(\tau)-\overline{\upsilon}(\tau)\|_\infty \;\leq\; \frac{1}{k}\,\max_j\,\|\upsilon(\tau)-\widehat{\upsilon}_{\varepsilon_j}(\tau)\|_\infty, \qquad j=0,\;1,\;...,\;k-1.$$

5. Error Analysis

In this section, the error analysis of approximate solution by the εMBPFs is studied. In the following theorems, for simplicity, we assume $R=1$ and $l=\dfrac{1}{n}$.

Theorem 5.1. *If* $\widehat{\upsilon}(\tau)=\sum_{j=0}^{n}\upsilon_j\vartheta_j(\tau)$ *and* $\upsilon_j=\dfrac{1}{\Delta(J_j)}\int_0^1\upsilon(\tau)\vartheta_j(\tau)d\tau$, $j=0,1,...,n$ *then:*
(i) $\delta=\int_0^1(\upsilon(\tau)-\sum_{j=0}^{n}\upsilon_j\vartheta_j(\tau))^2d\tau$, *achieves its minimum value.*
(ii) $\{\widehat{\upsilon}(\tau)\}$ *approaches* $\upsilon(\tau)$ *point wise.*
(iii) $\int_0^1\upsilon^2(\tau)d\tau=\sum_{j=0}^{\infty}\upsilon_j^2\|\vartheta_j\|^2$.

Proof. The proof is like similar theorem which was proved by Jiang and Schaufelberger (1992) but intervals of integrations have to redefine as J_j, $j=0,1,...,n$. □

Theorem 5.2. *Suppose* $\upsilon(\tau)$ *is continuous in* J, *differentiable in* $(0,1)$, *and there is a number* M *such that* $|\upsilon'(\tau)|\leqslant M$ *for every* $\tau\in J$. *Then*

$$|\upsilon(b)-\upsilon(a)|\leqslant M|b-a|,$$

for all $a,b\in J$.

Proof. See [13]. □

Theorem 5.3. *Assume that,*
(i) $\upsilon(\tau)$ *is continuous and differentiable in* $[\dfrac{-1}{n},1+\dfrac{1}{n}]$, *with bounded derivative; that is,* $|\upsilon'(\tau)|\leqslant M$,
(ii) $\widehat{\upsilon}_{\varepsilon_i}(\tau)$ *where* $\varepsilon_i=\dfrac{il}{k}$, $i=0,1,...,k-1$ *are correspondingly MBPFs* (ε_0) *=BPFs, MBPFs*(ε_1), *..., MBPFs*(ε_{k-1}), *expansions of* $\upsilon(\tau)$ *base on* $n+1$ ε*MBPFs over interval* $[0,1)$,
(iii) $\overline{\upsilon}(\tau)=\dfrac{1}{k}\sum_{i=0}^{k-1}\widehat{\upsilon}_{\varepsilon_i}(\tau)$.
Then

$$\|\upsilon(\tau)-\widehat{\upsilon}_{\varepsilon_i}(\tau)\|=O(\frac{1}{n}), \qquad \|\upsilon(\tau)-\overline{\upsilon}(\tau)\|=O(\frac{1}{nk}).$$

Proof. We define the error between $\upsilon(\tau)$ and its BPFs expansion as follows:

$$e(\tau)=|\upsilon(\tau)-\widehat{\upsilon}(\tau)|$$

where $\widehat{\upsilon}(\tau)=\sum_{j=0}^{n-1}\upsilon_j\vartheta_j(\tau)$. Now, over every subinterval J_j, we have

$$e_j(\tau)=|\upsilon(\tau)-\upsilon_j\vartheta_j(\tau)|=|\upsilon(\tau)-\upsilon_j|, \qquad \tau\in J_j=[\frac{j}{n},\frac{j+1}{n}),$$

where $\upsilon_j = \dfrac{1}{l}\int_{\frac{j}{n}}^{\frac{j+1}{n}} \upsilon(\tau)d\tau$. Using the mean value theorem and using Theorem 5.2, we have:

$$\begin{aligned}\|e_j\|^2 &= \int_{\frac{j}{n}}^{\frac{j+1}{n}} e_j^2(\tau)d\tau = \int_{\frac{j}{n}}^{\frac{j+1}{n}} |\upsilon(\tau)-\upsilon_j|^2 d\tau = \frac{1}{n}|\upsilon(\xi)-\upsilon(\eta)|^2 \\ &\leq \frac{1}{n}M^2|\xi-\eta|^2 \leq \frac{M^2}{n^3}, \qquad \xi,\eta \in J_j. \end{aligned} \tag{5. 26}$$

