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Pythagorean m-polar Fuzzy Soft Sets with TOPSIS Method for MCGDM

Muhammad Riaz
Department of Mathematics,
University of the Punjab Lahore, Pakistan,
Email: mriaz.math@pu.edu.pk

Khalid Naeem
Department of Mathematics & Statistics,
The University of Lahore, Pakistan,
Email: khalidnaeem333@gmail.com

Deeba Afzal
Department of Mathematics & Statistics,
The University of Lahore, Pakistan,
Email: deebafzal@gmail.com

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Abstract. In this article, we introduce the concept of Pythagorean m-polar fuzzy soft sets (PmFSSs). This set reduces to Pythagorean fuzzy soft set for $m = 1$. We define algebraic operations and some characteristics of PmFSSs. We define some linguistic terms utilizing the notion of product of PmFSSs ($\otimes$) by assigning different numeric values to the constant $k \in [0, \infty[$ and present an illustration to determine whether the traits of being well-dressed and attractive personality is possessed by a person or not and up to what extent. We present an application of PmFSSs in multi-criteria group decision making (MCGDM) problem of appraisal of employee for promotion making use of the well-distinguished tool TOPSIS.

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1. Introduction

Logic and set theory are believed to be the foundation stones of modern mathematics. A meticulous exploration of set theory belongs to the brass tacks of mathematical logic. Indeed, these two notions are interrelated for many logical operations like $\vee$, $\wedge$, $\neg$, $\oplus$, $\otimes$, $\rightarrow$ and $\leftrightarrow$ etc. decode into set theory and vice versa. On the same token; relations, functions, paradoxes, numbers, probability theory, algebra and modern measure theory etc. all rely on theory of sets. Logicians have investigated set theory in excessive details, articulating an assortment of axioms that affords an all-encompassing and sufficiently strong footing for mathematical reasoning. The customary system of axiomatic set theory is Zermelo-Fraenkel set theory, along with the axioms of choice. Each axiom incorporated in this theory states a property of sets which is far and wide acknowledged around the world by mathematicians. Beyond its foundational role, modern set theory owns a gigantic number of full of zip researchers.

At the primary phases of the development of set theory (traditionally accredited as classical set theory, developed by Cantor and Dedekind), a constituent was understood to be either a member of some given set or not. In other terms, a characteristic function is associated with each set which assigns a value of 1 to the element if it is member of the set and 0 otherwise. There was not any other moderate option for an element regarding belongingness to a set. Zadeh [60], in the second half of 20^{th} century, made public the perception of a new species of sets famous as fuzzy set by coupling a membership function with each member of the set whose values range from 0 to 1. According to Zadeh, an element may belong partially to a set. Besides exploring different mathematical operations on fuzzy sets Zadeh also introduced the concept of a linguistic variable and its application to approximate reasoning [61]. The philosophies commenced by Zadeh proved like a revolution in the ecosphere of dynamic mathematicians.

Subsequent upon the model rendered by Zadeh, mathematicians around the globe began meditative upon other sort of sets. Atanassov believed that if there exists a membership function to measure uncertainty against a set, then there must be a non-membership function associated with each element of that set too. Ensuing his work, Atanassov presented [15, 16] a fresh category of sets entitled intuitionistic fuzzy sets (IF-sets) by striking the constraint that the values of membership and its counterpart non-membership functions not only lie from 0 to 1 but their sum also must fall in the same interval. At the end of 20^{th} century, Molodstov [34] furnished a parameterized collection of sets, known as soft sets. Soft sets acquiescently designate a number of attributes for clarifying and reconnoitering a problem holding ambiguity and uncertainty. Molodstov also rendered some useful applications from everyday life. The soft set theory discovers varied range of applications in management economics, medical sciences, engineering, and social sciences predominantly due to its tractability without restraints on imprecise description of the situation.

With the study of innovative set structures, the science of set theory took a fresh twist. Increasingly hybrid set structures gave the impression on the canvas. Yager [57]-[59], by modifying the constraint on the parameters, presented the notion of

Pythagorean fuzzy sets (PF-sets) as an extension of IF-sets. Peng *et al.* [39]-[43] studied some results for PF-sets along with their applications. Naz *et al.* [37] employed the idea on Pythagorean fuzzy graphs and presented some decision making applications. Olgun *et al.* [38] introduced the notion of Pythagorean fuzzy topological spaces employing the notion of fuzzy topological spaces. Qamar and Hassan [1] presented an approach toward a Q-neutrosophic soft set and its application in decision making. Qamar and Hassan [2] also presented Q-neutrosophic soft relation and its application in decision making.

Akram *et al.* [3]-[7], by utilizing PF-sets and m-polar fuzzy sets, furnished some beguiling applications in decision making problems (DMPs). Akram and Adeel [8] presented m-polar fuzzy labeling graphs with application. Akram and Sarwar [9] presented novel applications of m-polar fuzzy hypergraphs. Akram *et al.* [10] presented notion of m-polar fuzzy lie subalgebras. Later, Akram and Farooq [11] extended this notion to m-polar fuzzy lie ideals of lie algebras. Farooq *et al.* [18] rendered the notion of m-polar fuzzy groups. Masarwah and Ahmad [12]-[14] presented m-polar fuzzy ideals of BCK/BCI-algebras, m-polar (α, β)-fuzzy ideals of BCK/BCI-algebras and (complete) normality of m-pF subalgebras in BCK/BCI-algebras. Garg [22]-[25] presented a new generalized Pythagorean fuzzy information aggregation (GPFIA) using Einstein operations and applied it to multi-criteria decision making (MCDM) by introducing various decision-making techniques employing aggregation operators. In contemporary times, Jana *et al.* [29]-[31] presented fascinating applications in decision making using different sorts of Dombi's aggregation operators. Feng *et al.* [19]-[21] presented some attention-grabbing results on soft sets, rough sets, generalized intuitionistic fuzzy soft sets and Lexicographic orders of intuitionistic fuzzy values and their relationships. Hashmi *et al.* [26] introduced the notion of m-polar neutrosophic set and m-polar neutrosophic topology and their applications to multi-criteria decision-making (MCDM) in medical diagnosis and clustering analysis. Hashmi and Riaz [27] introduced a novel approach to censuses process by using Pythagorean m-polar fuzzy Dombi's aggregation operators. Naeem *et al.* [35] introduced Pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators. Riaz and Naeem [44] introduced the concept of measurable Soft Mappings and related results. Riaz *et al.* [45, 46, 47] introduced N-soft topology and soft rough topology with its applications to group decision making. Riaz and Hashmi [48] introduced the concept of cubic m-polar fuzzy set and presented multi-attribute group decision making (MAGDM) method for agribusiness in the environment of various cubic m-polar fuzzy averaging aggregation operators. Riaz and Hashmi [49] introduced the notion of linear Diophantine fuzzy Set (LDFS) and its Applications towards multi-attribute decision making problems. Linear Diophantine fuzzy Set (LDFS) is superior than IFSs, PFSs and q-ROFSs. Riaz and Hashmi [50] introduced novel concepts of soft rough Pythagorean m-Polar fuzzy sets and Pythagorean m-polar fuzzy soft rough sets with application to decision-making. Riaz and Tehrim [51]-[55] established the idea of bipolar fuzzy soft topology, cubic bipolar fuzzy set and cubic bipolar fuzzy ordered weighted geometric aggregation operators, bipolar fuzzy soft mappings, TOPSIS method with bipolar neutrosophic soft topology, and their applications to aplications to multi-criteria group decision

making (MCGDM).
Zhang [62] proposed bipolar fuzzy sets as extension of fuzzy sets in 1994. Lee [32], in 2000, proposed an extension of fuzzy sets titled bipolar-valued fuzzy sets and presented two kinds of its representation. Chen *et al.* [17] generalized the notion of bipolar fuzzy sets to m-polar fuzzy sets and rendered some applications of m-polar fuzzy sets in real world problems. Smarandache [56] presented the notion of neutrosophic sets for coping with problems possessing imprecision, indeterminacy and inconsistency. Getting inspired by Smarandache, Maji [33] presented neutrosophic soft set along with peculiar concepts and characteristics. Qamar and Hassan [1] presented an approach toward a Q-neutrosophic soft set and its application in decision making. Qamar and Hassan [2] also presented Q-neutrosophic soft relation and its application in decision making.
TOPSIS, initially presented by Hwang and Yoon [28], is a useful model for tackling DMPs of the real world. This technique assists the decision makers to reach at some final decision and analyzing the conclusion in a system manner without any partiality. A number of varied versions of TOPSIS including adjusted TOPSIS, extended TOPSIS and modified TOPSIS may be found in literature. In recent years, this technique has been successfully applied in the fields of medical sciences, water management, business, transportation analysis, quality control, human resources management, and product design etc. which may be found in literature.
The prime ambition behind this research is to extend the notion of Pythagorean m-polar fuzzy sets (PmFSs) presented by Naeem *et al.* [36] to Pythagorean m-polar fuzzy soft sets (PmFSSs) along with algebraic operations on these sets and explore some idiosyncratic characteristics of this hybrid structure. PmFSSs have natural applications in multiple-valued logic, multi-sensor, multi-source and multi-process information fusion. PmFSSs provide a strong mathematical model to take in hand MCGDM problems. In order to tackle real world problems where intuitionistic fuzzy soft sets cannot deal with the situation when sum of membership and non-membership degrees of the parameter exceeds 1 making MCGDM demarcated and hence affecting the optimum decision, PmFSSs do not leave us alone and unassisted. PmFSSs provide a large number of applications to MCGDM problems in artificial intelligence, image processing, medical diagnosis, forecasting, recruitment problems and many other real life problems.
The article is prescribed as follows: In Section 1, account of different sorts of sets along with decision making technique of TOPSIS is presented with brevity. Section 2 is devoted to cover concise but comprehensive definitions of different sorts of sets that would be assisting in remnant part of the paper. The next segment i.e. Section 3 of this article serves as the main organ of this paper. In this section, we present the notion of PmFSSs along with associated mathematical operations and related results on these sets. In the very next segment of this article i.e. in Section 4, we exhibited how PmFSSs may be utilized in handling everyday problems using TOPSIS method. We ended with a concrete conclusion and some future directions in Section 5.

