Pythagorean $m$-polar Fuzzy Soft Sets with TOPSIS Method for MCGDM

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Abstract. In this article, we introduce the concept of Pythagorean $m$-polar fuzzy soft sets ($P_m$FSSs). This set reduces to Pythagorean fuzzy soft set for $m = 1$. We define algebraic operations and some characteristics of $P_m$FSSs. We define some linguistic terms utilizing the notion of product of $P_m$FSSs ($\otimes$) by assigning different numeric values to the constant $k \in [0, \infty]$ and present an illustration to determine whether the traits of being well-dressed and attractive personality is possessed by a person or not and up to what extent. We present an application of $P_m$FSSs in multi-criteria group decision making (MCGDM) problem of appraisal of employee for promotion making use of the well-distinguished tool TOPSIS.

AMS (MSC): 54A05, 54A40, 11B05, 90D50, 03E72  
Key Words: Pythagorean $m$-polar fuzzy soft sets, Algebraic operations on $P_m$FSSs,  
Linguistic terms associated with $P_m$FSSs, MCGDM, TOPSIS.
1. Introduction

Logic and set theory are believed to be the foundation stones of modern mathematics. A meticulous exploration of set theory belongs to the brass tacks of mathematical logic. Indeed, these two notions are interrelated for many logical operations like $\lor$, $\land$, $\neg$, $\oplus$, $\ominus$, $\to$ and $\iff$ etc. decode into set theory and vice versa. On the same token; relations, functions, paradoxes, numbers, probability theory, algebra and modern measure theory etc. all rely on theory of sets. Logicians have investigated set theory in excessive details, articulating an assortment of axioms that affords an all-encompassing and sufficiently strong footing for mathematical reasoning. The customary system of axiomatic set theory is Zermelo-Fraenkel set theory, along with the axioms of choice. Each axiom incorporated in this theory states a property of sets which is far and wide acknowledged around the world by mathematicians. Beyond its foundational role, modern set theory owns a gigantic number of full of zip researchers.

At the primary phases of the development of set theory (traditionally accredited as classical set theory, developed by Cantor and Dedekind), a constituent was understood to be either a member of some given set or not. In other terms, a characteristic function is associated with each set which assigns a value of 1 to the element if it is member of the set and 0 otherwise. There was not any other moderate option for an element regarding belongingness to a set. Zadeh [60], in the second half of 20th century, made public the perception of a new species of sets famous as fuzzy set by coupling a membership function with each member of the set whose values range from 0 to 1. According to Zadeh, an element may belong partially to a set. Besides exploring different mathematical operations on fuzzy sets Zadeh also introduced the concept of a linguistic variable and its application to approximate reasoning [61]. The philosophies commenced by Zadeh proved like a revolution in the ecosphere of dynamic mathematicians.

Subsequent upon the model rendered by Zadeh, mathematicians around the globe began meditative upon other sort of sets. Atanassov believed that if there exists a membership function to measure uncertainty against a set, then there must be a non-membership function associated with each element of that set too. Ensuing his work, Atanassov presented [15, 16] a fresh category of sets entitled intuitionistic fuzzy sets (IF-sets) by striking the constraint that the values of membership and its counterpart non-membership functions not only lie from 0 to 1 but their sum also must fall in the same interval. At the end of 20th century, Molodstov [34] furnished a parameterized collection of sets, known as soft sets. Soft sets acquiescently designate a number of attributes for clarifying and reconnoitering a problem holding ambiguity and uncertainty. Molodstov also rendered some useful applications from everyday life. The soft set theory discovers varied range of applications in management economics, medical sciences, engineering, and social sciences predominantly due to its tractability without restraints on imprecise description of the situation. With the study of innovative set structures, the science of set theory took a fresh twist. Increasingly hybrid set structures gave the impression on the canvas. Yager [57]-[59], by modifying the constraint on the parameters, presented the notion of
Pythagorean fuzzy sets (PF-sets) as an extension of IF-sets. Peng et al. [39]-[43] studied some results for PF-sets along with their applications. Naz et al. [37] employed the idea on Pythagorean fuzzy graphs and presented some decision making applications. Olgun et al. [38] introduced the notion of Pythagorean fuzzy topological spaces employing the notion of fuzzy topological spaces. Qamar and Hassan [1] presented an approach toward a Q-neutrosophic soft set and its application in decision making. Qamar and Hassan [2] also presented Q-neutrosophic soft relation and its application in decision making.


TOPSIS, initially presented by Hwang and Yoon [28], is a useful model for tackling DMPs of the real world. This technique assists the decision makers to reach at some final decision and analyzing the conclusion in a system manner without any partiality. A number of varied versions of TOPSIS including adjusted TOPSIS, extended TOPSIS and modified TOPSIS may be found in literature. In recent years, this technique has been successfully applied in the fields of medical sciences, water management, business, transportation analysis, quality control, human resources management, and product design etc. which may be found in literature.

The prime ambition behind this research is to extend the notion of Pythagorean $m$-polar fuzzy sets (PmFSSs) presented by Naeem et al. [36] to Pythagorean $m$-polar fuzzy soft sets (PmFSSs) along with algebraic operations on these sets and explore some idiosyncratic characteristics of this hybrid structure. PmFSSs have natural applications in multiple-valued logic, multi-sensor, multi-source and multi-process information fusion. PmFSSs provide a strong mathematical model to take in hand MCGDM problems. In order to tackle real world problems where intuitionistic fuzzy soft sets cannot deal with the situation when sum of membership and non-membership degrees of the parameter exceeds 1 making MCGDM demarcated and hence affecting the optimum decision, PmFSSs do not leave us alone and unassisted. PmFSSs provide a large number of applications to MCGDM problems in artificial intelligence, image processing, medical diagnosis, forecasting, recruitment problems and many other real life problems.

The article is prescribed as follows: In Section 1, account of different sorts of sets along with decision making technique of TOPSIS is presented with brevity. Section 2 is devoted to cover concise but comprehensive definitions of different sorts of sets that would be assisting in remnant part of the paper. The next segment i.e. Section 3 of this article serves as the main organ of this paper. In this section, we present the notion of PmFSSs along with associated mathematical operations and related results on these sets. In the very next segment of this article i.e. in Section 4, we exhibited how PmFSSs may be utilized in handling everyday problems using TOPSIS method. We ended with a concrete conclusion and some future directions in Section 5.
2. Preliminaries

In this section, we call to mind some fundamentals of different kinds of sets with brevity that would be ready to lend a hand in the remnant part of this article.

Definition 2.1. [60] Presume that $X$ is a non-void set of objects. A fuzzy set $A$ in $X$ comprises ordered doublets in which abscissa is member of $X$ and the ordinate is a mapping (termed as membership function of fuzzy set $A$) that drags elements of $X$ to the unit closed interval $[0,1]$.

Definition 2.2. [16] An intuitionistic fuzzy set (IF-set in brief) over the underlying set $X$ is defined as

$$A = \{< \zeta, \mu_A(\zeta), \nu_A(\zeta) > : \zeta \in X\}$$

The mappings $\mu_A$ and $\nu_A$ in order are acknowledged as the degrees of membership and non-membership of the element $\zeta \in X$ to the set $A$ and send elements of $X$ to unit closed interval along with the constraint that their sum must not exceed unity.

Definition 2.3. [57] A Pythagorean fuzzy set, abbreviated as PF-set, over $X$ is a family of the form

$$\mathbb{P} = \{< \zeta, \mu_P(\zeta), \nu_P(\zeta) > : \zeta \in X\}$$

where $\mu_P$ and $\nu_P$ are mappings from some crisp set $X$ to the unit closed interval with the restriction that sum of their squared values should not exceed unity i.e. $0 \leq \mu_P^2(\zeta) + \nu_P^2(\zeta) \leq 1$, called correspondingly the grade of association and non-association of $\zeta \in X$ to the set $\mathbb{P}$. The pair $(\mu_P, \nu_P)$ is called Pythagorean fuzzy number (PFN). The number $\gamma_P(\zeta) = \sqrt{1 - \mu_P^2(\zeta) - \nu_P^2(\zeta)}$ is called the hesitation margin.

Definition 2.4. [17] An $m$-polar fuzzy set on the reference set $X$ is characterized by a mapping $\mathfrak{A} : X \mapsto [0,1]^m$, where $m$ is any arbitrary natural number.

Definition 2.5. [34] Let $X$ be a reference set and $E$ a non-void collection of attributes with $A \subseteq E$. A soft set is a parameterized collection designated as $(\psi, A)$ where $\psi$ is a map that drives elements of $A$ to the power set $2^X$ of $X$.

