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New Hermite-Hadamard Type Inequalities for Exponentially GA and GG-Convex Functions

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Abstract. In the article, we present several new Hermite-Hadamard type inequalities for the exponentially GA and GG-convex functions by use of an integral identity. Our results are the refinements and improvements of some previously results.

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1. Introduction

Integral inequalities are a fabulous instrument for developing the qualitative and quantitative properties of convexity. There has been a continuous growth of interest in such an area of research in order to meet the needs of applications of these inequalities. Such inequalities have been studied by many researchers who in turn used diverse techniques for the sake of exploring and proposing these inequalities [1, 3, 4, 6, 8, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20]. One of the most important inequalities is the distinguished Hermite-Hadamard inequality.

Let $\psi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$. Then

$$\psi\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \leq \frac{1}{\kappa_2-\kappa_1}\int\limits_{\kappa_1}^{\kappa_2}\psi(x)dx \leq \frac{\psi(\kappa_1)+\psi(\kappa_2)}{2}. \tag{1. 1}$$

Both inequalities hold in the reversed direction for ψ to be concave.
In recent years, numerous generalizations, extensions and variants of Hermite- Hadamard inequality (1. 1) were studied extensively by many researchers and appeared in a number of papers, see [21, 22, 23, 24, 25, 27, 28, 29, 30, 33, 37, 38, 39, 40, 42, 47, 48, 50, 51].

In parallel to increasing interests in the convexity theory, many researchers have extended these mathematical inequalities to exponentially convex functions. Another approach is efficient to obtain the integral inequalities by utilizing convex functions. It is known that the subclass of convex functions is closely related to $\log$-convex functions referred to as exponentially convex functions. Exponentially convex function explored by Bernstein [9] in covariance formation then Avriel [7] contemplated and investigated this concept by imposing the condition of r-convex functions. Rashid et al. [43] and Noor et al. [31, 32] explored exponentially convex functions while studying the paper of Antczak [5]. For more features concerning to exponentially convex functions, see [36, 41, 42] and the references therein. Pal [34] provided the fertile application of exponentially convex functions in information theory, optimization theory, and statistical theory. For observing various other kinds of exponentially convex functions and their generalizations, see [44, 45, 46, 49]. Following this tendency, we provide a new version for Hermite Hadamard inequality in the frame of the exponentially convex function.
Dragomir and Gomm [13] proved that a function ψ is exponentially convex if and only if satisfies

$$e^{\psi\left(\frac{\kappa_1+\kappa_2}{2}\right)} \leq \frac{1}{\kappa_2-\kappa_1}\int\limits_{\kappa_1}^{\kappa_2}e^{\psi(x)}dx \leq \frac{e^{\psi(\kappa_1)}+e^{\psi(\kappa_2)}}{2}. \tag{1. 2}$$

The inequality (1. 2) is called the Hermite-Hadamard inequality and provides the upper and lower estimates for the exponential integral, (see[36, 41, 42, 43] and the references there in).

Alirezai and Mathar [2] have considered some properties of exponentially convex functions along with their potential applications in statistics and information theory, see [2, 34]. Due to its importance, in [8], Awan et al and also in [35], Pecaric and Jaksetic defined another variant of exponentially convex functions, have shown that the class of exponentially convex functions unifies various concepts in different manners.

In this article, we aim to define new class of geometrically convex functions, which is called exponentially GA- and GG-convex functions. We also prove a new integral identity in order to prove several new integral inequalities by using these new classes of functions. Innovative ideas and techniques of this paper may stimulate further research in this dynamic field.

2. PRELIMINARIES

We set forth some terminologies, definitions, and essential details that will be used throughout the remaining part of the paper.
In [26], Niculescu mentioned the following considerable definitions:

Definition 2.1. *The class of all $GA-$convex functions is constituted by all functions ψ : $K \to \mathbb{R}$(acting on subintervals of $(0,\infty)$) such that*

$$\psi\big(\kappa_1^{1-\xi}\kappa_2^{\xi}\big) \le (1-\xi)\psi(\kappa_1)+\xi\psi(\kappa_2), \quad \forall \kappa_1,\kappa_2 \in K,\ \xi\in[0,1].$$

Definition 2.2. *The $GG-$convex functions are those functions $\psi : K \to J$ (acting on subintervals of $(0,\infty)$) such that*

$$\psi\big(\kappa_1^{1-\xi}\kappa_2^{\xi}\big) \le \big(\psi(\kappa_1)\big)^{1-\xi}\big(\psi(\kappa_2)\big)^{\xi}, \quad \forall \kappa_1,\kappa_2 \in K,\ \xi\in[0,1].$$

We now define the concept of exponentially convex function, which is mainly due to Antczak [5], Dragomir and Gomm [13] and Rashid et al. [43]:

Definition 2.3. *(* [5, 13, 43]*) A positive real-valued function $\psi : K \subseteq \mathbb{R} \longrightarrow (0,\infty)$ is said to be exponentially convex on K if the inequality*

$$e^{\psi((1-\xi)\kappa_1+\xi\kappa_2)} \le (1-\xi)e^{\psi(\kappa_1)}+\xi e^{\psi(\kappa_2)}, \quad \forall \kappa_1,\kappa_2 \in K,\ \xi\in[0,1].$$

3. MAIN RESULTS

Firstly, we will start two new classes of functions that is called exponentially GA- and GG-convex functions, which are the main motivation of this paper. We denote $I=[\kappa_1,\kappa_2]$, unless otherwise specified.