2. Preliminaries

In this section, we call to mind some fundamentals of different kinds of sets with brevity that would be ready to lend a hand in the remnant part of this article.

Definition 2.1. [60] Presume that X is a non-void set of objects. A *fuzzy set* $\mathcal{A}$ in X comprises ordered doublets in which abscissa is member of X and the ordinate is a mapping (termed as membership function of fuzzy set $\mathcal{A}$) that drags elements of X to the unit closed interval $[0,1]$.

Definition 2.2. [16] An *intuitionistic fuzzy set* (IF-set in brief) over the underlying set X is defined as

$$\mathbb{A} = \{< \zeta, \mu_{\mathbb{A}}(\zeta), \nu_{\mathbb{A}}(\zeta) >: \zeta \in X\}$$

The mappings $\mu_{\mathbb{A}}$ and $\nu_{\mathbb{A}}$ in order are acknowledged as the degrees of membership and non-membership of the element $\zeta \in X$ to the set $\mathbb{A}$ and send elements of X to unit closed interval along with the constraint that their sum must not exceed unity.

Definition 2.3. [57] A *Pythagorean fuzzy set*, abbreviated as PF-set, over X is a family of the form

$$\mathbb{P} = \Big\{ < \zeta, \mu_{\mathbb{P}}(\zeta), \nu_{\mathbb{P}}(\zeta) >: \zeta \in X\Big\}$$

where $\mu_{\mathbb{P}}$ and $\nu_{\mathbb{P}}$ are mappings from some crisp set X to the unit closed interval with the restriction that sum of their squared values should not exceed unity i.e. $0 \leq \mu^2_{\mathbb{P}}(\zeta) + \nu^2_{\mathbb{P}}(\zeta) \leq 1$, called correspondingly the grade of association and non-association of $\zeta \in X$ to the set $\mathbb{P}$. The pair $(\mu_{\mathbb{P}}, \nu_{\mathbb{P}})$ is called Pythagorean fuzzy number (PFN). The number $\gamma_{\mathbb{P}}(\zeta) = \sqrt{1 - \mu^2_{\mathbb{P}}(\zeta) - \nu^2_{\mathbb{P}}(\zeta)}$ is called the hesitation margin.

Definition 2.4. [17] An m-polar fuzzy set on the reference set X is characterized by a mapping $\mathfrak{A} : X \mapsto [0,1]^m$, where m is any arbitrary natural number.

Definition 2.5. [34] Let X be a reference set and E a non-void collection of attributes with $A \sqsubseteq E$. A *soft set* is a parameterized collection designated as (ψ, A) where ψ is a map that drives elements of A to the power set 2^X of X.

Definition 2.6. [36] Assume that m is a natural number. A *Pythagorean m-polar fuzzy set* (a PmFS for short) $\mathcal{P}$ over an underlying set X is characterized by two mappings $\mu^{(i)}_{\mathcal{P}} : X \mapsto [0,1]$ (traditionally acknowledged membership functions) and $\nu^{(i)}_{\mathcal{P}} : X \mapsto [0,1]$ (conventionally called non-membership functions) with the constraint that sum of their squared values should not exceed unity i.e. $0 \leq \Big(\mu^{(i)}_{\mathcal{P}}(\zeta)\Big)^2 + \Big(\nu^{(i)}_{\mathcal{P}}(\zeta)\Big)^2 \leq 1$, for $i = 1, 2, \cdots, m$.
A PmFS may be expressed in set-builder notation as

$$\mathcal{P} = \Big\{\big\langle \zeta, \big(\mu^{(i)}_{\mathcal{P}}(\zeta), \nu^{(i)}_{\mathcal{P}}(\zeta)\big)_i \big\rangle : \zeta \in X; i = 1, 2, \cdots, m\Big\}$$

where

$$\big(\mu^{(i)}_{\mathcal{P}}(\zeta), \nu^{(i)}_{\mathcal{P}}(\zeta)\big)_i = \Big(\big(\mu^{(1)}_{\mathcal{P}}(\zeta), \nu^{(2)}_{\mathcal{P}}(\zeta)\big), \big(\mu^{(2)}_{\mathcal{P}}(\zeta), \nu^{(2)}_{\mathcal{P}}(\zeta)\big), \cdots, \big(\mu^{(m)}_{\mathcal{P}}(\zeta), \nu^{(m)}_{\mathcal{P}}(\zeta)\big)\Big)$$

3. Pythagorean m-polar Fuzzy Soft Sets

We devote this section to introduce novel concepts of a new hybrid structure Pythagorean m-polar fuzzy soft sets.

Definition 3.1. Assume that m is a natural number. For some non-void collection of attributes E, let $A = \{e_1, e_2, \cdots, e_n\}$ be a subset of E i.e $A \sqsubseteq E$. A *Pythagorean m-polar fuzzy soft set* (a PmFSS for short) ψ_A over an underlying set X is characterized by the mapping $\psi : A \mapsto PmFS(X)$, where PmFS(X) denotes the collection of all Pythagorean m-polar fuzzy sets over X.
A PmFSS may be expressed in set-builder notation as

$$\psi_A = \Big\{\Big(e, \big\langle\zeta, \big(\mu^{(i)}_{\mathcal{P}}(e)(\zeta), \nu^{(i)}_{\mathcal{P}}(e)(\zeta)\big)_i\big\rangle\Big) : e \in A, \zeta \in X; i = 1, 2, \cdots, m\Big\}$$

or more conveniently as

$$\begin{aligned}\psi_A &= \Bigg\{ e, \Bigg\{ \frac{\zeta}{\mu^{(1)}_{\mathcal{P}}(e)(\zeta), \nu^{(1)}_{\mathcal{P}}(e)(\zeta)\ ,\ \mu^{(2)}_{\mathcal{P}}(e)(\zeta), \nu^{(2)}_{\mathcal{P}}(e)(\zeta)\ , \cdots,\ \mu^{(m)}_{\mathcal{P}}(e)(\zeta), \nu^{(m)}_{\mathcal{P}}(e)(\zeta)} \Bigg\}\Bigg) \\ &: \quad e \in A, \zeta \in X \Bigg\} \\ &= \Bigg\{ e, \Bigg\{ \frac{\zeta}{\mu^{(i)}_{\mathcal{P}}(e)(\zeta), \nu^{(i)}_{\mathcal{P}}(e)(\zeta)\ _i} \Bigg\}\Bigg) : e \in A, \zeta \in X; i = 1, 2, \cdots, m \Bigg\}\end{aligned}$$

If cardinality of X is k, then tabular formation of ψ_A is

ψ_A	e_1	e_2	$\cdots$	e_n
ζ_1	$\mu^{(i)}_{\mathcal{P}}(e_1)(\zeta_1), \nu^{(i)}_{\mathcal{P}}(e_1)(\zeta_1)\ _i$	$\mu^{(i)}_{\mathcal{P}}(e_2)(\zeta_1), \nu^{(i)}_{\mathcal{P}}(e_2)(\zeta_1)\ _i$	$\cdots$	$\mu^{(i)}_{\mathcal{P}}(e_n)(\zeta_1), \nu^{(i)}_{\mathcal{P}}(e_n)(\zeta_1)\ _i$
ζ_2	$\mu^{(i)}_{\mathcal{P}}(e_1)(\zeta_2), \nu^{(i)}_{\mathcal{P}}(e_1)(\zeta_2)\ _i$	$\mu^{(i)}_{\mathcal{P}}(e_2)(\zeta_2), \nu^{(i)}_{\mathcal{P}}(e_2)(\zeta_2)\ _i$	$\cdots$	$\mu^{(i)}_{\mathcal{P}}(e_n)(\zeta_2), \nu^{(i)}_{\mathcal{P}}(e_n)(\zeta_2)\ _i$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ζ_k	$\mu^{(i)}_{\mathcal{P}}(e_1)(\zeta_k), \nu^{(i)}_{\mathcal{P}}(e_1)(\zeta_k)\ _i$	$\mu^{(i)}_{\mathcal{P}}(e_2)(\zeta_k), \nu^{(i)}_{\mathcal{P}}(e_2)(\zeta_k)\ _i$	$\cdots$	$\mu^{(i)}_{\mathcal{P}}(e_n)(\zeta_k), \nu^{(i)}_{\mathcal{P}}(e_n)(\zeta_k)\ _i$

and in matrix format as

$$\psi_A = \begin{bmatrix} \mu^{(i)}_{\mathcal{P}}(e_1)(\zeta_1), \nu^{(i)}_{\mathcal{P}}(e_1)(\zeta_1)\ _i & \mu^{(i)}_{\mathcal{P}}(e_2)(\zeta_1), \nu^{(i)}_{\mathcal{P}}(e_2)(\zeta_1)\ _i & \cdots & \mu^{(i)}_{\mathcal{P}}(e_n)(\zeta_1), \nu^{(i)}_{\mathcal{P}}(e_n)(\zeta_1)\ _i \\ \mu^{(i)}_{\mathcal{P}}(e_1)(\zeta_2), \nu^{(i)}_{\mathcal{P}}(e_1)(\zeta_2)\ _i & \mu^{(i)}_{\mathcal{P}}(e_2)(\zeta_2), \nu^{(i)}_{\mathcal{P}}(e_2)(\zeta_2)\ _i & \cdots & \mu^{(i)}_{\mathcal{P}}(e_n)(\zeta_2), \nu^{(i)}_{\mathcal{P}}(e_n)(\zeta_2)\ _i \\ \vdots & \vdots & \ddots & \vdots \\ \mu^{(i)}_{\mathcal{P}}(e_1)(\zeta_k), \nu^{(i)}_{\mathcal{P}}(e_1)(\zeta_k)\ _i & \mu^{(i)}_{\mathcal{P}}(e_2)(\zeta_k), \nu^{(i)}_{\mathcal{P}}(e_2)(\zeta_k)\ _i & \cdots & \mu^{(i)}_{\mathcal{P}}(e_n)(\zeta_k), \nu^{(i)}_{\mathcal{P}}(e_n)(\zeta_k)\ _i \end{bmatrix}$$

This $k \times n$ matrix is reckoned as *PmFS-matrix.* The collection of all PmFSSs defined over X will be designated by PmFSS(X).