This leads to

$$\|e(\tau)\|^2 = \int_0^1 e^2(\tau)d\tau = \int_0^1 \left(\sum_{j=0}^{n-1} e_j(\tau)\right)^2 d\tau = \int_0^1 \left(\sum_{j=0}^{n-1} e_j^2(\tau)\right) d\tau + 2\sum_{j\leq i}\int_0^1 e_j(\tau)e_i(\tau)d\tau.$$

Since for $i \neq j$, $Ji \cap Jj = \varnothing$, then

$$\|e(\tau)\|^2 = \sum_{j=0}^{n-1}\left(\int_0^1 e_j^2(\tau)d\tau\right) = \sum_{j=0}^{n-1}\|e_j\|^2. \tag{5. 27}$$

Substituting (5. 26) into (5. 27), we have

$$\|e(\tau)\|^2 \leq \frac{M^2}{n^2},$$

hence, $\|e(\tau)\| = O(\frac{1}{n})$.

Now to prove the next part of the theorem, according to the previous part, the error when $\upsilon(\tau)$ is represented in a series of εMBPFs is

$$e_j(\tau) = \upsilon(\tau) - \upsilon_j, \qquad \tau \in J_j = [\frac{j}{n}-\varepsilon, \frac{j+1}{n}-\varepsilon),$$

where

$$\upsilon_j = \frac{1}{\Delta(J_j)}\int_0^1 \upsilon(\tau)\vartheta_j(\tau)d\tau = \frac{1}{\Delta(J_j)}\int_{\frac{j}{n}-\varepsilon}^{\frac{j+1}{n}-\varepsilon} \upsilon(\tau)d\tau.$$

Now, using Trapezoidal rule for integral, we have

$$\upsilon_j = \frac{1}{2}\left(\upsilon(\frac{j}{n}-\varepsilon) + \upsilon(\frac{j+1}{n}-\varepsilon)\right) + E,$$

where E is the error of integration. Suppose n is so large that $\upsilon'(\tau)$ over interval J_j is approximately equal to a constant value. we use line $y = \tau$ instead of $\upsilon(\tau)$ over interval $J_j, j = 1, ..., n-1$. So $E = 0$ and we can write

$$\begin{aligned} e_j(\tau) = |\tau - \upsilon_j| &= |\tau - \frac{\sum_{i=0}^{k-1}\left(\frac{j}{n}-\varepsilon_i + \frac{j+1}{n}-\varepsilon_i\right)}{2k}| \\ &= |\tau - (\frac{2j+1}{2n}) + (\frac{k-1}{2kn})| \leq \frac{1}{2kn}, \end{aligned}$$

for J_0 we have

$$e_0(\tau) = |\tau - \frac{\sum_{i=0}^{k-1}\left(\frac{1}{n}-\varepsilon_i\right)}{2k}| = |\tau - \frac{1}{2n} + \frac{k-1}{4nk}| \leq \frac{1}{4kn}.$$

Similarly, for J_n we have $e_n(\tau) \leq \dfrac{1}{4kn}$. Now, for $j = 0, 1, ..., n-1$ we have

$$\|e_j\|^2 = \int_{J_j} e_j^2(\tau)d\tau = \int_{J_j} \frac{M^2}{4k^2n^2}d\tau \leq \frac{M^2}{4k^2n^3}.$$

This leads to

$$\|e(\tau)\|^2 = \sum_{j=0}^{n-1}\left(\int_0^1 e_j^2(\tau)d\tau\right) = \sum_{j=0}^{n-1}\|e_j\|^2 = n.\frac{M^2}{4k^2n^3} = \frac{M^2}{4k^2n^2}.$$

So, the error estimation for εMBPFs is $\|e(\tau)\| = O(\dfrac{1}{kn})$, where n shows the number of BPFs and k times of modifications. □

Lepik in [6] has introduced some methods to estimate the error when the exact solution is not available.