Definition 2.6. [36] Assume that $m$ is a natural number. A Pythagorean m-polar fuzzy set (a $\text{PMFS}$ for short) $\mathcal{P}$ over an underlying set $X$ is characterized by two mappings $\mu_P^{(i)} : X \mapsto [0,1]$ (traditionally acknowledged membership functions) and $\nu_P^{(i)} : X \mapsto [0,1]$ (conventionally called non-membership functions) with the constraint that sum of their squared values should not exceed unity i.e. $0 \leq \left(\mu_P^{(i)}(\zeta)\right)^2 + \left(\nu_P^{(i)}(\zeta)\right)^2 \leq 1$, for $i = 1, 2, \cdots, m$.

A $\text{PMFS}$ may be expressed in set-builder notation as

$$\mathcal{P} = \{< \zeta, (\mu_P^{(i)}(\zeta), \nu_P^{(i)}(\zeta)) > : \zeta \in X; i = 1, 2, \cdots, m\}$$

where

$$(\mu_P^{(i)}(\zeta), \nu_P^{(i)}(\zeta)) = \left((\mu_P^{(1)}(\zeta), \nu_P^{(2)}(\zeta)), (\mu_P^{(2)}(\zeta), \nu_P^{(2)}(\zeta)), \cdots, (\mu_P^{(m)}(\zeta), \nu_P^{(m)}(\zeta))\right)$$
3. Pythagorean $m$-polar Fuzzy Soft Sets

We devote this section to introduce novel concepts of a new hybrid structure Pythagorean $m$-polar fuzzy soft sets.

**Definition 3.1.** Assume that $m$ is a natural number. For some non-void collection of attributes $E$, let $A = \{e_1, e_2, \cdots, e_n\}$ be a subset of $E$ i.e $A \subseteq E$. A Pythagorean $m$-polar fuzzy soft set (a PmFSS for short) $\psi_A$ over an underlying set $X$ is characterized by the mapping $\psi: A \mapsto \text{PmFSS}(X)$, where PmFSS$(X)$ denotes the collection of all Pythagorean $m$-polar fuzzy soft sets over $X$.

A PmFSS may be expressed in set-builder notation as

$$\psi_A = \left\{ \left( e, \zeta, \left( \mu^{(i)}_p(e)(\zeta), \nu^{(i)}_p(e)(\zeta) \right) \right) : e \in A, \zeta \in X ; i = 1, 2, \cdots, m \right\}$$

or more conveniently as

$$\psi_A = \left\{ e, \left\{ \zeta, \left( \mu^{(i)}_p(e)(\zeta), \nu^{(i)}_p(e)(\zeta) \right) \right\} : e \in A, \zeta \in X \right\}$$

$$= \left\{ e, \left\{ \zeta, \left( \mu^{(i)}_p(e)(\zeta) \right) \right\} : e \in A, \zeta \in X ; i = 1, 2, \cdots, m \right\}$$

If cardinality of $X$ is $k$, then tabular formation of $\psi_A$ is

<table>
<thead>
<tr>
<th>$\psi_A$</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>\cdots</th>
<th>$e_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta_1$</td>
<td>$\mu^{(i)}_p(e_1)(\zeta_1), \nu^{(i)}_p(e_1)(\zeta_1)$</td>
<td>$\mu^{(i)}_p(e_2)(\zeta_1), \nu^{(i)}_p(e_2)(\zeta_1)$</td>
<td>\cdots</td>
<td>$\mu^{(i)}_p(e_n)(\zeta_1), \nu^{(i)}_p(e_n)(\zeta_1)$</td>
</tr>
<tr>
<td>$\zeta_2$</td>
<td>$\mu^{(i)}_p(e_1)(\zeta_2), \nu^{(i)}_p(e_1)(\zeta_2)$</td>
<td>$\mu^{(i)}_p(e_2)(\zeta_2), \nu^{(i)}_p(e_2)(\zeta_2)$</td>
<td>\cdots</td>
<td>$\mu^{(i)}_p(e_n)(\zeta_2), \nu^{(i)}_p(e_n)(\zeta_2)$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ddots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$\zeta_k$</td>
<td>$\mu^{(i)}_p(e_1)(\zeta_k), \nu^{(i)}_p(e_1)(\zeta_k)$</td>
<td>$\mu^{(i)}_p(e_2)(\zeta_k), \nu^{(i)}_p(e_2)(\zeta_k)$</td>
<td>\cdots</td>
<td>$\mu^{(i)}_p(e_n)(\zeta_k), \nu^{(i)}_p(e_n)(\zeta_k)$</td>
</tr>
</tbody>
</table>

and in matrix format as

$$\psi_A = \begin{bmatrix}
\mu^{(i)}_p(e_1)(\zeta_1), \nu^{(i)}_p(e_1)(\zeta_1) & \mu^{(i)}_p(e_2)(\zeta_1), \nu^{(i)}_p(e_2)(\zeta_1) & \cdots & \mu^{(i)}_p(e_n)(\zeta_1), \nu^{(i)}_p(e_n)(\zeta_1) \\
\mu^{(i)}_p(e_1)(\zeta_2), \nu^{(i)}_p(e_1)(\zeta_2) & \mu^{(i)}_p(e_2)(\zeta_2), \nu^{(i)}_p(e_2)(\zeta_2) & \cdots & \mu^{(i)}_p(e_n)(\zeta_2), \nu^{(i)}_p(e_n)(\zeta_2) \\
\vdots & \vdots & \ddots & \vdots \\
\mu^{(i)}_p(e_1)(\zeta_k), \nu^{(i)}_p(e_1)(\zeta_k) & \mu^{(i)}_p(e_2)(\zeta_k), \nu^{(i)}_p(e_2)(\zeta_k) & \cdots & \mu^{(i)}_p(e_n)(\zeta_k), \nu^{(i)}_p(e_n)(\zeta_k)
\end{bmatrix}$$

This $k \times n$ matrix is reckoned as $\text{PmFSS}$-matrix. The collection of all PmFSSs defined over $X$ will be designated by $\text{PmFSS}(X)$.

**Example 3.2.** Let $X = \{b, s, c, z\}$ be a crisp set and $A = \{e_1, e_2\} \subseteq E$, then

$$\psi_A = \left\{ \begin{array}{l}
\psi_1 = \left( e_1, \left\{ \begin{array}{l}
\mu_1(0.19,0.74),\nu_1(0.28,0.79);(0.04,0.97) \cdot (0.38,0.62);(0.74,0.36);(0.88,0.31) \\
\mu_1(0.19,0.74),\nu_1(0.28,0.79);(0.04,0.97) \cdot (0.38,0.62);(0.74,0.36);(0.88,0.31)
\end{array} \right\} \right)
\end{array} \right\}$$

is a P3FSS. In tabular array, we may represent this set as shown in Table 1:
Let $\psi_A$ be a $P_m$-FSS over $X$ with $e \in A \subseteq E$. The aggregate of those points $\zeta$ of $X$ for which $\mu_P^{(i)}(e)(\zeta) \neq 0$ or $\nu_P^{(i)}(e)(\zeta) \neq 1$, for at least one $i = 1, 2, \cdots, m$, is called support of $\psi_A$ i.e.

$$\text{supp}(\psi_A) = \{\zeta \in X : \mu_P^{(i)}(e)(\zeta) \neq 0 \text{ or } \nu_P^{(i)}(e)(\zeta) \neq 1 \text{ for at least one } i = 1, 2, \cdots, m\}$$

**Example 3.4.** For the $P_m$FSS represented in Table 3, defined over $X = \{t, b, j, k\}$, $\text{supp}(\psi_A) = \{t, b, k\}$.