Definition 3.1. *A function $\psi : K \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be exponentially GA-convex function, if*

$$e^{\psi\left(\kappa_1^{1-\xi}\kappa_2^{\xi}\right)} \le (1-\xi)e^{\psi(\kappa_1)}+\xi e^{\psi(\kappa_2)},\ \ \forall \kappa_1,\kappa_2 \in K,\ \ \xi\in[0,1].$$

Also note that for $\xi=\frac{1}{2}$ in Definition 3.1, we have Jensen type exponentially $GA-$convex functions.

$$e^{\psi\left(\sqrt{\kappa_1\kappa_2}\right)} \le \frac{1}{2}[e^{\psi(\kappa_1)}+e^{\psi(\kappa_2)}],\ \ \forall \kappa_1,\kappa_2 \in K.$$

Definition 3.2. *A function $\psi : K \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be exponentially GG-convex function, if*

$$e^{\psi\left(\kappa_1^{1-\xi}\kappa_2^{\xi}\right)} \le \big(e^{\psi(\kappa_1)}\big)^{(1-\xi)}\big(e^{\psi(\kappa_2)}\big)^{\xi},\ \ \forall \kappa_1,\kappa_2 \in K,\ \ \xi\in[0,1].$$

Also note that for $\xi = \frac{1}{2}$ in Definition 3.1, we have Jensen type exponentially $GG-$ convex functions.

$$e^{\psi\left(\sqrt{\kappa_1\kappa_2}\right)} \leq \sqrt{e^{\psi(\kappa_1)}e^{\psi(\kappa_2)}}, \ \forall \kappa_1, \kappa_2 \in K.$$

In order to establish our main results, we need a lemma which we present in this section. Making use of integration by parts, we can obtain the following Lemma immediately.

Lemma 3.3. *Let $\psi : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on the interior I° of I, where $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$ and $(e^\psi)' \in L[\kappa_1, \kappa_2]$. Then*

$$\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)} dx$$
$$= \frac{\ln\kappa_2 - \ln\kappa_1}{2}\Bigg[\int_0^1 \left(\kappa_2^{1+\xi}\kappa_1^{1-\xi}\right) e^{\psi\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right)} \psi'\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right) d\xi$$
$$+ \int_0^1 \left(\kappa_1^{1+\xi}\kappa_2^{1-\xi}\right) e^{\psi\left(\kappa_1^{\frac{1+\xi}{2}}\kappa_2^{\frac{1-\xi}{2}}\right)} \psi'\left(\kappa_1^{\frac{1+\xi}{2}}\kappa_2^{\frac{1-\xi}{2}}\right) d\xi\Bigg].$$

Our first result is given in the following theorem.

Theorem 3.4. *Let $\psi : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on the interior I° of I, where $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$ and $(e^\psi)' \in L([\kappa_1, \kappa_2])$. If $|(e^\psi)'|^q$ is GA-convex on I for $q \geq 1$, then*

$$\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)} dx \ \leq \frac{(\ln\kappa_2 - \ln\kappa_1)^{1-\frac{2}{q}} L(\kappa_1,\kappa_2)^{1-\frac{1}{q}}}{2^{1+\frac{1}{q}}} \tag{3. 3}$$
$$\times \ \kappa_2 \quad 2(L(\kappa_1,\kappa_2) - \kappa_1) - \kappa_1(\ln\kappa_2 - \ln\kappa_1) \ |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q + \ (4\kappa_2 - \kappa_1)(\ln\kappa_2 - \ln\kappa_1)$$
$$+2 \ L(\kappa_1,\kappa_2) + \kappa_1 - 2\kappa_2 \quad |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q + \ 2(\kappa_2 - L(\kappa_1,\kappa_2)) - \kappa_1(\ln\kappa_2 - \ln\kappa_1) \ \Delta_1(\kappa_1,\kappa_2) \quad ^{\frac{1}{q}}$$
$$+\kappa_1 \quad (4\kappa_1 - \kappa_2)(\ln\kappa_1 - \ln\kappa_2) - 2 \ (2\kappa_1 - \kappa_2) - L(\kappa_1,\kappa_2) \quad |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q + \ 2 \ L(\kappa_1,\kappa_2) - \kappa_2$$
$$-\kappa_2(\ln\kappa_1 - \ln\kappa_2) \ |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q + \ 2 \ \kappa_1 - L(\kappa_1,\kappa_2) \ - \kappa_2(\ln\kappa_1 - \ln\kappa_2) \ \Delta_1(\kappa_1,\kappa_2) \quad ^{\frac{1}{q}} \ ,$$

where

$$\Delta_1(\kappa_1,\kappa_2) = |e^{\psi(\kappa_1)}\psi'(\kappa_2)|^q + |e^{\psi(\kappa_2)}\psi'(\kappa_1)|^q, \tag{3. 4}$$

and

$$L(x,y) = \frac{x-y}{\ln x - \ln y}. \tag{3. 5}$$

Proof. From Lemma 3.3, using the property of modulus and GA-convexity of $|(e^{\psi})'|^q$, we can write