Example 3.2. Let $X = \{b, s, c, z\}$ be a crisp set and $A = \{e_1, e_2\} \sqsubseteq E$, then

$$\psi_A = \Bigg\{\Bigg(e_1, \Bigg\{\frac{b}{(0.19,0.74),(0.28,0.79),(0.04,0.97)}, \frac{s}{(0.38,0.62),(0.74,0.36),(0.88,0.31)}\Bigg\}\Bigg), \Bigg(e_2, \Bigg\{\frac{b}{(0.62,0.28),(0.59,0.47),(0.26,0.11)}, \frac{s}{(0.37,0.69),(0.01,0.58),(0.72,0.71)}\Bigg\}\Bigg)\Bigg\}$$

is a P3FSS. In tabular array, we may represent this set as shown in Table 1:

ψ_A	e_1	e_2
b	$\{(0.19, 0.74), (0.28, 0.79), (0.04, 0.97)\}$	$\{(0.62, 0.28), (0.59, 0.47), (0.26, 0.11)\}$
s	$\{(0.38, 0.62), (0.74, 0.36), (0.88, 0.31)\}$	$\{(0.37, 0.69), (0.01, 0.58), (0.72, 0.71)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 1. Tabular representation of ψ_A

In more brief form, Table 3.2 may be re-expressed as in Table 2:

ψ_A	e_1	e_2
b	$\{(0.19, 0.74), (0.28, 0.79), (0.04, 0.97)\}$	$\{(0.62, 0.28), (0.59, 0.47), (0.26, 0.11)\}$
s	$\{(0.38, 0.62), (0.74, 0.36), (0.88, 0.31)\}$	$\{(0.37, 0.69), (0.01, 0.58), (0.72, 0.71)\}$

TABLE 2. Brief tabular representation of ψ_A

The matrix form of ψ_A is

$$\psi_A = \begin{bmatrix} \{(0.19, 0.74), (0.28, 0.79), (0.04, 0.97)\} & \{(0.62, 0.28), (0.59, 0.47), (0.26, 0.11)\} \\ \{(0.38, 0.62), (0.74, 0.36), (0.88, 0.31)\} & \{(0.37, 0.69), (0.01, 0.58), (0.72, 0.71)\} \\ \{(0, 1), (0, 1), (0, 1)\} & \{(0, 1), (0, 1), (0, 1)\} \\ \{(0, 1), (0, 1), (0, 1)\} & \{(0, 1), (0, 1), (0, 1)\} \end{bmatrix}$$

Definition 3.3. Let ψ_A be a PmFSS over X with $e \in A \sqsubseteq E$. The aggregate of those points ζ of X for which $\mu_{\mathcal{P}}^{(i)}(e)(\zeta) \neq 0$ or $\nu_{\mathcal{P}}^{(i)}(e)(\zeta) \neq 1$, for at least one $i = 1, 2, \cdots, m$, is called *support* of ψ_A i.e.

$$supp(\psi_A) = \{\zeta \in X : \mu_{\mathcal{P}}^{(i)}(e)(\zeta) \neq 0 \text{ or } \nu_{\mathcal{P}}^{(i)}(e)(\zeta) \neq 1 \text{ for at least one } i = 1, 2, \cdots, m\}$$

Example 3.4. For the PmFSS represented in Table 3, defined over $X = \{t, b, j, k\}$, $supp(\psi_A) = \{t, b, k\}$.

ψ_A	e_1	e_2
t	$\{(0.32, 0.00), (0.49, 0.26), (0.37, 0.15)\}$	$\{(0.54, 0.13), (0.26, 0.18), (0.39, 0.51)\}$
b	$\{(0.33, 0.71), (1.00, 0.00), (0.27, 0.79)\}$	$\{(0.36, 0.28), (0.41, 0.52), (0.33, 0.78)\}$
j	$\{(0.00, 1.00), (0.00, 1.00), (0.00, 1.00)\}$	$\{(0.00, 1.00), (0.00, 1.00), (0.00, 1.00)\}$
k	$\{(0.83, 0.24), (0.71, 0.16), (0.45, 0.12)\}$	$\{(0.21, 0.43), (0.36, 0.18), (0.91, 0.23)\}$

TABLE 3. PmFSS ψ_A

Definition 3.5. Let ψ_A be a PmFSS over X with $e \in A \sqsubseteq E$. The aggregate of those points ζ of X for which $\mu_{\mathcal{P}}^{(i)}(e)(\zeta) = 1$ (and obviously $\nu_{\mathcal{P}}^{(i)}(e)(\zeta) = 0$), for at least one $i = 1, 2, \cdots, m$, is called *core* of ψ_A i.e.

$$core(\psi_A) = \{\zeta \in X : \mu_{\mathcal{P}}^{(i)}(e)(\zeta) = 1 \text{ for at least one } i = 1, 2, \cdots, m\}.$$

Example 3.6. For the PmFS ψ_A given in Example 3.4, $core(\psi_A) = \{b\}$.

Definition 3.7. Let ψ_A be a PmFSS over X with $e \in A \sqsubseteq E$. The maximum value attained by the membership function $\mu_{\mathcal{P}}^{(i)}(e)(\zeta)$, for any $\zeta \in X$ and any $i \in \{1, 2, \cdots, m\}$, is termed as *height* of ψ_A and is designated as $ht(\psi_A)$. A PmFSS ψ_A is said to be *normal* if $ht(\psi_A) = 1$ and is reckoned as *subnormal* otherwise.

Example 3.8. For the PmFSS ψ_A given in Example 3.2, $ht(\psi_A) = 0.88$ and for the PmFSS ψ_A given in Example 3.4, $ht(\psi_A) = 1$. Hence, the PmFSS ψ_A given in Example 3.4 is normal whereas the PmFSS ψ_A given in Example 3.2 is subnormal.

Definition 3.9. Let (ψ_1, A_1) and (ψ_2, A_2) be PmFSSs over X with $A_1, A_2 \sqsubseteq E$. We say that (ψ_1, A_1) is a *subset* of (ψ_2, A_2), written $(\psi_1, A_1)\widetilde{\sqsubseteq}(\psi_2, A_2)$ if

i. $A_1 \sqsubseteq A_2$
ii. $\mu_{\mathcal{P}_1}^{(i)}(e)(\zeta) \leq \mu_{\mathcal{P}_2}^{(i)}(e)(\zeta)$, and
iii. $\nu_{\mathcal{P}_1}^{(i)}(e)(\zeta) \geq \nu_{\mathcal{P}_2}^{(i)}(e)(\zeta)$

for all $e \in A_1$, $\zeta \in X$ and all admissible values of i.
(ψ_1, A_1) and (ψ_2, A_2) are said to be *equal* if and only if one of them is sandwiched between the other i.e. $(\psi_1, A_1)\widetilde{\sqsubseteq}(\psi_2, A_2)\widetilde{\sqsubseteq}(\psi_1, A_1)$.

Example 3.10. Let $A_1 = \{e_1\}, A_2 = \{e_1, e_2\} \sqsubseteq E$ and (ψ_1, A_1), (ψ_2, A_2) be PmFSSs, given in Tables 4 and 5 respectively, over some set $X = \{g, r, p\}$, then $(\psi_1, A_1)\widetilde{\sqsubseteq}(\psi_2, A_2)$.

(ψ_1, A_1)	e_1
g	$\{(0.29, 0.16), (0.43, 0.51), (0.33, 0.16)\}$
r	$\{(0.29, 0.42), (0.04, 0.86), (0.32, 0.24)\}$
p	$\{(0.26, 0.07), (0.21, 0.19), (0.00, 1.00)\}$

TABLE 4. PmFSS (ψ_1, A_1)

(ψ_2, A_2)	e_1	e_2
g	$\{(0.34, 0.11), (0.78, 0.37), (0.33, 0.10)\}$	$\{(0.36, 0.58), (0.30, 0.63), (0.52, 0.44)\}$
r	$\{(0.47, 0.30), (0.19, 0.52), (0.49, 0.24)\}$	$\{(0.11, 0.19), (0.28, 0.74), (0.49, 0.50)\}$
p	$\{(0.50, 0.02), (0.36, 0.13), (0.82, 0.26)\}$	$\{(0.83, 0.21), (0.44, 0.79), (0.69, 0.29)\}$

TABLE 5. PmFSS (ψ_2, A_2)

Remark. If (ψ_1, A_1) and (ψ_2, A_2) are PmFSSs over X, then $(\psi_1, A_1)\widetilde{\sqsubseteq}(\psi_2, A_2)$ implies $ht(\psi_1, A_1) \leq ht(\psi_2, A_2)$. The converse, however, may not hold.