<table>
<thead>
<tr>
<th>$\psi_A$</th>
<th>$e_1$</th>
<th>$e_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>${(0.19, 0.74), (0.28, 0.79), (0.04, 0.97)}$</td>
<td>${(0.62, 0.28), (0.59, 0.47), (0.26, 0.11)}$</td>
</tr>
<tr>
<td>$s$</td>
<td>${(0.38, 0.62), (0.74, 0.36), (0.88, 0.31)}$</td>
<td>${(0.37, 0.69), (0.01, 0.58), (0.72, 0.71)}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
</tr>
<tr>
<td>$z$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
</tr>
</tbody>
</table>

**Table 1.** Tabular representation of $\psi_A$

The matrix form of $\psi_A$ is

$$\psi_A = \begin{bmatrix}
(0.19, 0.74), (0.28, 0.79), (0.04, 0.97) & (0.62, 0.28), (0.59, 0.47), (0.26, 0.11) \\
(0.38, 0.62), (0.74, 0.36), (0.88, 0.31) & (0.37, 0.69), (0.01, 0.58), (0.72, 0.71) \\
(0, 1), (0, 1), (0, 1) & (0, 1), (0, 1), (0, 1) \\
(0, 1), (0, 1), (0, 1) & (0, 1), (0, 1), (0, 1)
\end{bmatrix}$$

**Definition 3.3.** Let $\psi_A$ be a $P_m$FSS over $X$ with $e \in A \subseteq E$. The aggregate of those points $\zeta$ of $X$ for which $\mu_P^{(i)}(e)(\zeta) = 1$ (and obviously $\nu_P^{(i)}(e)(\zeta) = 0$), for at least one $i = 1, 2, \cdots, m$, is called core of $\psi_A$ i.e.

$$\text{core}(\psi_A) = \{\zeta \in X : \mu_P^{(i)}(e)(\zeta) = 1 \text{ for at least one } i = 1, 2, \cdots, m\}$$

**Table 2.** Brief tabular representation of $\psi_A$

<table>
<thead>
<tr>
<th>$\psi_A$</th>
<th>$e_1$</th>
<th>$e_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>${(0.19, 0.74), (0.28, 0.79), (0.04, 0.97)}$</td>
<td>${(0.62, 0.28), (0.59, 0.47), (0.26, 0.11)}$</td>
</tr>
<tr>
<td>$s$</td>
<td>${(0.38, 0.62), (0.74, 0.36), (0.88, 0.31)}$</td>
<td>${(0.37, 0.69), (0.01, 0.58), (0.72, 0.71)}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
</tr>
<tr>
<td>$z$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
</tr>
</tbody>
</table>

**Table 3.** $P_m$FSS $\psi_A$

**Definition 3.5.** Let $\psi_A$ be a $P_m$FSS over $X$ with $e \in A \subseteq E$. The aggregate of those points $\zeta$ of $X$ for which $\mu_P^{(i)}(e)(\zeta) = 1$ (and obviously $\nu_P^{(i)}(e)(\zeta) = 0$), for at least one $i = 1, 2, \cdots, m$, is called core of $\psi_A$ i.e.

$$\text{core}(\psi_A) = \{\zeta \in X : \mu_P^{(i)}(e)(\zeta) = 1 \text{ for at least one } i = 1, 2, \cdots, m\}.$$
Example 3.6. For the PmFSS $\psi_A$ given in Example 3.4, core($\psi_A$) = \{b\}.

Definition 3.7. Let $\psi_A$ be a PmFSS over $X$ with $e \in \mathcal{A} \subseteq \mathcal{E}$. The maximum value attained by the membership function $\mu_{\mathcal{E}}(\mathcal{A})$, for any $\zeta \in \mathcal{X}$ and any $i \in \{1, 2, \cdots, m\}$, is termed as height of $\psi_A$ and is designated as $ht(\psi_A)$. A PmFSS $\psi_A$ is said to be normal if $ht(\psi_A) = 1$ and is reckoned as subnormal otherwise.

Example 3.8. For the PmFSS $\psi_A$ given in Example 3.2, $ht(\psi_A) = 0.88$ and for the PmFSS $\psi_A$ given in Example 3.4, $ht(\psi_A) = 1$. Hence, the PmFSS $\psi_A$ given in Example 3.4 is normal whereas the PmFSS $\psi_A$ given in Example 3.2 is subnormal.

Definition 3.9. Let $(\psi_1, A_1)$ and $(\psi_2, A_2)$ be PmFSSs over $X$ with $A_1, A_2 \subseteq \mathcal{E}$. We say that $(\psi_1, A_1)$ is a subset of $(\psi_2, A_2)$, written $(\psi_1, A_1) \subseteq (\psi_2, A_2)$ if

i. $A_1 \subseteq A_2$

ii. $\mu_{1}(\mathcal{A})(\zeta) \leq \mu_{2}(\mathcal{A})(\zeta)$, and

iii. $\nu_{1}(\mathcal{A}) (\zeta) \geq \nu_{2}(\mathcal{A}) (\zeta)$

for all $e \in A_1, \zeta \in X$ and all admissible values of $i$.

$(\psi_1, A_1)$ and $(\psi_2, A_2)$ are said to be equal if and only if one of them is sandwiched between the other i.e. $(\psi_1, A_1) \subseteq (\psi_2, A_2) \subseteq (\psi_1, A_1)$.

Example 3.10. Let $A_1 = \{e_1\}, A_2 = \{e_1, e_2\} \subseteq \mathcal{E}$ and $(\psi_1, A_1), (\psi_2, A_2)$ be PmFSSs, given in Tables 4 and 5 respectively, over some set $X = \{g, r, p\}$, then $(\psi_1, A_1) \subseteq (\psi_2, A_2)$.

<table>
<thead>
<tr>
<th>$(\psi_1, A_1)$</th>
<th>$e_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>{(0.29, 0.16), (0.43, 0.51), (0.33, 0.16)}</td>
</tr>
<tr>
<td>$r$</td>
<td>{(0.29, 0.42), (0.04, 0.86), (0.32, 0.24)}</td>
</tr>
<tr>
<td>$p$</td>
<td>{(0.26, 0.07), (0.21, 0.19), (0.00, 1.00)}</td>
</tr>
<tr>
<td><strong>Table 4. PmFSS $(\psi_1, A_1)$</strong></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(\psi_2, A_2)$</th>
<th>$e_1$</th>
<th>$e_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>{(0.34, 0.11), (0.78, 0.37), (0.33, 0.10)}</td>
<td>{(0.36, 0.58), (0.30, 0.63), (0.52, 0.44)}</td>
</tr>
<tr>
<td>$r$</td>
<td>{(0.47, 0.30), (0.19, 0.52), (0.49, 0.24)}</td>
<td>{(0.11, 0.19), (0.28, 0.74), (0.49, 0.50)}</td>
</tr>
<tr>
<td>$p$</td>
<td>{(0.50, 0.02), (0.36, 0.13), (0.82, 0.26)}</td>
<td>{(0.83, 0.21), (0.44, 0.79), (0.69, 0.29)}</td>
</tr>
<tr>
<td><strong>Table 5. PmFSS $(\psi_2, A_2)$</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Remark. If $(\psi_1, A_1)$ and $(\psi_2, A_2)$ are PmFSSs over $X$, then $(\psi_1, A_1) \subseteq (\psi_2, A_2)$ implies $ht(\psi_1, A_1) \leq ht(\psi_2, A_2)$. The converse, however, may not hold.

Definition 3.11. A PmFSS $(\psi, E)$ over $X$ is said to be a null PmFSS if $\mu_{1}(\mathcal{E}) (\zeta) = 0$ and $\nu_{1}(\mathcal{E}) (\zeta) = 1$, for all $e \in \mathcal{E}, \zeta \in X$ and all admissible values of $i$. It is denoted by $(\Phi, E)$ or $\Phi_E$. The tabular representation of $\Phi_E$ is as given in Table 6.
Notice that both support and core of \( \Phi_E \) are empty set. Further, height of \( \Phi_E \) is 0 and hence \( \Phi_E \) is a subnormal \( Pm\) FSS.

**Definition 3.12.** A \( Pm\) FSS \((\psi, E)\) over \( X\) is said to be an absolute \( Pm\) FSS if 
\[
\mu_\psi^{(i)}(e)(\zeta) = 1 \quad \text{and} \quad \nu_\psi^{(i)}(e)(\zeta) = 0, \quad \text{for all} \quad e \in E, \; \zeta \in X \quad \text{and all admissible values of} \quad i.
\]
It is denoted by \((\tilde{X}, E)\) or \(\tilde{X}_E\). The tabular representation of \(\tilde{X}_E\) is as given in Table 7.

Notice that both support and core of \(\tilde{X}_E\) are \( X \). Further, height of \(\tilde{X}_E\) is 1 and hence \(\tilde{X}_E\) is a normal \(Pm\) FSS.

**Proposition 3.13.** If \((\psi, E)\) is any \(Pm\) FSS over \( X\), then \((\Phi, E) \preceq (\psi, E) \preceq (\tilde{X}, E)\).

**Proof.** Straight forward. \(\square\)

**Remark.** It follows from Proposition 3.13 that \((\Phi, E)\) is the smallest and \((\tilde{X}, E)\) is the largest \(Pm\) FSS over \( X\).