$$
\begin{aligned}
&\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)}dx \qquad (3.\ 6)\\
&= \frac{\ln\kappa_2 - \ln\kappa_1}{2}\Bigg[\int_0^1 \kappa_2^{1+\xi}\kappa_1^{1-\xi}\, e^{\psi\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right)}\psi'\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right)d\xi \\
&\quad + \int_0^1 \kappa_2^{1-\xi}\kappa_1^{1+\xi}\, e^{\psi\left(\kappa_2^{\frac{1-\xi}{2}}\kappa_1^{\frac{1+\xi}{2}}\right)}\psi'\left(\kappa_2^{\frac{1-\xi}{2}}\kappa_1^{\frac{1+\xi}{2}}\right)d\xi\Bigg] \\
&= \frac{(\kappa_1\kappa_2)(\ln\kappa_2 - \ln\kappa_1)}{2}\Bigg[\left(\int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}\left|e^{\psi\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right)}\psi'(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}})\right|^q d\xi\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left(\frac{\kappa_1}{\kappa_2}\right)^{\xi}d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{\kappa_1}{\kappa_2}\right)^{\xi}\left|e^{\psi\left(\kappa_2^{\frac{1-\xi}{2}}\kappa_1^{\frac{1+\xi}{2}}\right)}\psi'(\kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}})\right|^q d\xi\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Now consider

$$
\begin{aligned}
I_1 &= \int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}\left|e^{\psi(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}})}\psi'(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}})\right|^q d\xi \qquad (3.\ 7)\\
&\le \int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}\left[|e^{\psi(\kappa_1)}|^q(\frac{1-\xi}{2}) + |e^{\psi(\kappa_2)}|^q(\frac{1+\xi}{2})\right]\left[|\psi'(\kappa_1)|^q(\frac{1-\xi}{2}) + |\psi'(\kappa_2)|^q(\frac{1+\xi}{2})\right]d\xi \\
&= |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q\int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}\frac{(1-\xi)^2}{4}d\xi + |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q\int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}\frac{(1+\xi)^2}{4}d\xi \\
&\quad + \left[|e^{\psi(\kappa_1)}\psi'(\kappa_2)|^q + |e^{\psi(\kappa_2)}\psi'(\kappa_1)|^q\right]\int_0^1 \frac{1-\xi^2}{4}\left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}d\xi \\
&= |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q\int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}\frac{(1-\xi)^2}{4}d\xi + |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q\int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}\frac{(1+\xi)^2}{4}d\xi \\
&\quad + \Delta_1(\kappa_1,\kappa_2)\int_0^1 \frac{1-\xi^2}{4}\left(\frac{\kappa_2}{\kappa_1}\right)^{\xi}d\xi \\
&= \frac{1}{4\kappa_1\left(\ln\kappa_2 - \ln\kappa_1\right)^2}\Bigg[\Big(2(L(\kappa_1,\kappa_2)-\kappa_1) - \kappa_1(\ln\kappa_2 - \ln\kappa_1)\Big)|e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q \\
&\quad + \Big[(4\kappa_2 - \kappa_1)(\ln\kappa_2 - \ln\kappa_1) + 2\left(L(\kappa_1,\kappa_2) + \kappa_1 - 2\kappa_2\right)\Big]|e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q \\
&\quad + \Big[2(\kappa_2 - L(\kappa_1,\kappa_2)) - \kappa_1(\ln\kappa_2 - \ln\kappa_1)\Big]\Delta_1(\kappa_1,\kappa_2)\Bigg].
\end{aligned}
$$

Similarly,we have

$$I_2 = \int_0^1 \frac{\kappa_1}{\kappa_2}\ \xi\ e^{\psi(\kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}})}\psi'(\kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}})\ ^q d\xi \leq \frac{1}{4\kappa_2\ \ln\kappa_2 - \ln\kappa_1\ ^2} \tag{3. 8}$$

$$(4\kappa_1-\kappa_2)(\ln\kappa_1-\ln\kappa_2) - 2\ (2\kappa_1-\kappa_2) - L(\kappa_1,\kappa_2)\ \ |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q + \ 2\ L(\kappa_1,\kappa_2) - \kappa_2$$

$$-\kappa_2(\ln\kappa_1-\ln\kappa_2)\ |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q + \ 2\ \kappa_1 - L(\kappa_1,\kappa_2)\ - \kappa_2(\ln\kappa_1-\ln\kappa_2)\ \Delta_1(\kappa_1,\kappa_2)\ .$$

Also

$$\int_0^1 \frac{\kappa_2}{\kappa_1}\ ^\xi d\xi\ ^{1-\frac{1}{q}} = \frac{L(\kappa_1,\kappa_2)}{\kappa_1}\ ^{\frac{1}{q}} \tag{3. 9}$$

and

$$\int_0^1 \frac{\kappa_1}{\kappa_2}\ ^\xi d\xi\ ^{1-\frac{1}{q}} = \frac{L(\kappa_1,\kappa_2)}{\kappa_2}\ ^{\frac{1}{q}}. \tag{3. 10}$$

Combining (3.6), (3.7), (3.8), (3.9) and (3.10), we get the desired inequality (3.3).
This completes the proof. □