Definition 3.11. A PmFSS (ψ, E) over X is said to be a *null PmFSS* if $\mu_{\psi}^{(i)}(e)(\zeta) = 0$ and $\nu_{\psi}^{(i)}(e)(\zeta) = 1$, for all $e \in E$, $\zeta \in X$ and all admissible values of i. It is denoted by (Φ, E) or Φ_E. The tabular representation of Φ_E is as given in Table 6.

Φ_E	e_1	e_2	$\cdots$	e_n
ζ_1	$\{(0,1),(0,1),\cdots,(0,1)\}$	$\{(0,1),(0,1),\cdots,(0,1)\}$	$\cdots$	$\{(0,1),(0,1),\cdots,(0,1)\}$
ζ_2	$\{(0,1),(0,1),\cdots,(0,1)\}$	$\{(0,1),(0,1),\cdots,(0,1)\}$	$\cdots$	$\{(0,1),(0,1),\cdots,(0,1)\}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ζ_m	$\{(0,1),(0,1),\cdots,(0,1)\}$	$\{(0,1),(0,1),\cdots,(0,1)\}$	$\cdots$	$\{(0,1),(0,1),\cdots,(0,1)\}$

TABLE 6. Null PmFSS Φ_E

Notice that both support and core of Φ_E are empty set. Further, height of Φ_E is 0 and hence Φ_E is a subnormal PmFSS.

Definition 3.12. A PmFSS (ψ, E) over X is said to be an *absolute PmFSS* if $\mu_\psi^{(i)}(e)(\zeta) = 1$ and $\nu_\psi^{(i)}(e)(\zeta) = 0$, for all $e \in E$, $\zeta \in X$ and all admissible values of i. It is denoted by $(\breve{X}, E)$ or $\breve{X}_E$. The tabular representation of $\breve{X}_E$ is as given in Table 7.

$\breve{X}_E$	e_1	e_2	$\cdots$	e_n
ζ_1	$\{(1,0),(1,0),\cdots,(1,0)\}$	$\{(1,0),(1,0),\cdots,(1,0)\}$	$\cdots$	$\{(1,0),(1,0),\cdots,(1,0)\}$
ζ_2	$\{(1,0),(1,0),\cdots,(1,0)\}$	$\{(1,0),(1,0),\cdots,(1,0)\}$	$\cdots$	$\{(1,0),(1,0),\cdots,(1,0)\}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ζ_m	$\{(1,0),(1,0),\cdots,(1,0)\}$	$\{(1,0),(1,0),\cdots,(1,0)\}$	$\cdots$	$\{(1,0),(1,0),\cdots,(1,0)\}$

TABLE 7. Absolute PmFSS $\breve{X}_E$

Notice that both support and core of $\breve{X}_E$ are X. Further, height of $\breve{X}_E$ is 1 and hence $\breve{X}_E$ is a normal PmFSS.

Proposition 3.13. *If (ψ, E) is any PmFSS over X, then $(\Phi, E)\widetilde{\sqsubseteq}(\psi, E)\widetilde{\sqsubseteq}(\breve{X}, E)$.*

Proof. Straight forward. □

Remark. It follows from Proposition 3.13 that (Φ, E) is the smallest and $(\breve{X}, E)$ is the largest PmFSS over X.

Definition 3.14. The *complement* of a PmFSS

$$\psi_E = \left\{ \left(e, \left\{ \frac{\zeta}{\left(\mu_\psi^{(i)}(e)(\zeta), \nu_\psi^{(i)}(e)(\zeta)\right)_i} \right\} \right) : e \in E, \zeta \in X; i = 1, 2, \cdots, m \right\}$$

over X is defined as

$$\psi_E^c = \left\{ \left(e, \left\{ \frac{\zeta}{\left(\nu_\psi^{(i)}(e)(\zeta), \mu_\psi^{(i)}(e)(\zeta)\right)_i} \right\} \right) : e \in E, \zeta \in X; i = 1, 2, \cdots, m \right\}$$

Notice that $\Phi_E^c = \breve{X}_E$ and $\breve{X}_E^c = \Phi_E$. Moreover, $\left(\psi_E^c\right)^c = \psi_E$.

Example 3.15. For the PmFSS given in Example 3.2, the complement of ψ_A is as given in Table 8:

ψ_A^c	e_1	e_2
b	$\{(0.74, 0.19), (0.79, 0.28), (0.97, 0.04)\}$	$\{(0.28, 0.62), (0.47, 0.59), (0.11, 0.26)\}$
s	$\{(0.62, 0.38), (0.36, 0.74), (0.31, 0.88)\}$	$\{(0.69, 0.37), (0.58, 0.01), (0.71, 0.72)\}$
c	$\{(1, 0), (1, 0), (1, 0)\}$	$\{(1, 0), (1, 0), (1, 0)\}$
z	$\{(1, 0), (1, 0), (1, 0)\}$	$\{(1, 0), (1, 0), (1, 0)\}$

TABLE 8. PmFSS ψ_A^c

Definition 3.16. The *union* of two PmFSSs (ψ_1, A_1) and (ψ_2, A_2) defined over the same universe X is defined as

$$(\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2) = \left\{ e, \left\{ \frac{\zeta}{\max\ \mu_{\psi_1}^{(i)}(e)(\zeta), \mu_{\psi_2}^{(i)}(e)(\zeta)\ , \min\ \nu_{\psi_1}^{(i)}(e)(\zeta), \nu_{\psi_2}^{(i)}(e)(\zeta)} \right\}_i \right) : e \in A_1 \sqcup A_2, \zeta \in X; i = 1, 2, \cdots, m \right\}$$

Definition 3.17. The *intersection* of two PmFSSs (ψ_1, A_1) and (ψ_2, A_2) defined over the same universe X is defined as

$$(\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2) = \left\{ e, \left\{ \frac{\zeta}{\min\ \mu_{\psi_1}^{(i)}(e)(\zeta), \mu_{\psi_2}^{(i)}(e)(\zeta)\ , \max\ \nu_{\psi_1}^{(i)}(e)(\zeta), \nu_{\psi_2}^{(i)}(e)(\zeta)} \right\}_i \right) : e \in A_1 \sqcap A_2, \zeta \in X; i = 1, 2, \cdots, m \right\}$$

Example 3.18. Let $X = \{y, d, g\}$ be a crisp set and $A_1 = \{e_1, e_2\}, A_2 = \{e_2, e_3\} \sqsubseteq E$. Let (ψ_1, A_1) and (ψ_2, A_2) be as given in Tables 9 and 10, respectively.

(ψ_1, A_1)	e_1	e_2
y	$\{(0.62, 0.17), (0.31, 0.82), (0.12, 0.06)\}$	$\{(0.10, 0.53), (0.84, 0.36), (0.13, 0.14)\}$
d	$\{(0.02, 0.28), (0.16, 0.39), (0.30, 0.80)\}$	$\{(0.29, 0.54), (0.36, 0.11), (0.03, 0.99)\}$
g	$\{(0.51, 0.52), (0.39, 0.42), (0.52, 0.53)\}$	$\{(0.00, 1.00), (0.38, 0.62), (0.81, 0.26)\}$

TABLE 9. PmFSS (ψ_1, A_1)

(ψ_2, A_2)	e_2	e_3
y	$\{(0.54, 0.11), (0.14, 0.15), (0.81, 0.17)\}$	$\{(0.44, 0.24), (0.43, 0.36), (0.11, 0.31)\}$
d	$\{(0.29, 0.56), (0.18, 0.05), (0.26, 0.67)\}$	$\{(0.24, 0.22), (0.38, 0.35), (0.40, 0.63)\}$
g	$\{(0.29, 0.46), (0.37, 0.37), (0.51, 0.02)\}$	$\{(0.25, 0.83), (0.52, 0.58), (1.00, 0.00)\}$

TABLE 10. PmFSS (ψ_2, A_2)

Then union and intersection of (ψ_1, A_1) and (ψ_2, A_2) are represented in Tables 11 and 12, respectively.