**Definition 3.14.** The complement of a \(Pm\) FSS 
\[
\psi_E^c = \left\{ e, \left\{ \zeta, \left( \mu_\psi^{(i)}(e)(\zeta), \nu_\psi^{(i)}(e)(\zeta) \right) \right\} : e \in E, \zeta \in X; i = 1, 2, \cdots, m \right\}
\]

over \( X\) is defined as 
\[
\psi_E^c = \left\{ e, \left\{ \zeta, \left( \nu_\psi^{(i)}(e)(\zeta), \mu_\psi^{(i)}(e)(\zeta) \right) \right\} : e \in E, \zeta \in X; i = 1, 2, \cdots, m \right\}
\]
Notice that \( \Phi_E = \tilde{X}_E \) and \( \tilde{X}_E^c = \Phi_E \). Moreover, \((\psi_E^c)^c = \psi_E\).
Example 3.15. For the PmFSS given in Example 3.2, the complement of $\psi_A$ is as given in Table 8:

<table>
<thead>
<tr>
<th>$\psi_A^c$</th>
<th>$e_1$</th>
<th>$e_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$(0.74, 0.19), (0.79, 0.28), (0.97, 0.04)$</td>
<td>$(0.28, 0.62), (0.47, 0.59), (0.11, 0.26)$</td>
</tr>
<tr>
<td>$s$</td>
<td>$(0.62, 0.38), (0.36, 0.74), (0.31, 0.88)$</td>
<td>$(0.69, 0.37), (0.58, 0.01), (0.71, 0.72)$</td>
</tr>
<tr>
<td>$c$</td>
<td>$(1, 0), (1, 0), (1, 0)$</td>
<td>$(1, 0), (1, 0), (1, 0)$</td>
</tr>
<tr>
<td>$z$</td>
<td>$(1, 0), (1, 0), (1, 0)$</td>
<td>$(1, 0), (1, 0), (1, 0)$</td>
</tr>
</tbody>
</table>

Table 8. PmFSS $\psi_A^c$

Definition 3.16. The **union** of two PmFSSs $(\psi_1, A_1)$ and $(\psi_2, A_2)$ defined over the same universe $X$ is defined as

$$(\psi_1, A_1) \cup (\psi_2, A_2) = \left\{ e_i \left\{ \max \mu_{\psi_1}^{(i)}(e)(\zeta), \mu_{\psi_2}^{(i)}(e)(\zeta), \min \mu_{\psi_1}^{(i)}(e)(\zeta), \mu_{\psi_2}^{(i)}(e)(\zeta) \right\} : e \in A_1 \cup A_2, \zeta \in X; i = 1, 2, \ldots, m \right\}$$

Definition 3.17. The **intersection** of two PmFSSs $(\psi_1, A_1)$ and $(\psi_2, A_2)$ defined over the same universe $X$ is defined as

$$(\psi_1, A_1) \cap (\psi_2, A_2) = \left\{ e_i \left\{ \min \mu_{\psi_1}^{(i)}(e)(\zeta), \mu_{\psi_2}^{(i)}(e)(\zeta), \max \mu_{\psi_1}^{(i)}(e)(\zeta), \mu_{\psi_2}^{(i)}(e)(\zeta) \right\} : e \in A_1 \cap A_2, \zeta \in X; i = 1, 2, \ldots, m \right\}$$

Example 3.18. Let $X = \{y, d, g\}$ be a crisp set and $A_1 = \{e_1, e_2\}, A_2 = \{e_2, e_3\} \subseteq E$. Let $(\psi_1, A_1)$ and $(\psi_2, A_2)$ be as given in Tables 9 and 10, respectively.

<table>
<thead>
<tr>
<th>$(\psi_1, A_1)$</th>
<th>$e_1$</th>
<th>$e_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$(0.62, 0.17), (0.31, 0.82), (0.12, 0.06)$</td>
<td>$(0.10, 0.53), (0.84, 0.36), (0.13, 0.14)$</td>
</tr>
<tr>
<td>$d$</td>
<td>$(0.02, 0.28), (0.16, 0.39), (0.30, 0.80)$</td>
<td>$(0.29, 0.54), (0.36, 0.11), (0.03, 0.99)$</td>
</tr>
<tr>
<td>$g$</td>
<td>$(0.51, 0.52), (0.39, 0.42), (0.52, 0.53)$</td>
<td>$(0.00, 1.00), (0.38, 0.62), (0.81, 0.26)$</td>
</tr>
</tbody>
</table>

Table 9. PmFSS $(\psi_1, A_1)$

<table>
<thead>
<tr>
<th>$(\psi_2, A_2)$</th>
<th>$e_2$</th>
<th>$e_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$(0.54, 0.11), (0.14, 0.15), (0.81, 0.17)$</td>
<td>$(0.44, 0.24), (0.43, 0.36), (0.11, 0.31)$</td>
</tr>
<tr>
<td>$d$</td>
<td>$(0.29, 0.56), (0.18, 0.05), (0.26, 0.67)$</td>
<td>$(0.24, 0.22), (0.38, 0.35), (0.40, 0.63)$</td>
</tr>
<tr>
<td>$g$</td>
<td>$(0.29, 0.46), (0.37, 0.37), (0.51, 0.02)$</td>
<td>$(0.25, 0.83), (0.52, 0.58), (1.00, 0.00)$</td>
</tr>
</tbody>
</table>

Table 10. PmFSS $(\psi_2, A_2)$

Then union and intersection of $(\psi_1, A_1)$ and $(\psi_2, A_2)$ are represented in Tables 11 and 12, respectively.
Proposition 3.19. If $(\psi, A), (\psi_1, A_1), (\psi_2, A_2)$ and $(\psi_3, A_3)$ are PmFSSs over $X$, then

(i) $(\Phi, A)\bigcap(\psi, A) = (\psi, A)$.
(ii) $(\Phi, A)\bigcap(\psi, A) = (\Phi, A)$.
(iii) $(X, E)\bigcap(\psi, A) = (X, E)$.
(iv) $(X, E)\bigcap(\psi, A) = (\psi, A)$.
(v) $(\psi, A)\bigcup(\psi, A) = (\psi, A)$.
(vi) $(\psi, A)\bigcap(\psi, A) = (\psi, A)$.
(vii) $(\psi, A_1)\bigcup(\psi_2, A_2) = (\psi_2, A_2)\bigcap(\psi_1, A_1)$.
(viii) $(\psi, A_1)\bigcap(\psi_2, A_2) = (\psi_2, A_2)\bigcap(\psi_1, A_1)$.
(ix) $(\psi_1, A_1)\bigcup(\psi_2, A_2)\bigcup(\psi_3, A_3) = \{(\psi_1, A_1)\bigcup(\psi_2, A_2)\}\bigcup(\psi_3, A_3)$.
(x) $(\psi_1, A_1)\bigcap(\psi_2, A_2)\bigcap(\psi_3, A_3) = \{(\psi_1, A_1)\bigcap(\psi_2, A_2)\}\bigcap(\psi_3, A_3)$.
(xi) $(\psi_1, A_1)\bigcap(\psi_2, A_2)\bigcap(\psi_3, A_3) = \{(\psi_1, A_1)\bigcap(\psi_2, A_2)\}\bigcap(\psi_3, A_3)$.
(xii) $(\psi_1, A_1)\bigcap(\psi_2, A_2)\bigcap(\psi_3, A_3) = \{(\psi_1, A_1)\bigcap(\psi_2, A_2)\}\bigcap(\psi_3, A_3)$.

Proof. Follows directly from definition. □

Corollary 3.20. (i) $\Phi_E \bigcup \tilde{X}_E = \tilde{X}_E$.
(ii) $\Phi_E \bigcap \tilde{X}_E = \Phi_E$.

Proposition 3.21. If $(\psi_1, A_1)$ and $(\psi_2, A_2)$ are PmFSSs over $X$, then any one of them may be sandwiched between $(\psi_1, A_1)\bigcap(\psi_2, A_2)$ and $(\psi_1, A_1)\bigcup(\psi_2, A_2)$ i.e.

(i) $(\psi, A_1)\bigcap(\psi_2, A_2)\bigcap(\psi_1, A_1)\bigcup(\psi_2, A_2) = \bigcap(\psi_1, A_1)\bigcup(\psi_2, A_2)$.
(ii) $(\psi_1, A_1)\bigcap(\psi_2, A_2)\bigcup(\psi_1, A_1)\bigcup(\psi_2, A_2) = \bigcap(\psi_1, A_1)\bigcup(\psi_2, A_2)$.