Theorem 3.5. *Let $\psi : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior I° of I, where $\kappa_1,\kappa_2 \in I$ with $\kappa_1 < \kappa_2$ and $(e^\psi)' \in L[\kappa_1,\kappa_2]$. If $|(e^\psi)'|^q$ is GA-convex on I for $q \geq 1$, then*

$$\left|\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)}dx\right| \leq \frac{\big(\ln\kappa_2-\ln\kappa_1\big)^{1-\frac{2}{q}}}{2.(2q)^{\frac{2}{q}}} \tag{3. 11}$$

$$\times\Bigg[\kappa_2\Bigg\{\big\{2\big(L(\kappa_1^q,\kappa_2^q)-\kappa_1^q\big) - q\kappa_1^q\big(\ln\kappa_2-\ln\kappa_1\big)\big\}|e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q$$

$$+\big\{2\big(L(\kappa_1^q,\kappa_2^q)-(2\kappa_2^q-\kappa_1^q)\big) + q(4\kappa_2^q-\kappa_1^q)\big(\ln\kappa_2-\ln\kappa_1\big)\big\}|e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q$$

$$+\big\{2\big(\kappa_2^q - L(\kappa_1^q,\kappa_2^q)\big) - q\kappa_1^q\big(\ln\kappa_2-\ln\kappa_1\big)\big\}\Delta_1(\kappa_1,\kappa_2)\Bigg\}^{\frac{1}{q}}$$

$$+\kappa_1\Bigg\{\big\{2\big(L(\kappa_1^q,\kappa_2^q)-\kappa_2^q-\kappa_2^q q\big(\ln\kappa_1-\ln\kappa_2\big)\big)\big\}|e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q$$

$$+\big\{(4\kappa_1^q-\kappa_2^q)q\big(\ln\kappa_1-\ln\kappa_2\big) - 2\big(2\kappa_1^q-\kappa_2^q-L(\kappa_1^q,\kappa_2^q)\big)\big\}|e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q$$

$$+\big\{2\big(\kappa_1^q - L(\kappa_1^q,\kappa_2^q)\big) - q\kappa_2^q\big(\ln\kappa_1-\ln\kappa_2\big)\big\}\Delta_1(\kappa_1,\kappa_2)\Bigg\}^{\frac{1}{q}}\Bigg],$$

where $\Delta_1(\kappa_1,\kappa_2)$ and $L(x,y)$ are given in (3. 4) *and* (3. 5)*, respectively.*

Proof. From Lemma 3.3, using the property of modulus and GA-convexity of $|(e^{\psi})'|^q$, we can write

$$\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)}dx$$

$$= \frac{\ln\kappa_2 - \ln\kappa_1}{2}[\int_0^1 \; \kappa_2^{1+\xi}\kappa_1^{1-\xi} \;\; e^{\psi} \;\; \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \;\; \psi' \;\; \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \;\; d\xi$$

$$+ \int_0^1 \; \kappa_2^{1-\xi}\kappa_1^{1+\xi} \;\; e^{\psi} \;\; \kappa_2^{\frac{1-\xi}{2}}\kappa_1^{\frac{1+\xi}{2}} \;\; \psi' \;\; \kappa_2^{\frac{1-\xi}{2}}\kappa_1^{\frac{1+\xi}{2}} \;\; d\xi$$

$$= \frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2} \;\; \int_0^1 1d\xi^{\;\;1-\frac{1}{q}} \;\; \int_0^1 \; \frac{\kappa_2}{\kappa_1}^{\;q\xi} e^{\psi} \;\; \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \;\; \psi' \;\; \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \;\; {}^{q}d\xi^{\;\;\frac{1}{q}}$$

$$+ \;\; \int_0^1 1d\xi^{\;\;1-\frac{1}{q}} \;\; \int_0^1 \; \frac{\kappa_1}{\kappa_2}^{\;q\xi} e^{\psi} \;\; \kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}} \;\; \psi' \;\; \kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}} \;\; {}^{q}d\xi^{\;\;\frac{1}{q}} \;\; .$$

Now consider

$$I_1 = \int_0^1 \; \frac{\kappa_2}{\kappa_1}^{\;q\xi} e^{\psi} \;\; \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \;\; \psi' \;\; \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \;\; {}^{q}d\xi \tag{3. 12}$$

$$\leq \int_0^1 \; \frac{\kappa_2}{\kappa_1}^{\;q\xi} \;\; \frac{1-\xi}{2}|e^{\psi(\kappa_1)}|^q + \frac{1+\xi}{2}|e^{\psi(\kappa_2)}|^q \;\; \frac{1-\xi}{2}|\psi'(\kappa_1)|^q + \frac{1+\xi}{2}|\psi'(\kappa_2)|^q \;\; d\xi$$

$$= \int_0^1 \; \frac{\kappa_2}{\kappa_1}^{\;q\xi} \;\; \frac{(1-\xi)^2}{4}|e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q + \frac{(1+\xi)^2}{4}|e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q + \frac{1-\xi^2}{4}\Delta_1(\kappa_1,\kappa_2)$$

$$= \frac{1}{4\kappa_1^q q^2 \;\; \ln\kappa_2 - \ln\kappa_1 \;\;^2} \;\; 2 \;\; L(\kappa_1^q,\kappa_2^q) - \kappa_1^q \;\; - q\kappa_1^q \;\; \ln\kappa_2 - \ln\kappa_1 \;\; |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q$$

$$+ \;\; 2 \;\; L(\kappa_1^q,\kappa_2^q) - (2\kappa_2^q - \kappa_1^q) + q(4\kappa_2^q - \kappa_1^q) \;\; \ln\kappa_2 - \ln\kappa_1 \;\; |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q$$

$$+ \;\; 2 \;\; \kappa_2^q - L(\kappa_1^q,\kappa_2^q) \;\; - q\kappa_1^q \;\; \ln\kappa_2 - \ln\kappa_1 \;\; \Delta_1(\kappa_1,\kappa_2) \;\; .$$