6. Numerical Results

We consider some examples which are reviewed by the program written in the Matlab. The numerical results are shown in Tables 1 to 3 that n represents the number of MBPFs and k indicates the number of modifications.

Example 6.1. *Consider the following equation with the exact solution $\upsilon(\tau) = \sin\tau$ for $\beta = 2$ and $\tau \in [0,\ 1]$*

$$(D^\beta\upsilon)(\tau) = \tau^3(-1+e^{\sin\tau})-\sin\tau-\int_0^\tau \tau^3\cos\eta\ e^{\upsilon(\eta)}\ d\eta, \quad 1 < \beta \leq 2, \quad \upsilon(0) = 0, \quad \upsilon'(0) = 1.$$

In Table 1 and Table 2, we show the approximate solutions for $n = 20$ and the error bound for $n = 20$ and $n = 40$. In the following, ρ_n as the empirical convergence rate symbol, is calculated by the following formula,

$$\rho_n = \frac{\log(\frac{error_\Delta}{error_{\Delta/2}})}{\log(2)}.$$

Table 1. Approximate solutions of Example 6.1 with MBPFs

τ	$n=20$			*Exact solution*
	k=1	*k=2*	*k=3*	
0.2	*0.198037*	*0.198324*	*0.198415*	*0.198669*
0.4	*0.387194*	*0.388248*	*0.388590*	*0.389418*
0.6	*0.559925*	*0.562202*	*0.562947*	*0.564642*
0.8	*0.709352*	*0.713258*	*0.714544*	*0.717356*

Table 2. Error bound and convergence rate for Example 6.1

	$n=20$				$n=40$			
	k=1	*k=2*	*k=3*	*k=4*	*k=1*	*k=2*	*k=3*	*k=4*
	0.008004	*0.004097*	*0.002173*	*0.001217*	*0.003896*	*0.001972*	*0.001017*	*0.000542*
ρ_n	–	–	–	–	*1.038727*	*1.054908*	*1.095368*	*1.166964*

Example 6.2. *Consider*

$$(D^{\frac{1}{2}}\upsilon)(\tau) = g(\tau) + \int_0^\tau \tau\eta\, \upsilon(\eta)\, d\eta, \qquad 0 \le \tau \le 1, \quad \upsilon(0) = 0,$$

where

$$g(\tau) = \frac{\sqrt{\pi}}{2}\, BesselJ[0, \sqrt{\tau}] - \tau(-2(-6+\tau)\sqrt{\tau}\cos\sqrt{\tau} + 6(-2+\tau)\sin\sqrt{\tau}),$$

the exact solution is $\upsilon(\tau) = \sin(\sqrt{\tau})$, *where* $BesselJ[n,x]$, *obtained the first kind Bessel function. In Table* 3, *we show the error bound for* $n = 20$ *and* $n = 40$.

Table 3. Error bound and convergence rate for Example 6.2

	$n=20$				$n=40$			
	k=1	*k=2*	*k=3*	*k=4*	*k=1*	*k=2*	*k=3*	*k=4*
	0.023972	*0.012486*	*0.007234*	*0.005261*	*0.012281*	*0.006141*	*0.003455*	*0.002383*
ρ_n	–	–	–	–	*0.964922*	*1.02377*	*1.06611*	*1.14256*

Example 6.3. *Consider the integro-differential equation*

$$(D^\beta\upsilon)(\tau) - \int_0^\tau e^\eta(\upsilon(\eta))^2 d\eta = 1, \quad 0 \le \tau < 1, \quad 3 < \beta \le 4,$$

where

$$\upsilon(0) = \upsilon'(0) = \upsilon''(0) = \upsilon'''(0) = 1.$$

The exact solution of this example for $\beta = 4$ *is* $\upsilon(\tau) = e^\tau$. *The numerical results for* $\beta = 3.25$ *are presented in Table* 4. *This Table shows that the obtained results by the proposed method are similar to* [11, 14].