Proof. (i) follows from the fact that $\min\{\mu_{\psi_1}^{(i)}, \mu_{\psi_2}^{(i)}\} \leq \mu_{\psi_1}^{(i)} \leq \max\{\mu_{\psi_1}^{(i)}, \mu_{\psi_2}^{(i)}\}$ and $\max\{\nu_{\psi_1}^{(i)}, \nu_{\psi_2}^{(i)}\} \geq \nu_{\psi_1}^{(i)} \geq \min\{\nu_{\psi_1}^{(i)}, \nu_{\psi_2}^{(i)}\}$. The proof of (ii) is similar. □

Proposition 3.22. If $(\psi_1, A_1)$ and $(\psi_2, A_2)$ are PmFSSs over $X$, then contrary to crisp sets, De Morgan laws do not hold i.e.

(i) $\left((\psi_1, A_1)\bigcup(\psi_2, A_2)\right)^c \neq (\psi_1, A_1)^c\bigcap(\psi_2, A_2)^c$. 

Example 3.23. Consider the PmFSSs $\psi_A$ given in Example 3.2. The tabular representations of $\psi_A \uplus \psi_A^c$ and $\psi_A \cap \psi_A^c$ are given in Tables 13 and 14, respectively.

\[
\begin{array}{c|c|c}
\psi_A \uplus \psi_A^c & e_1 & e_2 \\
\hline
b & \{(0.74, 0.19), (0.79, 0.28), (0.97, 0.04)\} & \{(0.62, 0.28), (0.59, 0.47), (0.26, 0.11)\} \\
s & \{(0.62, 0.38), (0.74, 0.36), (0.88, 0.31)\} & \{(0.69, 0.37), (0.58, 0.01), (0.72, 0.71)\} \\
c & \{(1, 0), (1, 0), (1)\} & \{(1, 0), (1, 0), (1, 0)\} \\
z & \{(1, 0), (1, 0), (1)\} & \{(1, 0), (1, 0), (1, 0)\} \\
\end{array}
\]

Table 13. $\psi_A \uplus \psi_A^c$

\[
\begin{array}{c|c|c}
\psi_A \cap \psi_A^c & e_1 & e_2 \\
\hline
b & \{(0.19, 0.74), (0.28, 0.79), (0.04, 0.97)\} & \{(0.28, 0.62), (0.47, 0.59), (0.11, 0.26)\} \\
s & \{(0.38, 0.62), (0.36, 0.74), (0.31, 0.88)\} & \{(0.37, 0.69), (0.01, 0.58), (0.71, 0.72)\} \\
c & \{(0, 1), (0, 1), (0, 1)\} & \{(0, 1), (0, 1), (0, 1)\} \\
z & \{(0, 1), (0, 1), (0, 1)\} & \{(0, 1), (0, 1), (0, 1)\} \\
\end{array}
\]

Table 14. $\psi_A \cap \psi_A^c$

We observe, keeping in view Tables 13 and 14, that $\psi_A \uplus \psi_A^c \neq \bar{X}_A$ and $\psi_A \cap \psi_A^c \neq \Phi_A$. Hence, we have the following proposition.

Proposition 3.24. If $\psi_A$ is a PmFSS over $X$, then unlike in crisp sets

(i) $\psi_A \uplus \psi_A^c \neq \bar{X}_A$.
(ii) $\psi_A \cap \psi_A^c \neq \Phi_A$.

Definition 3.25. The difference of two PmFSSs $(\psi_1, A_1)$ and $(\psi_2, A_2)$ defined over the same universe $X$ is defined as

\[(\psi_1, A_1) \setminus (\psi_2, A_2) = \left\{ e, \left\{ \max_{\mu_{\psi_1}(\epsilon) \in \Sigma} \mu_{\psi_1}(\epsilon) \cdot \min_{\mu_{\psi_2}(\epsilon) \in \Sigma} \mu_{\psi_2}(\epsilon) : e \in A_1 \setminus A_2, \zeta \in X : i = 1, 2, \ldots, m \right\} : e \in A_1 \setminus A_2, \zeta \in X \right\} \]

Example 3.26. For the PmFSSs $\psi_1$ and $\psi_2$ given in Example 3.18, $(\psi_1, A_1) \setminus (\psi_2, A_2)$ is exhibited in Table 15.

\[
\begin{array}{c|c}
(\psi_1, A_1) \setminus (\psi_2, A_2) & e_1 \\
\hline
y & \{(0.62, 0.17), (0.31, 0.82), (0.12, 0.06)\} \\
d & \{(0.02, 0.28), (0.16, 0.39), (0.30, 0.80)\} \\
g & \{(0.51, 0.52), (0.39, 0.42), (0.52, 0.53)\} \\
\end{array}
\]

Table 15. $(\psi_1, A_1) \setminus (\psi_2, A_2)$
Definition 3.27. If \((\psi, A)\) is a PmFSS extracted from \(X\), then the necessity operator \(\hat{\circ}\) on \((\psi, A)\) is defined as
\[
\hat{\circ}(\psi, A) = \left\{ \left( e, \left\{ \frac{\zeta}{\left[ \sqrt{1 - \left( \mu_{\psi}^{i}(e)(\zeta) \right)^{2}} \right]} \right\} \right) : e \in A, \zeta \in X; i = 1, 2, \ldots, m \right\}
\]

Definition 3.28. If \((\psi, A)\) is a PmFSS extracted from \(X\), then the possibility operator \(\hat{\ast}\) on \((\psi, A)\) is defined as
\[
\hat{\ast}(\psi, A) = \left\{ \left( e, \left\{ \frac{\zeta}{\sqrt{1 - \left( \nu_{\psi}^{i}(e)(\zeta) \right)^{2}}} \right\} \right) : e \in A, \zeta \in X; i = 1, 2, \ldots, m \right\}
\]

Example 3.29. For the PmFS \(\psi\) given in Example 3.2, \(\hat{\circ}\psi_A\) and \(\hat{\ast}\psi_A\) are given in Tables 16 and 17, respectively:

<table>
<thead>
<tr>
<th>(\hat{\circ}\psi_A)</th>
<th>(e_1)</th>
<th>(e_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>{(0.19, 0.98), (0.28, 0.96), (0.04, 0.99)}</td>
<td>{(0.62, 0.78), (0.59, 0.81), (0.26, 0.96)}</td>
</tr>
<tr>
<td>(s)</td>
<td>{(0.38, 0.92), (0.74, 0.67), (0.88, 0.47)}</td>
<td>{(0.37, 0.93), (0.01, 0.99), (0.72, 0.69)}</td>
</tr>
<tr>
<td>(c)</td>
<td>{(0.1), (0.1), (0.1)}</td>
<td>{(0.1), (0.1), (0.1)}</td>
</tr>
<tr>
<td>(z)</td>
<td>{(0.1), (0.1), (0.1)}</td>
<td>{(0.1), (0.1), (0.1)}</td>
</tr>
</tbody>
</table>

Table 16. \(\hat{\circ}\psi_A\)

<table>
<thead>
<tr>
<th>(\hat{\ast}\psi_A)</th>
<th>(e_1)</th>
<th>(e_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>{(0.67, 0.74), (0.61, 0.79), (0.24, 0.97)}</td>
<td>{(0.96, 0.28), (0.88, 0.47), (0.09, 0.11)}</td>
</tr>
<tr>
<td>(s)</td>
<td>{(0.78, 0.62), (0.93, 0.36), (0.95, 0.31)}</td>
<td>{(0.72, 0.69), (0.81, 0.58), (0.70, 0.71)}</td>
</tr>
<tr>
<td>(c)</td>
<td>{(0.1), (0.1), (0.1)}</td>
<td>{(0.1), (0.1), (0.1)}</td>
</tr>
<tr>
<td>(z)</td>
<td>{(0.1), (0.1), (0.1)}</td>
<td>{(0.1), (0.1), (0.1)}</td>
</tr>
</tbody>
</table>

Table 17. \(\hat{\ast}\psi_A\)

Remark. The necessity and possibility operator defined above in definitions 3.27 and 3.28 transform any PmFSS \(\psi_A\) to \(m\)-polar fuzzy set.

Proposition 3.30. For any PmFSS \(\psi_A\) defined over \(X\), \(\hat{\circ}\psi_A\subseteq\hat{\ast}\psi_A\).