Similarly, we have

$$I_2 = \int_0^1 \; \frac{\kappa_1}{\kappa_2}^{\;q\xi} e^{\psi} \;\; \kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}} \;\; \psi' \;\; \kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}} \;\; {}^{q}d\xi \tag{3. 13}$$

$$\leq \frac{\kappa_1}{4\kappa_2^q q^2 \;\; \ln\kappa_2 - \ln\kappa_1 \;\;^2} \;\; 2 \;\; L(\kappa_1^q,\kappa_2^q) - \kappa_2^q - \kappa_2^q q \;\; \ln\kappa_1 - \ln\kappa_2 \;\; |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q$$

$$+ \;\; (4\kappa_1^q - \kappa_2^q)q \;\; \ln\kappa_1 - \ln\kappa_2 \;\; - 2 \;\; 2\kappa_1^q - \kappa_2^q - L(\kappa_1^q,\kappa_2^q) \;\; |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q$$

$$+ \;\; 2 \;\; \kappa_1^q - L(\kappa_1^q,\kappa_2^q) \;\; - q\kappa_2^q \;\; \ln\kappa_1 - \ln\kappa_2 \;\; \Delta_1(\kappa_1,\kappa_2) \;\; .$$

Substituting (3.12) and (3.13) in (3.12). This completes the proof. □

Theorem 3.6. *Let $\psi : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior I° of I, where $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$ and $(e^\psi)' \in L[\kappa_1,\kappa_2]$. If $|(e^\psi)'|^q$ is $GA-$convex on I for $q \geq 1$, then*

$$\left|\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)}dx\right| \leq \frac{(\ln\kappa_2 - \ln\kappa_1)^{1-\frac{2}{q}} L\big(\kappa_1^{\frac{(q-p)}{q-1}}, \kappa_2^{\frac{(q-p)}{q-1}}\big)^{1-\frac{1}{q}}}{2(2p)^{\frac{2}{q}}} \quad (3.\ 14)$$
$$\times\Bigg[\kappa_2\Bigg\{2\big(L(\kappa_1^p,\kappa_2^p) - \kappa_1^p\big) - p\kappa_1^p\big(\ln\kappa_2 - \ln\kappa_1\big)\Big\}|e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q$$
$$+\big\{2\big(L(\kappa_1^p,\kappa_2^p) - (2\kappa_2^p - \kappa_1^p) + p(4\kappa_2^p - \kappa_1^p\big(\ln\kappa_2 - \ln\kappa_1\big))\big)\big\}|e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q$$
$$+\big\{2\big(\kappa_2^p - L(\kappa_1^p,\kappa_2^p) - p\kappa_1^p\big(\ln\kappa_2 - \ln\kappa_1\big)\big)\big\}\Delta_1(\kappa_1,\kappa_2)\Bigg\}^{\frac{1}{q}}$$
$$+\kappa_1\Bigg\{\big\{2\big(L(\kappa_1^p,\kappa_2^p) - (2\kappa_1^p - \kappa_2^p)\big) + p(4\kappa_1^p - \kappa_2^p)\big(\ln\kappa_2 - \ln\kappa_1\big)\big\}|e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q$$
$$+\big\{2\big(L(\kappa_1^p,\kappa_2^p) - \kappa_2^p\big) - p\kappa_2^p\big(\ln\kappa_2 - \ln\kappa_1\big)\big\}|e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q$$
$$+\big\{2\big(L(\kappa_1^p,\kappa_2^p) - (2\kappa_1^p - \kappa_2^p)\big) - p\kappa_2^p\big(\ln\kappa_2 - \ln\kappa_1\big)\big\}\Delta_1(\kappa_1,\kappa_2)\Bigg\}^{\frac{1}{q}}\Bigg],$$

where $\Delta_1(\kappa_1,\kappa_2)$ and $L(x,y)$ are given in (3. 4) *and* (3. 5), *respectively.*

Proof. From Lemma 3.3, using the property of modulus and GA-convexity of $|(e^\psi)'|^q$, we can write

$$\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)}dx \quad (3.\ 15)$$
$$= \frac{\ln\kappa_2 - \ln\kappa_1}{2}[\int_0^1 \kappa_2^{1+\xi}\kappa_1^{1-\xi} \quad e^{\psi} \ \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \quad \psi' \ \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \quad d\xi$$
$$+\int_0^1 \kappa_2^{1-\xi}\kappa_1^{1+\xi} \quad e^{\psi} \ \kappa_2^{\frac{1-\xi}{2}}\kappa_1^{\frac{1+\xi}{2}} \quad \psi' \ \kappa_2^{\frac{1-\xi}{2}}\kappa_1^{\frac{1+\xi}{2}} \quad d\xi$$
$$= \frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2} \quad \int_0^1 \ \frac{\kappa_2}{\kappa_1}^{\frac{(q-p)\xi}{q-1}} \ {}^{1-\frac{1}{q}} \quad \int_0^1 \ \frac{\kappa_2}{\kappa_1}^{p\xi} \ e^{\psi} \ \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \quad \psi' \ \kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}} \quad {}^{\frac{1}{q}}$$
$$+ \quad \int_0^1 \ \frac{\kappa_1}{\kappa_2}^{\frac{(q-p)\xi}{q-1}} \ {}^{1-\frac{1}{q}} \quad \int_0^1 \ \frac{\kappa_1}{\kappa_2}^{p\xi} \ e^{\psi} \ \kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}} \quad \psi' \ \kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}} \quad {}^{\frac{1}{q}} \quad .$$