Union	e_1	e_2	e_3
y	$\{(0.62, 0.17), (0.31, 0.82), (0.12, 0.06)\}$	$\{(0.54, 0.11), (0.84, 0.15), (0.81, 0.14)\}$	$\{(0.44, 0.24), (0.43, 0.36), (0.11, 0.31)\}$
d	$\{(0.02, 0.28), (0.16, 0.39), (0.30, 0.80)\}$	$\{(0.29, 0.54), (0.36, 0.05), (0.26, 0.67)\}$	$\{(0.24, 0.22), (0.38, 0.35), (0.40, 0.63)\}$
g	$\{(0.51, 0.52), (0.39, 0.42), (0.52, 0.53)\}$	$\{(0.29, 0.46), (0.38, 0.37), (0.81, 0.02)\}$	$\{(0.25, 0.83), (0.52, 0.58), (1.00, 0.00)\}$

TABLE 11. PmFSS $(\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2)$

Intersection	e_2
y	$\{(0.10, 0.53), (0.14, 0.36), (0.13, 0.17)\}$
d	$\{(0.29, 0.56), (0.18, 0.11), (0.03, 0.99)\}$
g	$\{(0.00, 1.00), (0.37, 0.62), (0.51, 0.26)\}$

TABLE 12. PmFSS $(\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2)$

Proposition 3.19. *If (ψ, A), (ψ_1, A_1), (ψ_2, A_2) and (ψ_3, A_3) are PmFSSs over X, then*

(i) $(\Phi, A)\widetilde{\sqcup}(\psi, A) = (\psi, A)$.
(ii) $(\Phi, A)\widetilde{\sqcap}(\psi, A) = (\Phi, A)$.
(iii) $(\check{X}, E)\widetilde{\sqcup}(\psi, A) = (\check{X}, E)$.
(iv) $(\check{X}, E)\widetilde{\sqcap}(\psi, A) = (\psi, A)$.
(v) $(\psi, A)\widetilde{\sqcup}(\psi, A) = (\psi, A)$.
(vi) $(\psi, A)\widetilde{\sqcap}(\psi, A) = (\psi, A)$.
(vii) $(\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2) = (\psi_2, A_2)\widetilde{\sqcup}(\psi_1, A_1)$.
(viii) $(\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2) = (\psi_2, A_2)\widetilde{\sqcap}(\psi_1, A_1)$.
(ix) $(\psi_1, A_1)\widetilde{\sqcup}\{(\psi_2, A_2)\widetilde{\sqcup}(\psi_3, A_3)\} = \{(\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2)\}\widetilde{\sqcup}(\psi_3, A_3)$.
(x) $(\psi_1, A_1)\widetilde{\sqcap}\{(\psi_2, A_2)\widetilde{\sqcap}(\psi_3, A_3)\} = \{(\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2)\}\widetilde{\sqcap}(\psi_3, A_3)$.
(xi) $(\psi_1, A_1)\widetilde{\sqcup}\{(\psi_2, A_2)\widetilde{\sqcap}(\psi_3, A_3)\} = \{(\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2)\}\widetilde{\sqcap}\{(\psi_1, A_1)\widetilde{\sqcup}(\psi_3, A_3)\}$.
(xii) $(\psi_1, A_1)\widetilde{\sqcap}\{(\psi_2, A_2)\widetilde{\sqcup}(\psi_3, A_3)\} = \{(\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2)\}\widetilde{\sqcup}\{(\psi_1, A_1)\widetilde{\sqcap}(\psi_3, A_3)\}$.

Proof. Follows directly from definition. □

Corollary 3.20. *(i)* $\Phi_E\widetilde{\sqcup}\check{X}_E = \check{X}_E$.
(ii) $\Phi_E\widetilde{\sqcap}\check{X}_E = \Phi_E$.

Proposition 3.21. *If (ψ_1, A_1) and (ψ_2, A_2) are PmFSSs over X, then any one of them may be sandwiched between $(\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2)$ and $(\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2)$ i.e.*

(i) $(\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2)\widetilde{\sqsubseteq}(\psi_1, A_1)\widetilde{\sqsubseteq}(\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2)$.
(ii) $(\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2)\widetilde{\sqsubseteq}(\psi_2, A_2)\widetilde{\sqsubseteq}(\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2)$.

Proof. (i) follows from the fact that $\min\left\{\mu^{(i)}_{\psi_1}, \mu^{(i)}_{\psi_2}\right\} \leq \mu^{(i)}_{\psi_1} \leq \max\left\{\mu^{(i)}_{\psi_1}, \mu^{(i)}_{\psi_2}\right\}$ and $\max\left\{\nu^{(i)}_{\psi_1}, \nu^{(i)}_{\psi_2}\right\} \geq \nu^{(i)}_{\psi_1} \geq \min\left\{\nu^{(i)}_{\psi_1}, \nu^{(i)}_{\psi_2}\right\}$. The proof of (ii) is similar. □

Proposition 3.22. *If (ψ_1, A_1) and (ψ_2, A_2) are PmFSSs over X, then contrary to crisp sets, De Morgan laws do not hold i.e.*

(i) $\left((\psi_1, A_1)\widetilde{\sqcup}(\psi_2, A_2)\right)^c \neq (\psi_1, A_1)^c\widetilde{\sqcap}(\psi_2, A_2)^c$.

(ii) $\left((\psi_1, A_1)\widetilde{\sqcap}(\psi_2, A_2)\right)^c \neq (\psi_1, A_1)^c\widetilde{\sqcup}(\psi_2, A_2)^c$.

Example 3.23. Consider the PmFSSs ψ_A given in Example 3.2. The tabular representations of $\psi_A\widetilde{\sqcup}\psi_A^c$ and $\psi_A\widetilde{\sqcap}\psi_A^c$ are given in Tables 13 and 14, respectively.

$\psi_A\widetilde{\sqcup}\psi_A^c$	e_1	e_2
b	$\{(0.74, 0.19), (0.79, 0.28), (0.97, 0.04)\}$	$\{(0.62, 0.28), (0.59, 0.47), (0.26, 0.11)\}$
s	$\{(0.62, 0.38), (0.74, 0.36), (0.88, 0.31)\}$	$\{(0.69, 0.37), (0.58, 0.01), (0.72, 0.71)\}$
c	$\{(1, 0), (1, 0), (1, 0)\}$	$\{(1, 0), (1, 0), (1, 0)\}$
z	$\{(1, 0), (1, 0), (1, 0)\}$	$\{(1, 0), (1, 0), (1, 0)\}$

TABLE 13. $\psi_A\widetilde{\sqcup}\psi_A^c$

$\psi_A\widetilde{\sqcap}\psi_A^c$	e_1	e_2
b	$\{(0.19, 0.74), (0.28, 0.79), (0.04, 0.97)\}$	$\{(0.28, 0.62), (0.47, 0.59), (0.11, 0.26)\}$
s	$\{(0.38, 0.62), (0.36, 0.74), (0.31, 0.88)\}$	$\{(0.37, 0.69), (0.01, 0.58), (0.71, 0.72)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 14. $\psi_A\widetilde{\sqcap}\psi_A^c$

We observe, keeping in view Tables 13 and 14, that $\psi_A\widetilde{\sqcup}\psi_A^c \neq \check{X}_A$ and $\psi_A\widetilde{\sqcap}\psi_A^c \neq \Phi_A$. Hence, we have the following proposition.

Proposition 3.24. *If ψ_A is a PmFSS over X, then unlike in crisp sets*

(i) $\psi_A\widetilde{\sqcup}\psi_A^c \neq \check{X}_A$.
(ii) $\psi_A\widetilde{\sqcap}\psi_A^c \neq \Phi_A$.

Definition 3.25. The *difference* of two PmFSSs (ψ_1, A_1) and (ψ_2, A_2) defined over the same universe X is defined as

$$(\psi_1, A_1)\widetilde{\setminus}(\psi_2, A_2) =$$
$$\left\{ e, \left\{ \frac{\zeta}{\max\ \mu^{(i)}_{\psi_1}(e)(\zeta), \nu^{(i)}_{\psi_2}(e)(\zeta)\ , \min\ \nu^{(i)}_{\psi_1}(e)(\zeta), \mu^{(i)}_{\psi_2}(e)(\zeta)\qquad {}_i} \right\} \right) : e \in A_1\setminus A_2, \zeta \in X; i = 1, 2, \cdots, m \right\}$$

Example 3.26. For the PmFSSs ψ_1 and ψ_2 given in Example 3.18, $(\psi_1, A_1)\widetilde{\setminus}(\psi_2, A_2)$ is exhibited in Table 15.

$(\psi_1, A_1)\widetilde{\setminus}(\psi_2, A_2)$	e_1
y	$\{(0.62, 0.17), (0.31, 0.82), (0.12, 0.06)\}$
d	$\{(0.02, 0.28), (0.16, 0.39), (0.30, 0.80)\}$
g	$\{(0.51, 0.52), (0.39, 0.42), (0.52, 0.53)\}$

TABLE 15. $(\psi_1, A_1)\widetilde{\setminus}(\psi_2, A_2)$

Definition 3.27. If (ψ, A) is a PmFSS extracted from X, then the *necessity operator* $\widetilde{\Box}$ on (ψ, A) is defined as

$$\widetilde{\Box}(\psi, A) = \left\{ \left(e, \left\{ \frac{\zeta}{\left(\mu_{\psi}^{(i)}(e)(\zeta), \sqrt{1-\left(\mu_{\psi}^{(i)}(e)(\zeta)\right)^2}\right)_i} \right\} \right) : e \in A, \zeta \in X; i = 1, 2, \cdots, m \right\}$$

Definition 3.28. If (ψ, A) is a PmFSS extracted from X, then the *possibility operator* $\widetilde{\Diamond}$ on (ψ, A) is defined as

$$\widetilde{\Diamond}(\psi, A) = \left\{ \left(e, \left\{ \frac{\zeta}{\left(\sqrt{1-\left(\nu_{\psi}^{(i)}(e)(\zeta)\right)^2}, \nu_{\psi}^{(i)}(e)(\zeta)\right)_i} \right\} \right) : e \in A, \zeta \in X; i = 1, 2, \cdots, m \right\}$$

Example 3.29. For the PmFS ψ given in Example 3.2, $\widetilde{\Box}\psi_A$ and $\widetilde{\Diamond}\psi_A$ are given in Tables 16 and 17, respectively:

$\widetilde{\Box}\psi_A$	e_1	e_2
b	$\{(0.19, 0.98), (0.28, 0.96), (0.04, 0.99)\}$	$\{(0.62, 0.78), (0.59, 0.81), (0.26, 0.96)\}$
s	$\{(0.38, 0.92), (0.74, 0.67), (0.88, 0.47)\}$	$\{(0.37, 0.93), (0.01, 0.99), (0.72, 0.69)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

Table 16. $\widetilde{\Box}\psi_A$

$\widetilde{\Diamond}\psi_A$	e_1	e_2
b	$\{(0.67, 0.74), (0.61, 0.79), (0.24, 0.97)\}$	$\{(0.96, 0.28), (0.88, 0.47), (0.99, 0.11)\}$
s	$\{(0.78, 0.62), (0.93, 0.36), (0.95, 0.31)\}$	$\{(0.72, 0.69), (0.81, 0.58), (0.70, 0.71)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

Table 17. $\widetilde{\Diamond}\psi_A$

Remark. The necessity and possibility operator defined above in definitions 3.27 and 3.28 transform any PmFSS ψ_A to m-polar fuzzy soft set.