Proof. Since for each \(\zeta \in X, e \in A\) and all admissible values of \(i\), we have
\[
\left( \mu_{\psi}^{i}(e)(\zeta) \right)^{2} + \left( \nu_{\psi}^{i}(e)(\zeta) \right)^{2} \leq 1
\]
\[
\therefore \mu_{\psi}^{i}(e)(\zeta) \leq \sqrt{1 - \left( \nu_{\psi}^{i}(e)(\zeta) \right)^{2}}
\]
\[
\& \nu_{\psi}^{i}(e)(\zeta) \leq \sqrt{1 - \left( \mu_{\psi}^{i}(e)(\zeta) \right)^{2}}
\]
so the result follows. \qed

**Corollary 3.31.** For any PmFSS \(\psi_A\), we have

(i) \(\tilde{\tilde{\psi}}_A \tilde{\tilde{\psi}}_A = \tilde{\psi}_A\)

(ii) \(\tilde{\psi}_A \tilde{\psi}_A = \tilde{\tilde{\psi}}_A\)

**Definition 3.32.** The sum of two PmFSSs \((\psi_1, A_1)\) and \((\psi_2, A_2)\) extracted from the same universe is defined as

\[
(\psi_1, A_1) \oplus (\psi_2, A_2) = \left\{ \left( e, \left\{ \sum_{i=1}^{m} \left( \mu_{\psi_1}^{(i)}(e) \xi + \mu_{\psi_2}^{(i)}(e) \xi \right) - \left( \mu_{\psi_1}^{(i)}(e) \xi \right) \right) \right) : e \in A_1 \cup A_2, \xi \in X \right\}
\]

**Example 3.33.** For the PmFSSs \((\psi_1, A_1)\) and \((\psi_2, A_2)\) given in Example 3.18, \((\psi_1, A_1) \oplus (\psi_2, A_2)\) is given in Table 18.

<table>
<thead>
<tr>
<th>((\psi_1, A_1) \oplus (\psi_2, A_2))</th>
<th>(e_1)</th>
<th>(e_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>((0.75, 0.02), (0.34, 0.12), (0.81, 0.01))</td>
<td>((0.45, 0.13), (0.87, 0.13), (0.17, 0.04))</td>
</tr>
<tr>
<td>(d)</td>
<td>((0.29, 0.16), (0.24, 0.02), (0.39, 0.54))</td>
<td>((0.37, 0.12), (0.51, 0.04), (0.40, 0.62))</td>
</tr>
<tr>
<td>(g)</td>
<td>((0.57, 0.24), (0.52, 0.16), (0.68, 0.01))</td>
<td>((0.25, 0.83), (0.61, 0.36), (1.00, 0.00))</td>
</tr>
</tbody>
</table>

**Table 18.** \((\psi_1, A_1) \oplus (\psi_2, A_2)\)

**Definition 3.34.** The product of two PmFSSs \((\psi_1, A_1)\) and \((\psi_2, A_2)\) extracted from the same universe is defined as

\[
(\psi_1, A_1) \odot (\psi_2, A_2) = \left\{ \left( e, \left\{ \prod_{i=1}^{m} \mu_{\psi_1}^{(i)}(e) \xi \right\} \right) : e \in A_1 \cup A_2, \xi \in X \right\}
\]

**Example 3.35.** For the PmFSSs \((\psi_1, A_1)\) and \((\psi_2, A_2)\) given in Example 3.18, \((\psi_1, A_1) \odot (\psi_2, A_2)\) is given in Table 19.

<table>
<thead>
<tr>
<th>((\psi_1, A_1) \odot (\psi_2, A_2))</th>
<th>(e_1)</th>
<th>(e_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>((0.33, 0.20), (0.04, 0.82), (0.10, 0.18))</td>
<td>((0.04, 0.57), (0.36, 0.49), (0.01, 0.34))</td>
</tr>
<tr>
<td>(d)</td>
<td>((0.00, 0.61), (0.03, 0.39), (0.08, 0.90))</td>
<td>((0.07, 0.57), (0.14, 0.36), (0.01, 0.99))</td>
</tr>
<tr>
<td>(g)</td>
<td>((0.15, 0.65), (0.14, 0.54), (0.26, 0.53))</td>
<td>((0.00, 1.00), (0.20, 0.77), (0.81, 0.26))</td>
</tr>
</tbody>
</table>

**Table 19.** \((\psi_1, A_1) \odot (\psi_2, A_2)\)
Definition 3.36. If \((\psi_1, A_1) = (\psi_2, A_2)\) in Definition 3.34, then we express \((\psi_1, A_1)\overset{\circ}{\otimes}(\psi_2, A_2)\) by \((\psi_1, A_1)^2\). Thus,

\[
(\psi, A)^2 = \left\{ e, \left\{ \frac{\zeta}{\mu_\psi(e)(\zeta) \sqrt{2 \nu_\psi(e)(\zeta)}^2 - \nu_\psi(e)(\zeta)} : e \in A, \zeta \in X; i = 1, \cdots, m \right\} \right.
\]

The set \((\psi, A)^2\) is termed as concentration of \((\psi, A)\), designated as \(con(\psi, A)\). In general, if \(k \in [0, \infty)\), then

\[
(\psi, A)^k = \left\{ e, \left\{ \frac{\zeta}{\mu_\psi(e)(\zeta) \sqrt{2 \nu_\psi(e)(\zeta)}^2 - \nu_\psi(e)(\zeta)} : e \in A, \zeta \in X; i = 1, \cdots, m \right\} \right.
\]

The set \((\psi, A)^\frac{1}{k}\) is called dilation of \((\psi, A)\), designated as \(dil(\psi, A)\).

Example 3.37. For PmFSS \(\psi_A\) given in Example 3.2, \(con(\psi_A)\) and \(dil(\psi_A)\) are given in Tables 20 and 21, respectively.

<table>
<thead>
<tr>
<th>(con(\psi_A))</th>
<th>(e_1)</th>
<th>(e_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>{(0.04, 0.89), (0.08, 0.93), (0.00, 0.99)}</td>
<td>{(0.38, 0.39), (0.35, 0.63), (0.07, 0.16)}</td>
</tr>
<tr>
<td>(s)</td>
<td>{(0.14, 0.79), (0.55, 0.49), (0.77, 0.43)}</td>
<td>{(0.14, 0.85), (0.00, 0.75), (0.52, 0.87)}</td>
</tr>
<tr>
<td>(c)</td>
<td>{(0, 1), (0, 1), (0, 1)}</td>
<td>{(0, 1), (0, 1), (0, 1)}</td>
</tr>
<tr>
<td>(z)</td>
<td>{(0, 1), (0, 1), (0, 1)}</td>
<td>{(0, 1), (0, 1), (0, 1)}</td>
</tr>
</tbody>
</table>

**Table 20. \(con(\psi_A)\)**

<table>
<thead>
<tr>
<th>(dil(\psi_A))</th>
<th>(e_1)</th>
<th>(e_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>{(0.44, 0.57), (0.53, 0.62), (0.20, 0.87)}</td>
<td>{(0.79, 0.20), (0.77, 0.34), (0.51, 0.08)}</td>
</tr>
<tr>
<td>(s)</td>
<td>{(0.62, 0.46), (0.86, 0.26), (0.94, 0.22)}</td>
<td>{(0.61, 0.52), (0.10, 0.43), (0.85, 0.54)}</td>
</tr>
<tr>
<td>(c)</td>
<td>{(0, 1), (0, 1), (0, 1)}</td>
<td>{(0, 1), (0, 1), (0, 1)}</td>
</tr>
<tr>
<td>(z)</td>
<td>{(0, 1), (0, 1), (0, 1)}</td>
<td>{(0, 1), (0, 1), (0, 1)}</td>
</tr>
</tbody>
</table>

**Table 21. \(dil(\psi_A)\)**

Remark. We may link linguistic terms like "very", "very very", "medium", "more or less" and "high" etc. with the set \(\psi_A^k\) given in Definition 3.36 by assigning
different non-negative real values to \( k \). For example,

\[
\begin{align*}
  k &= 2 \quad \Rightarrow \quad " \text{very} " \\
  k &= 2 \text{ twice} \quad \Rightarrow \quad " \text{very very} " \\
  k &= 0.5 \quad \Rightarrow \quad " \text{highly} " \\
  k &= 0.75 \quad \Rightarrow \quad " \text{more or less} " \\
  k &= 0.75 \text{ twice} \quad \Rightarrow \quad " \text{medium} "
\end{align*}
\]

**Example 3.38.** For the PmFSS \( \psi_A \) given in Example 3.2, \( \text{very}(\psi_A) \), \( \text{very very}(\psi_A) \), \( \text{highly}(\psi_A) \), \( \text{more or less}(\psi_A) \) and \( \text{medium}(\psi_A) \) are demonstrated in Tables 22, 23, 24, 25 and 26, respectively.