Now consider I_1

$$\begin{aligned} I_1 &= \int_0^1 \frac{\kappa_2}{\kappa_1}^{p\xi} e^{\psi}\ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \quad \psi'\ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \quad d\xi \qquad (3.16)\\ &\leq \int_0^1 \frac{\kappa_2}{\kappa_1}^{p\xi}\ \frac{1-\xi}{2}|e^{\psi(\kappa_1)}|^q + \frac{1+\xi}{2}|e^{\psi(\kappa_2)}|^q \quad \frac{1-\xi}{2}|\psi(\kappa_1)|^q + \frac{1+\xi}{2}|\psi(\kappa_2)\ \ d\xi \\ &= \int_0^1 \frac{\kappa_2}{\kappa_1}^{p\xi}\ \frac{(1-\xi)^2}{4}|e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q + \frac{(1+\xi)^2}{4}|e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q + \frac{1-\xi^2}{4}\Delta_1(\kappa_1,\kappa_2)\ \ d\xi \\ &= \frac{1}{\kappa_1^p p^2\ \ \ln\kappa_2 - \ln\kappa_1}\quad 2\ \ L(\kappa_1^p,\kappa_2^p) - \kappa_1^p\ \ - p\kappa_1^p\ \ \ln\kappa_2 - \ln\kappa_1 \quad |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q \\ &\quad +\ 2\ \ L(\kappa_1^p,\kappa_2^p) - (2\kappa_2^p - \kappa_1^p) + p(4\kappa_2^p - \kappa_1^p\ \ \ln\kappa_2 - \ln\kappa_1\ \ ) \quad |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q \\ &\quad +\ 2\ \ \kappa_2^p - L(\kappa_1^p,\kappa_2^p) - p\kappa_1^p\ \ \ln\kappa_2 - \ln\kappa_1 \quad \Delta_1(\kappa_1,\kappa_2)\ \ . \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &= \int_0^1 \frac{\kappa_1}{\kappa_2}^{p\xi} e^{\psi}\ \kappa_1^{\frac{1-\xi}{2}} b^{\frac{1+\xi}{2}} \quad \psi'\ \kappa_1^{\frac{1+\xi}{2}} \kappa_2^{\frac{1-\xi}{2}} \quad d\xi \qquad (3.17)\\ &\leq \frac{1}{\kappa_2^p p^2\ \ \ln\kappa_2 - \ln\kappa_1}\quad 2\ \ L(\kappa_1^p,\kappa_2^p) - (2\kappa_1^p - \kappa_2^p)\ \ + p(4\kappa_1^p - \kappa_2^p)\ \ \ln\kappa_2 - \ln\kappa_1 \quad |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^q \\ &\quad +\ 2\ \ L(\kappa_1^p,\kappa_2^p) - \kappa_2^p\ \ - p\kappa_2^p\ \ \ln\kappa_2 - \ln\kappa_1 \quad |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^q \\ &\quad +\ 2\ \ L(\kappa_1^p,\kappa_2^p) - (2\kappa_1^p - \kappa_2^p)\ \ - p\kappa_2^p\ \ \ln\kappa_2 - \ln\kappa_1 \quad \Delta_1(\kappa_1,\kappa_2)\ \ . \end{aligned}$$

Also

$$\begin{aligned} \int_0^1 \quad \frac{\kappa_2}{\kappa_1}^{\frac{(q-p)\xi}{q-1}}\ \ d\xi &= \frac{\kappa_2^{\frac{q-p}{q-1}} - \kappa_1^{\frac{q-p}{q-1}}}{(\frac{q-p}{q-1})\kappa_1^{\frac{q-p}{q-1}}\ \ \ln\kappa_2 - \ln\kappa_1} \qquad (3.18)\\ &= \frac{L\ \ \kappa_1^{\frac{q-p}{q-1}}, \kappa_2^{\frac{q-p}{q-1}}}{\kappa_1^{\frac{q-p}{q-1}}}. \end{aligned}$$

Substituting (3.16), (3.17) and (3.18) in (3.15). We get the inequality (3.14).
This completes the proof. □

Remark 3.7. *Under the assumption of Theorem 3.6, if $p \to q$, we have, using L'Hospital's rule, that*

$$L\big(\kappa_1^{\frac{q-p}{q-1}}, \kappa_2^{\frac{q-p}{q-1}}\big) \to 1.$$

Hence we get the inequality proved in Theorem 3.5.

In this section, we will proceed a similar argument for GG-exponentially convex functions to obtain some new inequalities.

Theorem 3.8. *Let $\psi : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior I° of I, where $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$ and $(e^\psi)' \in L[\kappa_1,\kappa_2]$. If $|(e^\psi)'|$ is GG-convex on*

I, then

$$\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)}dx \ \le \frac{\ln\kappa_2 - \ln\kappa_1}{2}$$
$$\times \ L \ \sqrt{|\kappa_1^2 e^{\psi(\kappa_1)}\psi'(\kappa_1)|}, \sqrt{|\kappa_2^2 e^{\psi(\kappa_2)}\psi'(\kappa_2)|} \qquad \sqrt{|\kappa_1^2 e^{\psi(\kappa_1)}\psi'(\kappa_1)|} + \sqrt{|\kappa_2^2 e^{\psi(\kappa_2)}\psi'(\kappa_2)|} \ .$$