Proposition 3.30. *For any PmFSS ψ_A defined over X, $\widetilde{\Box}\psi_A \widetilde{\sqsubseteq} \widetilde{\Diamond}\psi_A$.*

Proof. Since For each $\zeta \in X$, $e \in A$ and all admissible values of i, we have

$$\begin{aligned} \left(\mu_{\psi}^{(i)}(e)(\zeta)\right)^2 + \left(\nu_{\psi}^{(i)}(e)(\zeta)\right)^2 &\leq 1 \\ \therefore \mu_{\psi}^{(i)}(e)(\zeta) &\leq \sqrt{1-\left(\nu_{\psi}^{(i)}(e)(\zeta)\right)^2} \\ \& \ \nu_{\psi}^{(i)}(e)(\zeta) &\leq \sqrt{1-\left(\mu_{\psi}^{(i)}(e)(\zeta)\right)^2} \end{aligned}$$

so the result follows. □

Corollary 3.31. *For any PmFSS ψ_A, we have*

(i) $\widetilde{\Box}\psi_A \widetilde{\sqcup} \widetilde{\Diamond}\psi_A = \widetilde{\Diamond}\psi_A$
(ii) $\widetilde{\Box}\psi_A \widetilde{\sqcap} \widetilde{\Diamond}\psi_A = \widetilde{\Box}\psi_A$

Definition 3.32. The *sum* of two PmFSSs (ψ_1, A_1) and (ψ_2, A_2) extracted from the same universe X is defined as

$$(\psi_1, A_1)\widetilde{\oplus}(\psi_2, A_2) = \left\{\left(e, \left\{\frac{\zeta}{\left(\sqrt{\left(\mu^{(i)}_{\psi_1}(e)(\zeta)\right)^2 + \left(\mu^{(i)}_{\psi_2}(e)(\zeta)\right)^2 - \left(\mu^{(i)}_{\psi_1}(e)(\zeta)\mu^{(i)}_{\psi_2}(e)(\zeta)\right)^2}, \nu^{(i)}_{\psi_1}(e)(\zeta)\nu^{(i)}_{\psi_2}(e)(\zeta)\right)_i}\right\}\right) : e \in A_1 \sqcup A_2, \zeta \in X; i = 1, 2, \cdots, m\right\}$$

Example 3.33. For the PmFSSs (ψ_1, A_1) and (ψ_2, A_2) given in Example 3.18, $(\psi_1, A_1)\widetilde{\oplus}(\psi_2, A_2)$ is given in Table 18.

$(\psi_1, A_1)\widetilde{\oplus}(\psi_2, A_2)$	e_1	e_2
y	$\{(0.75, 0.02), (0.34, 0.12), (0.81, 0.01)\}$	$\{(0.45, 0.13), (0.87, 0.13), (0.17, 0.04)\}$
d	$\{(0.29, 0.16), (0.24, 0.02), (0.39, 0.54)\}$	$\{(0.37, 0.12), (0.51, 0.04), (0.40, 0.62)\}$
g	$\{(0.57, 0.24), (0.52, 0.16), (0.68, 0.01)\}$	$\{(0.25, 0.83), (0.61, 0.36), (1.00, 0.00)\}$

TABLE 18. $(\psi_1, A_1)\widetilde{\oplus}(\psi_2, A_2)$

Definition 3.34. The *product* of two PmFSSs (ψ_1, A_1) and (ψ_2, A_2) extracted from the same universe X is defined as

$$(\psi_1, A_1)\widetilde{\otimes}(\psi_2, A_2) = \left\{\left(e, \left\{\frac{\zeta}{\left(\mu^{(i)}_{\psi_1}(e)(\zeta)\mu^{(i)}_{\psi_2}(e)(\zeta), \sqrt{\left(\nu^{(i)}_{\psi_1}(e)(\zeta)\right)^2 + \left(\nu^{(i)}_{\psi_2}(e)(\zeta)\right)^2 - \left(\nu^{(i)}_{\psi_1}(e)(\zeta)\nu^{(i)}_{\psi_2}(e)(\zeta)\right)^2}\right)_i}\right\}\right) : e \in A_1 \sqcup A_2, \zeta \in X; i = 1, 2, \cdots, m\right\}$$

Example 3.35. For the PmFSSs (ψ_1, A_1) and (ψ_2, A_2) given in Example 3.18, $(\psi_1, A_1)\widetilde{\otimes}(\psi_2, A_2)$ is given in Table 19.

$(\psi_1, A_1)\widetilde{\otimes}(\psi_2, A_2)$	e_1	e_2
y	$\{(0.33, 0.20), (0.04, 0.82), (0.10, 0.18)\}$	$\{(0.04, 0.57), (0.36, 0.49), (0.01, 0.34)\}$
d	$\{(0.00, 0.61), (0.03, 0.39), (0.08, 0.90)\}$	$\{(0.07, 0.57), (0.14, 0.36), (0.01, 0.99)\}$
g	$\{(0.15, 0.65), (0.14, 0.54), (0.26, 0.53)\}$	$\{(0.00, 1.00), (0.20, 0.77), (0.81, 0.26)\}$

TABLE 19. $(\psi_1, A_1)\widetilde{\otimes}(\psi_2, A_2)$

Definition 3.36. If $(\psi_1, A_1) = (\psi_2, A_2)$ in Definition 3.34, then we express $(\psi_1, A_1)\widetilde{\otimes}(\psi_2, A_2)$ by $(\psi_1, A_1)^2$. Thus,

$$\begin{aligned}(\psi, A)^2 &= \left\{ e, \left\{ \frac{\zeta}{\left(\mu_\psi^{(i)}(e)(\zeta)\right)^2, \sqrt{2\left(\nu_\psi^{(i)}(e)(\zeta)\right)^2 - \left(\nu_\psi^{(i)}(e)(\zeta)\right)^4}} \right\}_i \right) : e \in A, \zeta \in X; i = 1, \cdots, m \Bigg\} \\ &= \left\{ e, \left\{ \frac{\zeta}{\left(\mu_\psi^{(i)}(e)(\zeta)\right)^2, \sqrt{1 - \left(1 - \left(\nu_\psi^{(i)}(e)(\zeta)\right)^2\right)^2}} \right\}_i \right) : e \in A, \zeta \in X; i = 1, \cdots, m \Bigg\}\end{aligned}$$

The set $(\psi, A)^2$ is termed as *concentration* of (ψ, A), designated as $con(\psi, A)$. In general, if $k \in [0, \infty)$, then

$$(\psi, A)^k = \left\{ e, \left\{ \frac{\zeta}{\left(\mu_\psi^{(i)}(e)(\zeta)\right)^k, \sqrt{1 - \left(1 - \left(\nu_\psi^{(i)}(e)(\zeta)\right)^2\right)^k}} \right\}_i \right) : e \in A, \zeta \in X; i = 1, \cdots, m \Bigg\}$$

The set

$$(\psi, A)^{\frac{1}{2}} = \left\{ e, \left\{ \frac{\zeta}{\sqrt{\mu_\psi^{(i)}(e)(\zeta)}, \sqrt{1 - \sqrt{1 - \left(\nu_\psi^{(i)}(e)(\zeta)\right)^2}}} \right\}_i \right) : e \in A, \zeta \in X; i = 1, \cdots, m \Bigg\}$$

is called *dilation* of (ψ, A), designated as $dil(\psi, A)$.

Example 3.37. For PmFSS ψ_A given in Example 3.2, $con(\psi_A)$ and $dil(\psi_A)$ are given in Tables 20 and 21, respectively.

$con(\psi_A)$	e_1	e_2
b	$\{(0.04, 0.89), (0.08, 0.93), (0.00, 0.99)\}$	$\{(0.38, 0.39), (0.35, 0.63), (0.07, 0.16)\}$
s	$\{(0.14, 0.79), (0.55, 0.49), (0.77, 0.43)\}$	$\{(0.14, 0.85), (0.00, 0.75), (0.52, 0.87)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 20. $con(\psi_A)$

$dil(\psi_A)$	e_1	e_2
b	$\{(0.44, 0.57), (0.53, 0.62), (0.20, 0.87)\}$	$\{(0.79, 0.20), (0.77, 0.34), (0.51, 0.08)\}$
s	$\{(0.62, 0.46), (0.86, 0.26), (0.94, 0.22)\}$	$\{(0.61, 0.52), (0.10, 0.43), (0.85, 0.54)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 21. $dil(\psi_A)$

Remark. We may link linguistic terms like "*very*", "*very very*", "*medium*", "*more or less*" and "*high*" etc. with the set ψ_A^k given in Definition 3.36 by assigning different non-negative real values to k. For example,

$$\begin{aligned} k = 2 &\Rightarrow \text{"}very\text{"} \\ k = 2 \text{ twice} &\Rightarrow \text{"}very\ very\text{"} \\ k = 0.5 &\Rightarrow \text{"}highly\text{"} \\ k = 0.75 &\Rightarrow \text{"}more\ or\ less\text{"} \\ k = 0.75 \text{ twice} &\Rightarrow \text{"}medium\text{"} \end{aligned}$$

Example 3.38. For the PmFSS ψ_A given in Example 3.2, $very(\psi_A)$, $very\ very(\psi_A)$, $highly(\psi_A)$, $more\ or\ less(\psi_A)$ and $medium(\psi_A)$ are demonstrated in Tables 22, 23, 24, 25 and 26, respectively.