<table>
<thead>
<tr>
<th>( \text{very}(\psi_A) )</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>((0.04, 0.89), (0.08, 0.93), (0.00, 0.99))</td>
<td>((0.38, 0.39), (0.35, 0.63), (0.07, 0.16))</td>
</tr>
<tr>
<td>( s )</td>
<td>((0.14, 0.79), (0.55, 0.49), (0.77, 0.43))</td>
<td>((0.14, 0.85), (0.00, 0.75), (0.52, 0.87))</td>
</tr>
<tr>
<td>( c )</td>
<td>((0.1), (0.1), (0.1))</td>
<td>((0.1), (0.1), (0.1))</td>
</tr>
<tr>
<td>( z )</td>
<td>((0.1), (0.1), (0.1))</td>
<td>((0.1), (0.1), (0.1))</td>
</tr>
</tbody>
</table>

**Table 22.** \( \text{very}(\psi_A) \)

<table>
<thead>
<tr>
<th>( \text{very very}(\psi_A) )</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>((0.00, 0.98), (0.01, 0.99), (0.00, 1.00))</td>
<td>((0.14, 0.53), (0.12, 0.80), (0.00, 0.22))</td>
</tr>
<tr>
<td>( s )</td>
<td>((0.02, 0.93), (0.30, 0.65), (0.59, 0.58))</td>
<td>((0.02, 0.96), (0.00, 0.90), (0.27, 0.97))</td>
</tr>
<tr>
<td>( c )</td>
<td>((0.1), (0.1), (0.1))</td>
<td>((0.1), (0.1), (0.1))</td>
</tr>
<tr>
<td>( z )</td>
<td>((0.1), (0.1), (0.1))</td>
<td>((0.1), (0.1), (0.1))</td>
</tr>
</tbody>
</table>

**Table 23.** \( \text{very very}(\psi_A) \)

<table>
<thead>
<tr>
<th>( \text{highly}(\psi_A) )</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>((0.44, 0.57), (0.53, 0.62), (0.20, 0.87))</td>
<td>((0.79, 0.20), (0.77, 0.34), (0.51, 0.08))</td>
</tr>
<tr>
<td>( s )</td>
<td>((0.62, 0.46), (0.86, 0.26), (0.94, 0.22))</td>
<td>((0.61, 0.52), (0.10, 0.43), (0.85, 0.54))</td>
</tr>
<tr>
<td>( c )</td>
<td>((0.1), (0.1), (0.1))</td>
<td>((0.1), (0.1), (0.1))</td>
</tr>
<tr>
<td>( z )</td>
<td>((0.1), (0.1), (0.1))</td>
<td>((0.1), (0.1), (0.1))</td>
</tr>
</tbody>
</table>

**Table 24.** \( \text{highly}(\psi_A) \)
We may interpret these figures as follows: Suppose that $b, s, c$ and $z$ are persons named Babar, Soneri, Chloe and Zunera respectively. Assume that $e_1$ denotes the trait “well-dressed” and $e_2$ stands for “attractive personality”. Further assume that $\psi_A$ is the initial data provided by three judges. Then with respect to the trait $e_1$ i.e. well-dressed, the rating of first judge changes to the PFN $(0.04, 0.89)$ to Babar in view of very well-dressed, the rating of second judge becomes $(0.08, 0.93)$ and that of the third judge becomes $(0.00, 0.09)$. On the same token, with respect to the trait $e_2$ i.e. attractive personality, the rating of first judge becomes PFN $(0.38, 0.39)$ to Babar in view of very attractive personality, the rating of second judge changes to $(0.35, 0.63)$ and that of the third judge becomes $(0.07, 0.16)$. The other figures may be interpreted on the parallel track.

### Table 25. more or less($\psi_A$)

<table>
<thead>
<tr>
<th>more or less($\psi_A$)</th>
<th>$e_1$</th>
<th>$e_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>${(0.29, 0.67), (0.38, 0.72), (0.09, 0.94)}$</td>
<td>${(0.70, 0.24), (0.67, 0.41), (0.36, 0.10)}$</td>
</tr>
<tr>
<td>$s$</td>
<td>${(0.48, 0.55), (0.80, 0.31), (0.91, 0.27)}$</td>
<td>${(0.47, 0.62), (0.03, 0.51), (0.78, 0.64)}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
</tr>
<tr>
<td>$z$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
</tr>
</tbody>
</table>

### Table 26. medium($\psi_A$)

<table>
<thead>
<tr>
<th>medium($\psi_A$)</th>
<th>$e_1$</th>
<th>$e_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>${(0.40, 0.60), (0.48, 0.65), (0.16, 0.89)}$</td>
<td>${(0.76, 0.21), (0.74, 0.36), (0.46, 0.09)}$</td>
</tr>
<tr>
<td>$s$</td>
<td>${(0.58, 0.49), (0.84, 0.27), (0.93, 0.23)}$</td>
<td>${(0.57, 0.55), (0.07, 0.45), (0.83, 0.57)}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
</tr>
<tr>
<td>$z$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
<td>${(0, 1), (0, 1), (0, 1)}$</td>
</tr>
</tbody>
</table>

4. **Selection of employee for promotion using PmFS TOPSIS**

TOPSIS is employed to decide the superlative alternative from the notions of compromise solution. The solution which is closest to the ideal solution and farthest from negative ideal solution is acknowledged as *compromise solution*. In this section, we study how PmFSSs may be utilized in multiple criteria group decision making (MCGDM) using TOPSIS. First of all we shall extend TOPSIS to PmFSSs and then shall consider a problem.
Table 27. Linguistic terms for judging alternatives

<table>
<thead>
<tr>
<th>Linguistic Terms</th>
<th>Fuzzy Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not necessary (NN)</td>
<td>[0, 0.20]</td>
</tr>
<tr>
<td>Necessary (N)</td>
<td>(0.20, 0.40]</td>
</tr>
<tr>
<td>More or less necessary (MN)</td>
<td>(0.40, 0.60]</td>
</tr>
<tr>
<td>Very necessary (VN)</td>
<td>(0.60, 0.80]</td>
</tr>
<tr>
<td>Extremely necessary (EN)</td>
<td>(0.80, 1]</td>
</tr>
</tbody>
</table>

We make an inception by explaining the technique step by step as follows:

### Decision Making Method

**Input:**

Step 1: Recognize the problem as what we have and what we have to do: Assume that \( V = \{ \zeta_i : i = 1, 2, \cdots, n \} \) is the finite aggregate of alternatives under consideration, \( D = \{ d_i : i = 1, 2, \cdots, m \} \) is the group of decision makers (DMs) and \( E = \{ e_i : i = 1, 2, \cdots, k \} \) is a finite family of attributes. Thus the \((i, j)^{th}\) entry of the PmFS matrix represents a set of \( m \) PFNs assigned to \( i^{th} \) alternative with respect to \( j^{th} \) attribute. Further, \( r^{th} \) PFN in the set at \((i, j)^{th}\) position yields the value of membership and non-membership functions, respectively, given by the \( r^{th} \) DM to \( i^{th} \) alternative with respect to \( j^{th} \) attribute.

Step 2: Construct weighted parameter matrix \( A \) as

\[
A = [w_{ij}]_{m \times k} = \begin{bmatrix}
w_{11} & w_{12} & \cdots & w_{1k} \\
w_{21} & w_{22} & \cdots & w_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
w_{i1} & w_{i2} & \cdots & w_{ik} \\
\vdots & \vdots & \ddots & \vdots \\
w_{m1} & w_{m2} & \cdots & w_{mk}
\end{bmatrix}
\]

where \( w_{ij} \) is the fuzzy weight assigned by the DM \( d_i \) to the attribute \( e_j \) by considering linguistic terms as given (for example) in Table 27.