Proof. From Lemma 3.3, using the property of the modulus and GG-convexity of $|(e^{\psi})'|$, we can write

$$\begin{aligned}
&\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)}dx \\
= \ &\frac{(\ln\kappa_2 - \ln\kappa_1)}{2} \int_0^1 \ \kappa_2^{1+\xi}\kappa_1^{1-\xi} \ \ e^{\psi} \ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \ \ \psi' \ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \ \ d\xi \\
&+ \int_0^1 \ \kappa_1^{1-\xi}\kappa_2^{1+\xi} \ \ e^{\psi} \ \kappa_1^{\frac{1-\xi}{2}} \kappa_2^{\frac{1+\xi}{2}} \ \ \psi' \ \kappa_1^{\frac{1-\xi}{2}} \kappa_2^{\frac{1+\xi}{2}} \ \ d\xi \\
= \ &\frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2} \int_0^1 \ \frac{\kappa_2}{\kappa_1} \ ^{\xi} \ e^{\psi} \ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \ \ \psi' \ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \ \ d\xi \\
&+ \int_0^1 \ \frac{\kappa_1}{\kappa_2} \ ^{\xi} \ e^{\psi} \ \kappa_1^{\frac{1-\xi}{2}} b^{\frac{1+\xi}{2}} \ \ \psi' \ \kappa_1^{\frac{1-\xi}{2}} \kappa_2^{\frac{1+\xi}{2}} \ \ d\xi \\
= \ &\frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2} \int_0^1 \ \frac{\kappa_2}{\kappa_1} \ ^{\xi} \ e^{\psi(\kappa_2)}\psi'(\kappa_2) \ ^{\frac{1+\xi}{2}} \ e^{\psi(\kappa_1)}\psi'(\kappa_1) \ ^{\frac{1-\xi}{2}} d\xi \\
&+ \int_0^1 \ \frac{\kappa_1}{\kappa_2} \ ^{\xi} \ e^{\psi(\kappa_1)}\psi'(\kappa_1) \ ^{\frac{1-\xi}{2}} \ e^{\psi(\kappa_2)}\psi'(\kappa_2) \ ^{\frac{1+\xi}{2}} d\xi \\
= \ &\frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2} \int_0^1 \ e^{\psi(\kappa_1)}e^{\psi(\kappa_2)}\psi'(\kappa_1)\psi'(\kappa_2) \ ^{\frac{1}{2}} \ \frac{|\kappa_2^2 e^{\psi(\kappa_2)}\psi'(\kappa_2)|}{|\kappa_1^2 e^{\psi(\kappa_1)}\psi'(\kappa_1)|} \ ^{\frac{\xi}{2}} d\xi \\
&+ \int_0^1 \ e^{\psi(\kappa_1)}e^{\psi(\kappa_2)}\psi'(\kappa_1)\psi'(\kappa_2) \ ^{\frac{1}{2}} \ \frac{|\kappa_1^2 e^{\psi(\kappa_1)}\psi'(\kappa_1)|}{|\kappa_2^2 e^{\psi(\kappa_2)}\psi'(\kappa_2)|} \ ^{\frac{\xi}{2}} d\xi \ .
\end{aligned}$$

If we evaluate the above integral, we get the desired identity. □

Theorem 3.9. *Let $\psi : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a function on the interior I° of I, where $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$ and $(e^{\psi})' \in L[\kappa_1, \kappa_2]$. If $|(e^{\psi})'|^q$ is GG-convex on I for $q > 1$ and $q > p > 0$, then*

$$\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)}dx \ \le \frac{(\ln\kappa_2 - \ln\kappa_1)L^{\frac{1}{p}}(\kappa_1^p, \kappa_2^p)}{2} \tag{3. 19}$$
$$\times \ L^{\frac{1}{q}} \ |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^{\frac{q}{2}}, |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^{\frac{q}{2}} \qquad \sqrt{\kappa_1^2|e^{\psi(\kappa_1)}\psi'(\kappa_1)|} + \sqrt{\kappa_2^2|e^{\psi(\kappa_2)}\psi'(\kappa_2)|} \ .$$

Proof. From Lemma 3.3, using the property of the modulus and GG-convexity of $|(e^{\psi})'|^q$ and Hölder integral inequality, we can write