$very(\psi_A)$	e_1	e_2
b	$\{(0.04, 0.89), (0.08, 0.93), (0.00, 0.99)\}$	$\{(0.38, 0.39), (0.35, 0.63), (0.07, 0.16)\}$
s	$\{(0.14, 0.79), (0.55, 0.49), (0.77, 0.43)\}$	$\{(0.14, 0.85), (0.00, 0.75), (0.52, 0.87)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 22. $very(\psi_A)$

$very\ very(\psi_A)$	e_1	e_2
b	$\{(0.00, 0.98), (0.01, 0.99), (0.00, 1.00)\}$	$\{(0.14, 0.53), (0.12, 0.80), (0.00, 0.22)\}$
s	$\{(0.02, 0.93), (0.30, 0.65), (0.59, 0.58)\}$	$\{(0.02, 0.96), (0.00, 0.90), (0.27, 0.97)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 23. $very\ very(\psi_A)$

$highly(\psi_A)$	e_1	e_2
b	$\{(0.44, 0.57), (0.53, 0.62), (0.20, 0.87)\}$	$\{(0.79, 0.20), (0.77, 0.34), (0.51, 0.08)\}$
s	$\{(0.62, 0.46), (0.86, 0.26), (0.94, 0.22)\}$	$\{(0.61, 0.52), (0.10, 0.43), (0.85, 0.54)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 24. $highly(\psi_A)$

more or less(ψ_A)	e_1	e_2
b	$\{(0.29, 0.67), (0.38, 0.72), (0.09, 0.94)\}$	$\{(0.70, 0.24), (0.67, 0.41), (0.36, 0.10)\}$
s	$\{(0.48, 0.55), (0.80, 0.31), (0.91, 0.27)\}$	$\{(0.47, 0.62), (0.03, 0.51), (0.78, 0.64)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 25. *more or less*(ψ_A)

medium(ψ_A)	e_1	e_2
b	$\{(0.40, 0.60), (0.48, 0.65), (0.16, 0.89)\}$	$\{(0.76, 0.21), (0.74, 0.36), (0.46, 0.09)\}$
s	$\{(0.58, 0.49), (0.84, 0.27), (0.93, 0.23)\}$	$\{(0.57, 0.55), (0.07, 0.45), (0.83, 0.57)\}$
c	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$
z	$\{(0, 1), (0, 1), (0, 1)\}$	$\{(0, 1), (0, 1), (0, 1)\}$

TABLE 26. *medium*(ψ_A)

We may interpret these figures as follows: Suppose that b, s, c and z are persons named Babar, Soneri, Chloe and Zunera respectively. Assume that e_1 denotes the trait "well-dressed" and e_2 stands for "attractive personality". Further assume that ψ_A is the initial data provided by three judges. Then with respect to the trait e_1 i.e. well-dressed, the rating of first judge changes to the PFN $(0.04, 0.89)$ to Babar in view of very well-dressed, the rating of second judge becomes $(0.08, 0.93)$ and that of the third judge becomes $(0.00, 0.09)$. On the same token, with respect to the trait e_2 i.e. attractive personality, the rating of first judge becomes PFN $(0.38, 0.39)$ to Babar in view of very attractive personality, the rating of second judge changes to $(0.35, 0.63)$ and that of the third judge becomes $(0.07, 0.16)$. The other figures may be interpreted on the parallel track.

4. Selection of employee for promotion using PmFS TOPSIS

TOPSIS is employed to decide the superlative alternative from the notions of compromise solution. The solution which is closest to the ideal solution and farthest from negative ideal solution is acknowledged as *compromise solution*. In this section, we study how PmFSSs may be utilized in multiple criteria group decision making (MCGDM) using TOPSIS. First of all we shall extend TOPSIS to PmFSSs and then shall consider a problem.

Linguistic Terms	Fuzzy Weights
Not necessary (NN)	[0, 0.20]
Necessary (N)	(0.20, 0.40]
More or less necessary (MN)	(0.40, 0.60]
Very necessary (VN)	(0.60, 0.80]
Extremely necessary (EN)	(0.80, 1]

TABLE 27. Linguistic terms for judging alternatives

We make an inception by explaining the technique step by step as follows:

Decision Making Method

Input:

Step 1: Recognize the problem as what we have and what we have to do: Assume that $V = \{\zeta_i : i = 1, 2, \cdots, n\}$ is the finite aggregate of alternatives under consideration, $D = \{d_i : i = 1, 2, \cdots, m\}$ is the group of decision makers (DMs) and $E = \{e_i : i = 1, 2, \cdots, k\}$ is a finite family of attributes. Thus the $(i, j)^{th}$ entry of the PmFS matrix represents a set of m PFNs assigned to i^{th} alternative with respect to j^{th} attribute. Further, r^{th} PFN in the set at $(i, j)^{th}$ position yields the value of membership and non-membership functions, respectively, given by the r^{th} DM to i^{th} alternative with respect to j^{th} attribute.

Step 2: Construct weighted parameter matrix $\mathcal{A}$ as

$$\mathcal{A} = [w_{ij}]_{m\times k} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & \cdots & w_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{mk} \end{bmatrix}$$

where w_{ij} is the fuzzy weight assigned by the DM d_i to the attribute e_j by considering linguistic terms as given (for example) in Table 27.

Step 3: Construct normalized weighted matrix

$$\hat{\mathcal{A}} = [\hat{w}_{ij}]_{m\times k} = \begin{bmatrix} \hat{w}_{11} & \hat{w}_{12} & \cdots & \hat{w}_{1k} \\ \hat{w}_{21} & \hat{w}_{22} & \cdots & \hat{w}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{w}_{i1} & \hat{w}_{i2} & \cdots & \hat{w}_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{w}_{m1} & \hat{w}_{m2} & \cdots & \hat{w}_{mk} \end{bmatrix}$$

where $\hat{w}_{ij} = \frac{w_{ij}}{\sqrt{\Sigma_{i=1}^{m} w_{ij}^2}}$ and obtaining the weighted vector $\mathcal{W} = (\mathfrak{w}_1, \mathfrak{w}_2, \cdots, \mathfrak{w}_k)$, where $\mathfrak{w}_i = \frac{w_i}{\Sigma w_i}$ and $w_j = \frac{\Sigma_{i=1}^{m} \hat{w}_{ij}}{m}$.

Step 4: Construct PmFS decision matrix

$$\mathcal{B} = [\zeta_{jk}]_{n\times k} = \begin{bmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1k} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{j1} & \zeta_{j2} & \cdots & \zeta_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n1} & \zeta_{n2} & \cdots & \zeta_{nk} \end{bmatrix}$$

where $(i,j)^{th}$ entry of the PmFS matrix i.e. ζ_{ij} represents a set of m PFNs assigned to i^{th} alternative with respect to j^{th} attribute. Further, r^{th} PFN in the set at $(i,j)^{th}$ position yields the value of membership and non-membership functions, respectively, given by the r^{th} DM to i^{th} alternative with respect to j^{th} attribute.

Computations:

Step 5: Construct weighted PmFS decision matrix $\mathcal{C}$ by multiplying each element in the j^{th} column by j^{th} weight from weight vector obtained at Step 3 above, for each value of j varying from 1 to k.

Step 6: Obtain PFS-valued positive ideal solution (PFSV-PIS) and PFS-valued negative ideal solution (PFSV-NIS), employing in order

$$\begin{aligned} \text{PFSV-PIS} &= \{\eta_1^+, \eta_2^+, \cdots, \eta_n^+\} \\ &= \big\{(\vee_j\ \mu_{jk}, \wedge_j\ \nu_{jk}) : j = 1, 2, \cdots, n\big\} \end{aligned}$$

and

$$\begin{aligned} \text{PFSV-NIS} &= \{\eta_1^-, \eta_2^-, \cdots, \eta_n^-\} \\ &= \big\{(\wedge_j\ \mu_{jk}, \vee_j\ \nu_{jk}) : j = 1, 2, \cdots, n\big\} \end{aligned}$$

where $\vee$ stands for PFS union and $\wedge$ represents PFS intersection.

Step 7: Compute PFS-Euclidean distances of each alternative from PFSV-PIS and PFSV-NIS, respectively, utilizing

$$d_j^+ = \sqrt{\Sigma_{k=1}^n \Big\{ \big(\mu_{jk} - \vee_j\ \mu_{jk}\big)^2 + \big(\nu_{jk} - \wedge_j\ \nu_{jk}\big)^2 \Big\}}$$

and

$$d_j^- = \sqrt{\Sigma_{k=1}^n \Big\{ \big(\mu_{jk} - \wedge_j\ \mu_{jk}\big)^2 + \big(\nu_{jk} - \vee_j\ \nu_{jk}\big)^2 \Big\}}$$

for each $j = 1, 2, \cdots, n$.

Step 8: Determine the closeness coefficient of each alternative with ideal solution utilizing

$$\mathcal{C}^*(\zeta_j) = \frac{d_j^-}{d_j^+ + d_j^-} \in [0,1]$$

Output:

Step 9: In order to obtain the preference order of the alternatives, rank the alternatives in descending (or ascending) order.