Step 3: Construct normalized weighted matrix

\[
\hat{A} = [\hat{w}_{ij}]_{m \times k} = \begin{bmatrix}
\hat{w}_{11} & \hat{w}_{12} & \cdots & \hat{w}_{1k} \\
\hat{w}_{21} & \hat{w}_{22} & \cdots & \hat{w}_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{w}_{i1} & \hat{w}_{i2} & \cdots & \hat{w}_{ik} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{w}_{m1} & \hat{w}_{m2} & \cdots & \hat{w}_{mk}
\end{bmatrix}
\]

where \( \hat{w}_{ij} = \frac{w_{ij}}{\sqrt{\sum_{i=1}^{m} w_{ij}^2}} \) and obtaining the weighted vector \( \mathbf{W} = (w_1, w_2, \cdots, w_k) \), where \( w_i = \frac{\sum_{j=1}^{k} w_{ij}}{\sum_{i=1}^{m} w_{ij}} \) and \( w_j = \frac{\sum_{i=1}^{m} w_{ij}}{m} \).
Step 4: Construct PmFS decision matrix

\[ \mathcal{B} = [\zeta_{jk}]_{n \times k} = \begin{bmatrix} 
\zeta_{11} & \zeta_{12} & \cdots & \zeta_{1k} \\
\zeta_{21} & \zeta_{22} & \cdots & \zeta_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_{j1} & \zeta_{j2} & \cdots & \zeta_{jk} \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_{n1} & \zeta_{n2} & \cdots & \zeta_{nk} 
\end{bmatrix} \]

where \((i, j)^{th}\) entry of the PmFS matrix i.e. \(\zeta_{ij}\) represents a set of \(m\) PFNs assigned to \(i^{th}\) alternative with respect to \(j^{th}\) attribute. Further, \(r^{th}\) PFN in the set at \((i, j)^{th}\) position yields the value of membership and non-membership functions, respectively, given by the \(r^{th}\) DM to \(i^{th}\) alternative with respect to \(j^{th}\) attribute.

Computations:

Step 5: Construct weighted PmFS decision matrix \(\mathcal{C}\) by multiplying each element in the \(j^{th}\) column by \(j^{th}\) weight from weight vector obtained at Step 3 above, for each value of \(j\) varying from 1 to \(k\).

Step 6: Obtain PFS-valued positive ideal solution (PFSV-PIS) and PFS-valued negative ideal solution (PFSV-NIS), employing in order

\[
\text{PFSV-PIS} = \{\eta_1^+, \eta_2^+, \cdots, \eta_n^+\} = \{(\lor_j \mu_{jk}, \land_j \nu_{jk}) : j = 1, 2, \cdots, n\}
\]

and

\[
\text{PFSV-NIS} = \{\eta_1^−, \eta_2^−, \cdots, \eta_n^−\} = \{(\land_j \mu_{jk}, \lor_j \nu_{jk}) : j = 1, 2, \cdots, n\}
\]

where \(\lor\) stands for PFS union and \(\land\) represents PFS intersection.

Step 7: Compute PFS-Euclidean distances of each alternative from PFSV-PIS and PFSV-NIS, respectively, utilizing

\[
d_j^+ = \sqrt{\sum_{k=1}^{n} \left( (\mu_{jk} - \lor_j \mu_{jk})^2 + (\nu_{jk} - \land_j \nu_{jk})^2 \right)}
\]

and

\[
d_j^- = \sqrt{\sum_{k=1}^{n} \left( (\mu_{jk} - \land_j \mu_{jk})^2 + (\nu_{jk} - \lor_j \nu_{jk})^2 \right)}
\]

for each \(j = 1, 2, \cdots, n\).

Step 8: Determine the closeness coefficient of each alternative with ideal solution utilizing

\[
C^*(\zeta_j) = \frac{d_j^-}{d_j^+ + d_j^-} \in [0, 1]
\]

Output:

Step 9: In order to obtain the preference order of the alternatives, rank the alternatives in descending (or ascending) order.
The flowchart of the decision making method appears below in Figure 1 below:

![Flowchart representation of decision making method](image)

**Figure 1.** Flow chart representation of decision making method

We apply the proposed flow chart representation of decision making method by using presumptive data in the forthcoming example as follows:

**Example 4.1.** Assume that a firm wants to promote one of its employees to a higher position. To cope with the competitive environment prevailing, the firm wishes to choose the best of the best from the options available. The chief executive of the firm constitutes a panel of three decision makers (DMs) and gives them the task to select the best suitable employee for promotion. After a long discussion, the panel decides to consider five employees and focus on three traits to be required in selected person.

Step 1: Identifying the problem: Assume that $V = \{\zeta_i : i = 1, 2, \cdots, 5\}$ is the family of employees under consideration for promotion, $E = \{e_i : i = 1, 2, 3\}$ is the set of traits, and $D = \{d_i : i = 1, 2, 3\}$ is the group of DMs, where

- $e_1 = \text{Communication skills}$,
- $e_2 = \text{Hard working}$, and
- $e_3 = \text{Well aware of emerging technologies}$

Step 2: The weighted parameter matrix is

\[
A = [w_{ij}]_{3 \times 3} = \begin{bmatrix}
VN & EN & NN \\
N & EN & EN \\
EN & VN & MN \\
\end{bmatrix} = \begin{bmatrix}
0.70 & 0.90 & 0.10 \\
0.30 & 0.85 & 0.90 \\
0.90 & 0.70 & 0.50 \\
\end{bmatrix}
\]
The weighted PFS-valued positive ideal solution (PFSV-PIS) and PFS-valued negative ideal solution (PFSV-NIS), in order, are

\[
\{\eta_1^+, \eta_2^+, \ldots, \eta_n^+\} = (0.29, 0.09), (0.32, 0.03), (0.26, 0.03), (0.31, 0.01), (0.25, 0.04)
\]

and

\[
\{\eta_1^-, \eta_2^-, \ldots, \eta_n^-\} = (0.03, 0.28), (0.09, 0.12), (0.11, 0.18), (0.11, 0.20), (0.10, 0.18)
\]

Step 7, 8: The PFS-Euclidean distances of each alternative from PFSV-PIS and PFSV-NIS along with closeness coefficients are given in Table 28 below:

<table>
<thead>
<tr>
<th>Alternative (ζ_i)</th>
<th>d_i^+</th>
<th>d_i^-</th>
<th>C_i^*</th>
</tr>
</thead>
<tbody>
<tr>
<td>ζ_1</td>
<td>0.4972</td>
<td>0.6146</td>
<td>0.5528</td>
</tr>
<tr>
<td>ζ_2</td>
<td>0.4901</td>
<td>0.3731</td>
<td>0.4322</td>
</tr>
<tr>
<td>ζ_3</td>
<td>0.4580</td>
<td>0.3197</td>
<td>0.4236</td>
</tr>
<tr>
<td>ζ_4</td>
<td>0.4853</td>
<td>0.5103</td>
<td>0.5126</td>
</tr>
<tr>
<td>ζ_5</td>
<td>0.3736</td>
<td>0.3716</td>
<td>0.4986</td>
</tr>
</tbody>
</table>

Table 28. Distance measures & closeness coefficient of each alternative

Step 9: The preference order of the alternatives, therefore, is

\[ ζ_1 ∗ ζ_4 ∗ ζ_5 ∗ ζ_2 ∗ ζ_3 \]

This ranking is depicted with the help of 3D bar chart in Figure 2:
In view of above TOPSIS ranking, we may infer that the employee $\zeta_1$ is most meritorious for promotion on higher post.

5. Conclusion

We delivered an innovative crossbreed structure titled Pythagorean $m$-polar fuzzy soft sets in conjunction with some basic algebraic operations and features. We dig out crisp sets like support, core and height from $PmFSS$. A plenty of illustrations are also contained within to comprehend the notions effectively. We proposed a TOPSIS method for solving multiple criteria group decision making (MCGDM) problems accompanied by flowchart of the said decision making method. We employed the proposed algorithm to decide most appropriate person for the appraisal to higher position under Pythagorean $m$-polar fuzzy soft environment.

Since every intuitionistic fuzzy soft set is also a Pythagorean fuzzy soft set, so by familiarizing $PmFSS$, we have also wordlessly introduced intuitionistic $m$-polar fuzzy soft sets ($I_mFSSs$). Theoretically, the ideas presented in this article may be extended to develop algebraic structures like Pythagorean $m$-polar fuzzy soft groups, Pythagorean $m$-polar fuzzy soft rings, Pythagorean $m$-polar fuzzy soft ideals, Pythagorean $m$-polar fuzzy soft algebras, Pythagorean $m$-polar fuzzy soft topology and undoubtedly Pythagorean $m$-polar fuzzy soft graphs too. Above and beyond the theoretical side, the ideas presented have charge to be extended in handling day to day problems from the real world including business, life sciences, social sciences, economics, pattern recognition, human resource management, robotics and many other areas. We trust that this article will serve as a foundation pit for the researchers working in this field.

References


