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Families of Means-Based Modified Newtons Method for Solving Nonlinear Models

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Abstract.: In literature, the arithmetic mean of the two functions in the denominator of the second step of order three Weerakoon and Fernando (2000) iterative method have been replaced with other different means. However, these actions have not improve its order of convergence. To improve the order of convergence of these modified methods, a generic family of iterative methods that involve two weight functions and a generic consequential function for replacement of means is proposed. The analysis of convergence carried out on the families of methods, shows that they are of fourth order convergence and requires evaluation of three functions per iteration cycle. Further, the flexibility of the weight functions enables the re-discovery of some existing and construction of new families of iterative methods. Some concrete members of the family of methods are applied to solve some nonlinear equations and real life problems that are modeled into nonlinear equations.

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1. Introduction

The area of numerical mathematics that involved the development of new iterative methods (IM) for determining the solution of nonlinear equations (NE) have attracted researchers' interest for decades. This is because many real life phenomena are continued to be modelled into NE and requires the application of efficient IM to determine their solutions. The generic form of Nonlinear models (NLM) is:

$$f(x) = 0, \tag{1. 1}$$

where $x \in R,\ f : D \subset R \rightarrow R$ is functional.
The Newton method (NM)[24] is the most popular IM for determining the solution x^* of (1.1) and is given as:

$$x_{k+1} = x_k - s_k, \quad k = 0, 1, 2, \cdots \tag{1. 2}$$

where $s_k = \frac{f(x_k)}{f'(x_k)}$. The convergence order (CO) of the IM (1.2) is 2 and requires the evaluation of two distinct functions per iteration cycle.
Over the years, plethora of IM for determining the solution x^* of (1.1) have been developed in literature. These developed methods are usually variants of the method (1.2) and in most cases targeted at improving its CO and efficiency index (EI). The EI of a method with CO ρ is determined using $\rho^{\frac{1}{q}}$,[18]. In order to improve the CO and EI of (1.2), many authors have used different techniques. A good review on some of these techniques such as the geometric, functional, composition, sampling, Adomain decomposition, rational function and weight function techniques can be found in [1, 15, 16, 17] and reference therein.
The Weerakoon and Fernando (W-F) method in [25] is one modified form of the NM (1.2), put forward by replacing the function $f'(x)$ in (1.2) with the arithmetic mean of the functions $f(x)$ and $f'(w)$. It is given as

$$\begin{aligned} w_k &= x_k - s_k, \\ x_{k+1} &= x_k - \frac{2f(x_k)}{f'(x_k) + f'(w_k)}, \end{aligned} \tag{1. 3}$$

with CO three and $EI = 1.4422$. Since the W-F method was put forward, many authors have replaced the arithmetic mean used in [25] with other types of means to developed many IM of CO three. For instance, Harmonic mean was used in [19], geometric mean was employed in [14], Heronian mean and Quadratic mean were utilized in [23] and [26] respectively. Further, in [9, 10] the Stolarsky, Gini, Power, p-Logarithm, Heron, Contra-harmonic and Symmetric means where used to develop families of means-based CO three methods which contain methods from papers [25,19,14,23] and others.
To improve the CO and EI of the third orders means-based methods, many authors have introduced to the second step of the methods, a unique weight function that contain one additional new function evaluation which ensures the method attain higher CO and EI, see [13, 4, 11 , 3]. In quest for contribution to these research trends, we introduce two weight functions to the second step of means-based iterative function which led to the development of families of modified means-based modified NM for approximating the solution of (1.1).
The rest part of this paper is arranged in the following format. Section 2, presents the developmental stages of the new families of IM and their convergence analysis. While the numerical experience of the developed and compared methods on theoretical problems are presented in Section 3, the application to real life problems are presented in Section 4. The last section of this document, contains the conclusion and suggested further research areas.

2. Method Formulation

Consider the modified NM given as:

$$x_{k+1} = x_k - \frac{f(x_k)}{M_T\left[f'(x_k), f'(w_k)\right]}, k = 0, 1, 2.\cdots \quad (2.\ 4)$$

where w_k is Newton method iterative step and M_T is any type of mean of the functions derivatives $f'(x_k)$ and $f'(w_k)$.
Define a transformation Ω on the second step of the method in (2.4) such that

$$\Omega\left(\frac{f(x_k)}{M_T\left[f'(x_k), f'(w_k)\right]}\right) = \Phi_{M_T}(s_k, v_k), \quad (2.\ 5)$$

where $s = \frac{f(x_k)}{f'(x_k)}$, $v = \frac{f'(w_k)}{f'(x_k)}$ and Φ_{M_T} is a real valued bi-variate function of s and v which is the result of the transformation Ω when a type of mean is used in M_T. For this reason, Φ_{M_T} is referred to as the consequential transformation function when any type of mean is used as M_T. Consequently, (2.4) becomes

$$x_{k+1} = x_k - \Phi_{M_T}(s_k, v_k), k = 0, 1, 2, \cdots \quad (2.\ 6)$$

Suppose M_T is Arithmetic mean, that is $M_T = \frac{f'(x_k)+f'(w_k)}{2}$ and after subjecting it to the transformation in (2.5), the consequential IM in (2.4) becomes

$$x_{k+1} = x_k - \frac{2s_k}{1+v_k}, k = 0, 1, 2.\cdots \quad (2.\ 7)$$

The IM in (2.7) is the W-F method [25] with its corrector function expressed as a function of s_k and w_k.