The flowchart of the decision making method appears below in Figure 1 below:

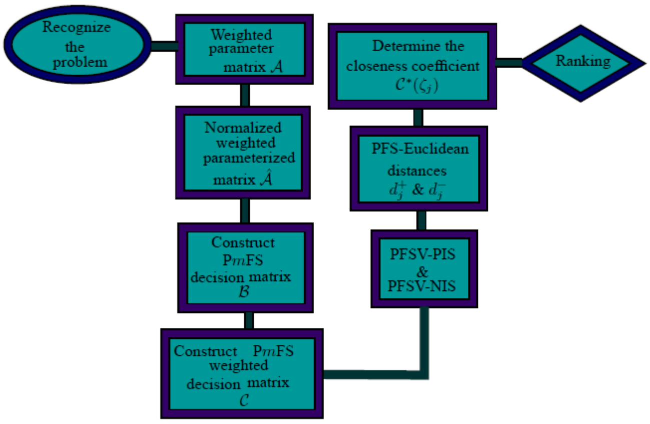

FIGURE 1. Flow chart representation of decision making method

We apply the proposed decision making method by using presumptive data in the forthcoming example as follows:

Example 4.1. Assume that a firm wants to promote one of its employees to higher position. To cope with the competitive environment prevailing, the firm wishes to choose the best of the best from the options available. The chief executive of the firm constitutes a panel of three decision makers (DMs) and gives them the task to select the best suitable employee for promotion. After a long discussion, the panel decides to consider five employees and focus on three traits to be required in selected person.

Step 1: Identifying the problem: Assume that $V = \{\zeta_i : i = 1, 2, \cdots, 5\}$ is the family of employees under consideration for promotion, $E = \{e_i : i = 1, 2, 3\}$ is the set of traits, and $D = \{d_i : i = 1, 2, 3\}$ is the group of DMs, where

$$\begin{aligned} e_1 &= \text{Communication skills,} \\ e_2 &= \text{Hard working, and} \\ e_3 &= \text{Well aware of emerging technologies} \end{aligned}$$

Step 2: The weighted parameter matrix is

$$\begin{aligned} \mathcal{A} &= [w_{ij}]_{3\times 3} \\ &= \begin{bmatrix} VN & EN & NN \\ N & EN & EN \\ EN & VN & MN \end{bmatrix} \\ &= \begin{bmatrix} 0.70 & 0.90 & 0.10 \\ 0.30 & 0.85 & 0.90 \\ 0.90 & 0.70 & 0.50 \end{bmatrix} \end{aligned}$$

where w_{ij} is the weight assigned by the DM d_i to the trait e_j by considering linguistic terms as given (for example) in Table 27.

Step 3: The normalized weighted matrix is

$$\begin{aligned}\hat{\mathcal{A}} &= [\hat{w}_{ij}]_{3\times 3}\\ &= \begin{bmatrix}0.59 & 0.63 & 0.10\\ 0.25 & 0.60 & 0.87\\ 0.76 & 0.49 & 0.48\end{bmatrix}\end{aligned}$$

and hence the weighted vector is $\mathcal{W} = (0.34, 0.36, 0.30)$.

Step 4: Assume that the four DMs provide the following PmFSS matrix in which the $(i,j)^{th}$ element represents m-polar PFN $\{(\mu,\nu)_m\}$, where alternatives are represented row-wise and traits are represented column-wise.

$$\mathcal{B} = \begin{bmatrix}\{(0.27,0.81),(0.47,0.51),(0.54,0.26)\} & \{(0.53,0.42),(0.81,0.28),(0.62,0.39)\} & \{(0.11,0.52),(0.61,0.37),(0.50,0.64)\}\\ \{(0.59,0.32),(0.93,0.24),(0.49,0.37)\} & \{(0.48,0.21),(0.58,0.32),(0.27,0.31)\} & \{(0.82,0.37),(0.61,0.26),(0.29,0.10)\}\\ \{(0.77,0.26),(0.48,0.32),(0.35,0.21)\} & \{(0.52,0.43),(0.61,0.51),(0.37,0.49)\} & \{(0.50,0.51),(0.58,0.40),(0.36,0.11)\}\\ \{(0.49,0.26),(0.88,0.27),(0.36,0.33)\} & \{(0.85,0.23),(0.52,0.04),(0.31,0.56)\} & \{(0.79,0.13),(0.91,0.23),(0.63,0.67)\}\\ \{(0.38,0.32),(0.59,0.52),(0.45,0.47)\} & \{(0.69,0.44),(0.57,0.11),(0.68,0.26)\} & \{(0.62,0.56),(0.33,0.37),(0.82,0.41)\}\end{bmatrix}$$

Step 5: The weighted PmFS decision matrix is

$$\mathcal{C} = \begin{bmatrix}\{(0.09,0.28),(0.16,0.17),(0.18,0.09)\} & \{(0.19,0.15),(0.29,0.10),(0.22,0.14)\} & \{(0.03,0.16),(0.18,0.11),(0.15,0.19)\}\\ \{(0.20,0.11),(0.32,0.08),(0.17,0.12)\} & \{(0.17,0.08),(0.21,0.12),(0.10,0.11)\} & \{(0.25,0.11),(0.18,0.08),(0.09,0.03)\}\\ \{(0.26,0.09),(0.16,0.11),(0.12,0.07)\} & \{(0.19,0.15),(0.22,0.18),(0.13,0.18)\} & \{(0.15,0.15),(0.17,0.12),(0.11,0.03)\}\\ \{(0.17,0.09),(0.30,0.09),(0.12,0.11)\} & \{(0.31,0.08),(0.19,0.01),(0.11,0.20)\} & \{(0.24,0.04),(0.27,0.07),(0.19,0.20)\}\\ \{(0.13,0.11),(0.20,0.18),(0.15,0.16)\} & \{(0.25,0.16),(0.20,0.04),(0.24,0.09)\} & \{(0.19,0.17),(0.10,0.11),(0.25,0.12)\}\end{bmatrix}$$

Step 6: The PFS-valued positive ideal solution (PFSV-PIS) and PFS-valued negative ideal solution (PFSV-NIS), in order, are

$$\begin{aligned}\text{PFSV-PIS} &= \{\eta_1^+, \eta_2^+, \cdots, \eta_5^+\}\\ &= (0.29,0.09),(0.32,0.03),(0.26,0.03),(0.31,0.01),(0.25,0.04)\end{aligned}$$

and

$$\begin{aligned}\text{PFSV-NIS} &= \{\eta_1^-, \eta_2^-, \cdots, \eta_5^-\}\\ &= (0.03,0.28),(0.09,0.12),(0.11,0.18),(0.11,0.20),(0.10,0.18)\end{aligned}$$

Step 7, 8: The PFS-Euclidean distances of each alternative from PFSV-PIS and PFSV-NIS along with closeness coefficients are given in Table 28 below:

Alternative (ζ_i)	d_i^+	d_i^-	$\mathcal{C}_i^*$
ζ_1	0.4972	0.6146	0.5528
ζ_2	0.4901	0.3731	0.4322
ζ_3	0.4350	0.3197	0.4236
ζ_4	0.4853	0.5103	0.5126
ζ_5	0.3736	0.3716	0.4986

Table 28. Distance measures & closeness coefficient of each alternative

Step 9: The preference order of the alternatives, therefore, is

$$\zeta_1 \succ \zeta_4 \succ \zeta_5 \succ \zeta_2 \succ \zeta_3$$

This ranking is depicted with the help of 3D bar chart in Figure 2:

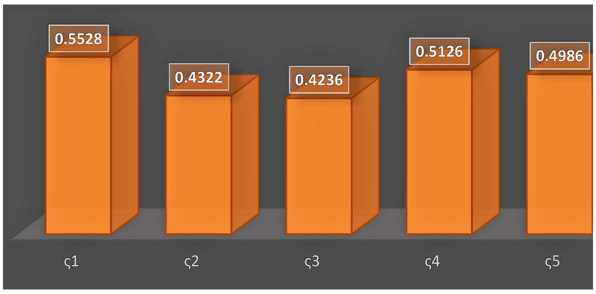

FIGURE 2. 3D bar chart of ranking of alternatives

In view of above TOPSIS ranking, we may infer that the employee ζ_1 is most meritorious for promotion on higher post.

5. Conclusion

We delivered an innovative crossbreed structure titled Pythagorean m-polar fuzzy soft sets in conjunction with some basic algebraic operations and features. We dig out crisp sets like support, core and height from PmFSS. A plenty of illustrations are also contained within to comprehend the notions effectively. We proposed a TOPSIS method for solving multiple criteria group decision making (MCGDM) problems accompanied by flowchart of the said decision making method. We employed the proposed algorithm to decide most appropriate person for the appraisal to higher position under Pythagorean m-polar fuzzy soft environment.
Since every intuitionistic fuzzy soft set is also a Pythagorean fuzzy soft set, so by familiarizing PmFSS, we have also wordlessly introduced intuitionistic m-polar fuzzy soft sets (ImFSSs). Theoretically, the ideas presented in this article may be extended to develop algebraic structures like Pythagorean m-polar fuzzy soft groups, Pythagorean m-polar fuzzy soft rings, Pythagorean m-polar fuzzy soft ideals, Pythagorean m-polar fuzzy soft algebras, Pythagorean m-polar fuzzy soft topology and undoubtedly Pythagorean m-polar fuzzy soft graphs too. Above and beyond the theoretical side, the ideas presented have charge to be extended in handling day to day problems from the real world including business, life sciences, social sciences, economics, pattern recognition, human resource management, robotics and many other areas. We trust that this article will serve as a foundation pit for the researchers working in this field.

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