$$
\begin{aligned}
& \kappa_1 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)} dx \\
= \ & \frac{\ln\kappa_2 - \ln\kappa_1}{2} \int_0^1 \kappa_2^{1+\xi}\kappa_1^{1-\xi} \ e^{\psi} \ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \ \psi' \ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \ d\xi \\
& + \int_0^1 \kappa_1^{1-\xi}\kappa_2^{1+\xi} \ e^{\psi} \ \kappa_1^{\frac{1-\xi}{2}} \kappa_2^{\frac{1+\xi}{2}} \ \psi' \ \kappa_1^{\frac{1-\xi}{2}} \kappa_2^{\frac{1+\xi}{2}} \ d\xi \\
\leq \ & \frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2} \ \int_0^1 \ \frac{\kappa_2}{\kappa_1} \ ^{p\xi} d\xi \ ^{\frac{1}{p}} \ \int_0^1 e^{\psi} \ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \ \psi' \ \kappa_2^{\frac{1+\xi}{2}} \kappa_1^{\frac{1-\xi}{2}} \ ^{q} d\xi \ ^{\frac{1}{q}} \\
& + \ \int_0^1 \ \frac{\kappa_1}{\kappa_2} \ ^{p\xi} d\xi \ ^{\frac{1}{p}} \ \int_0^1 e^{\psi} \ \kappa_1^{\frac{1-\xi}{2}} \kappa_2^{\frac{1+\xi}{2}} \ \psi' \ \kappa_1^{\frac{1-\xi}{2}} \kappa_2^{\frac{1+\xi}{2}} \ ^{q} d\xi \ ^{\frac{1}{q}} \\
= \ & \frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2} \ \frac{L^{\frac{1}{p}}(\kappa_1^p, \kappa_2^p)}{\kappa_1} \ \int_0^1 e^{\psi(\kappa_2)}\psi'(\kappa_2) \ ^{\frac{q(1+\xi)}{2}} \ e^{\psi(\kappa_1)}\psi'(\kappa_1) \ ^{\frac{q(1-\xi)}{2}} d\xi \ ^{\frac{1}{q}} \\
& + \frac{L^{\frac{1}{p}}(\kappa_1^p, \kappa_2^p)}{\kappa_2} \ \int_0^1 e^{\psi(\kappa_1)}\psi'(\kappa_1) \ ^{\frac{q(1-\xi)}{2}} \ e^{\psi(\kappa_2)}\psi'(\kappa_2) \ ^{\frac{q(1+\xi)}{2}} d\xi \ ^{\frac{1}{q}} \\
= \ & \frac{(\ln\kappa_2 - \ln\kappa_1)L^{\frac{1}{p}}(\kappa_1^p, \kappa_2^p)}{2} \ \kappa_2 \ \int_0^1 |e^{\psi(\kappa_1)}\psi'(\kappa_1)e^{\psi(\kappa_2)}\psi'(\kappa_2)|^{\frac{q}{2}} \ \frac{e^{\psi(\kappa_2)}\psi'(\kappa_2)}{e^{\psi(\kappa_1)}\psi'(\kappa_1)} \ ^{\frac{q\xi}{2}} d\xi \ ^{\frac{1}{q}} \\
& + \kappa_1 \ \int_0^1 |e^{\psi(\kappa_1)}\psi'(\kappa_1)e^{\psi(\kappa_2)}\psi'(\kappa_2)|^{\frac{q}{2}} \ \frac{e^{\psi(\kappa_2)}\psi'(\kappa_2)}{e^{\psi(\kappa_1)}\psi'(\kappa_1)} \ ^{\frac{q\xi}{2}} d\xi \ ^{\frac{1}{q}} \ .
\end{aligned}
$$

If we evaluate the above integral, we get the desired inequality. □

Theorem 3.10. *Let $\psi : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior I° of I, where $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$ and $(e^{\psi})' \in L[\kappa_1, \kappa_2]$. If $|(e^{\psi})'|^q$ is GG-convex on I for $q > 1$ and $q > p > 0$, then*

$$
\begin{aligned}
& \kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)} dx \\
\leq \ & \frac{\ln\kappa_2 - \ln\kappa_1}{2} L^{\frac{1}{q}} \ |\kappa_1^2 e^{\psi(\kappa_1)}\psi'(\kappa_1)|, |\kappa_2^2 e^{\psi(\kappa_2)}\psi'(\kappa_2)| \ \ \kappa_1 |e^{\psi(\kappa_1)}\psi'(\kappa_1)|^{\frac{q}{2}} + \kappa_2 |e^{\psi(\kappa_2)}\psi'(\kappa_2)|^{\frac{q}{2}} \ .
\end{aligned}
$$

Proof. From Lemma 3.3, using the property of the modulus and GG-convexity of $|(e^{\psi})'|^q$ and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\kappa_2 e^{\psi(\kappa_2)} - \kappa_1 e^{\psi(\kappa_1)} - \int_{\kappa_1}^{\kappa_2} e^{\psi(x)} dx\right| \\
= &\frac{\ln\kappa_2 - \ln\kappa_1}{2}\Bigg[\int_0^1 \left(\kappa_2^{1+\xi}\kappa_1^{1-\xi}\right)\left|e^{\psi\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right)}\psi'\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right)\right| d\xi \\
&+\int_0^1 \left(\kappa_1^{1-\xi}\kappa_2^{1+\xi}\right)\left|e^{\psi\left(\kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}}\right)}\psi'\left(\kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}}\right)\right| d\xi\Bigg] \\
\le &\frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2}\Bigg[\left(\int_0^1 1 d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{\kappa_2}{\kappa_1}\right)^{q\xi}\left|e^{\psi\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right)}\psi'\left(\kappa_2^{\frac{1+\xi}{2}}\kappa_1^{\frac{1-\xi}{2}}\right)\right|^q d\xi\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 1 d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{\kappa_1}{\kappa_2}\right)^{q\xi}\left|e^{\psi\left(\kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}}\right)}\psi'\left(\kappa_1^{\frac{1-\xi}{2}}\kappa_2^{\frac{1+\xi}{2}}\right)\right|^q d\xi\right)^{\frac{1}{q}}\Bigg] \\
= &\frac{\kappa_1\kappa_2(\ln\kappa_2 - \ln\kappa_1)}{2}\Bigg[\left(\int_0^1 |e^{\psi(\kappa_1)}\psi'(\kappa_1)e^{\psi(\kappa_2)}\psi'(\kappa_2)|^{\frac{q}{2}}\left|\frac{\kappa_2^2 e^{\psi(\kappa_2)}\psi'(\kappa_2)}{\kappa_1^2 e^{\psi(\kappa_1)}\psi'(\kappa_1)}\right|^{\frac{q\xi}{2}} d\xi\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |e^{\psi(\kappa_1)}\psi'(\kappa_1)e^{\psi(\kappa_2)}\psi'(\kappa_2)|^{\frac{q}{2}}\left|\frac{\kappa_1^2 e^{\psi(\kappa_1)}\psi'(\kappa_1)}{\kappa_2^2 e^{\psi(\kappa_2)}\psi'(\kappa_2)}\right|^{\frac{q\xi}{2}} d\xi\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

If we evaluate the above integral, we get the desired result. □

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