Table 4. Approximate solutions of Example 6.3 for $\beta = 3.25$ in agreement with [11, 14]

	Exact solution	$n=40$		$n=120$			
τ	for $\beta=4$	k=2	k=6	k=2	k=6	[11]	[14]
0.1	1.105171	1.105254	1.105244	1.105241	1.105238	1.106551	1.105258
0.2	1.221403	1.222095	1.222046	1.222039	1.222023	1.223932	1.221892
0.3	1.349859	1.352329	1.352204	1.352191	1.352150	1.353200	1.352313
0.4	1.491825	1.498007	1.497756	1.497735	1.497652	1.495601	1.496762
0.5	1.648721	1.661461	1.661022	1.660989	1.660845	1.652553	1.663409
0.6	1.822119	1.845392	1.844687	1.844638	1.844405	1.825655	1.843799
0.7	2.013753	2.052996	2.051921	2.051849	2.051494	2.016687	2.044381
0.8	2.225541	2.288135	2.286555	2.286453	2.285931	2.227634	2.227591
0.9	2.459603	2.555594	2.553329	2.553184	2.552435	2.460691	2.526496

7. Conclusion

In this article, using the iterative method under some suitable conditions, we studied the behavior of the solution of fractional equations of the Hammerstein type in the Banach space. Applying the ε-modified functions and the fractional integration operational matrix, we obtained an approximate solution with high accuracy, whose convergence speed is faster than the BPF-based method. The results show that with a relatively small value selection of k, we have obtained the good accuracy for finding the approximate solution of the fractional integro-differential equations. In future work, we can investigate the stability of the solution for fractional equations of the Hammerstein type using the techniques of noncompactness measures in an infinite interval.

References

[1] B. Ahmad, S.K. Ntouyas, R.P. Agarwal, and A. Alsaedi, *Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions*, Bound. Value. Probl. **205**, (2016) 1–16.

[2] K. Balachandran, and S. Kiruthika, *Existence results for fractional integrodifferential equations with nonlocal condition via resolvent operators*, Comput. Math. Appl. **62**, No. 3 (2011) 1350–1358.

[3] S. Das, *Functional fractional calculus for system identification and controls*, Springer-Verlag, Berlin, Heidelberg, 2008.

[4] Z.H. Jiang, and W. Schaufelberger, *Block pulse functions and their applications in control systems*, Berlin, Springer-Verlag, 1992.

[5] J. Klafter, S.C. Lim, and R. Metzler *Fractional dynamics in physics: Recent advances*, World Scientific, Singapore, 2011.

[6] U. Lepik, *Haar wavelet method for nonlinear integro-differential equations*, Appl. Math. Comput. **176**, (2006) 324–333.

[7] Y. Li, and N. Sun, *Numerical solution of fractional differential equations using the generalized block pulse operational matrix*, Comput. Math. Appl. **62**, (2011) 1054–1064.

[8] F. Mainard, *Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models*, Imperial College Press, London, 2010.

[9] K. Maleknejad, and B. Rahimi, *Modification of block pulse functions and their application to solve numerically Volterra integral equation of the first kind*, Commun. Nonlinear. Sci. Numer. Simulat. **16**, (2011) 2469–2477.

[10] F. Mirzaee, and E. Hadadiyan, *A Collocation method to the solution of nonlinear Fredholm-Hammerstein integral and integro-differential equations*, J. Hyperstrutures. **2**, No. 1 (2013) 72–86.

[11] S. Momani, and M. Noor, *Numerical methods for fourth order fractional integro-differential equations*, Appl. Math. Comput. **182**, (2006) 754–760.

[12] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1990.

[13] W. Rudin, *Principles of Mathematical Analysis*, Singapore: McGraw-Hill; 1976.

[14] H. Saeedi, and M. Mohseni Moghadam, *Numerical solution of nonlinear Volterra integrodifferential equations of arbitrary order by CAS wavelets*, Commun. Nonlinear. Sci. Numer. Simulat. **16**, (2011) 1216–1226.

[15] P.K. Sahu, and S. Saha Ray, *Hybrid Legendre block-pulse functions for the numerical solutions of system of nonlinear FredholmHammerstein integral equations*, Appl. Math. Comput. **270**, (2015) 871–878.

[16] F. Wang, *Existence and uniqueness of solutions for a nonlinear fractional differential equation*, J. Appl. Math. Comput. **39**, No. 1-2 (2012) 53–67.

[17] L. Zhang, B. Ahmad, G. Wang, and R.P. Agarwal, *Nonlinear fractional integro-differential equations on unbounded domains in a Banach space*, J. Comput. Appl. Math. **249**, (2013) 51-56-.