Other similar modified form of method (2.4) that are of CO three have been obtained by using different type of mean in M_T and then subjecting it to the transformation in (2.5). For example:

For Harmonic mean [19], $M_H = 2\left(\frac{1}{f'(x_k)} + \frac{1}{f'(w_k)}\right)^{-1}$, we get

$$\Phi_{M_H}(s_k, v_k) = \frac{s_k}{2}\left(1 + \frac{1}{v_k}\right), k = 0, 1, 2.\cdots \quad (2.\ 8)$$

For Geometric mean [14], $M_G = \sqrt{f'(x_k)\, f'(w_k)}$, we get

$$\Phi_{M_G}(s_k, v_k) = \frac{s_k}{\sqrt{v_k}}, k = 0, 1, 2.\cdots \quad (2.\ 9)$$

For Hearonian mean [23], $M_{He} = \frac{2}{3}\frac{f'(x_k)+f'(w_k)}{2} + \frac{1}{3}\sqrt{f'(x_k)\, f'(w_k)}$, yield

$$\Phi_{M_{He}}(s_k, v_k) = \frac{3s_k}{1 + v_k + \sqrt{v_k}}, k = 0, 1, 2.\cdots \quad (2.\ 10)$$

For Quadratic mean [26], $M_Q = \sqrt{\frac{1}{2}\left(f'^2(x_k) + f'^2(w_k)\right)}$, yield

$$\Phi_{M_Q}(s_k, v_k) = \frac{s_k}{\sqrt{\frac{1}{2} + \frac{1}{2}v_k^2}}, k = 0, 1, 2.\cdots \quad (2.\ 11)$$

2.1. **The families of the means-based methods.** In order to make the method (2.7) and all other methods developed from (2.4) by using the transformation of the types of mean given in Equations (2.8 - 2.11) attain CO four, the following family of IM is suggested.

$$\begin{aligned} y_k &= x_k - \alpha s_k; \\ x_{k+1} &= x_k - \Phi_{M_T}(s_k, v_k)\, H(s_k)\, G(v_k), \quad k = 0, 1, 2, \cdots \end{aligned} \tag{2. 12}$$

where $\alpha \subset R\backslash\{0\}$ a free parameter, $H(s_k)$ and $G(v_k)$ are real-valued weight functions that ensures the CO of (2.12) attain four by utilizing fixed number of functions evaluations per iteration cycle. If method (2.12) is established to converge with order four, then for any concrete method of it will require the evaluation of three functions per iteration cycle and consequently, its EI will be 1.5873.

Next we establish the convergence of the families of means-based IM given in (2.12) for different types of mean used in M_T. The Taylor's expansion technique also used in literature [15, 16, 17, 13] and some reference therein, was used in proving the convergence of the method. First, the Proposition 2.1 is considered.

Proposition 2.1. *Let $f : D \subset R \to R$ be a real valued function that is at least three-times differentiable and that $f'(\cdot) \neq 0$ in D. Then for every type of mean used in M_T, the function $\Phi_{M_T}(s_k, v_k)$ in (2.12) contributes an error to the iterative process given as:*

$$\begin{aligned} e_{\Phi_{M_{T_k}}} = e_k &+ (\alpha - 1)\, c_2 e_k^2 + \left(B_{M_T} c_2^2 + D_{M_T} c_3\right) e_k^3 \\ &+ \left(E_{M_T} c_2^3 + F_{M_T} c_2 c_3 + J_{M_T} c_4\right) e_k^4 + O(e_k^5) \end{aligned} \tag{2. 13}$$

where $B_{M_T}, D_{M_T}, E_{M_T}, F_{M_T}$ and J_{M_T} are constants expressed in terms of α.

Theorem 2.1. *Assume that $f : D \subset R \to R$ is a real valued function that is at least three-times differentiable such that $f'(\cdot) \neq 0$ in D. If x_0 is an initial guess close to $x^* \subset D$, then the sequence of approximations $\{x_k\}_{k \geq 0}, (x_k \in D)$ generated using the families of methods (2.12) for any mean type M_T used, converges to x^* with order four and generic error equation*

$$\begin{aligned} e_{k+1} =& \frac{1}{162G(1)}(54c_2c_3(-2 + 12B_{M_T} - 3F_{M_T})G(1) - 2c_2^3((-327 + 540B_{M_T} \\ &+ 81E_{M_T})G(1) - 32G''(1)) - 3G(1)(9G(1)H'''(0) + c_4(52 + 54J_{M_T})))e_k^4 \\ &+ O(e_k^5) \end{aligned} \tag{2. 14}$$

where $c_n = \frac{1}{n!}\frac{f^n(x^*)}{f'(x^*)}$, $n \geq 2$, provided the weight functions $H(s_k)$ and $G(v_k)$ satisfy the following conditions:
$\alpha = \frac{2}{3}$, $D_{M_T} = -\alpha$, $H(0) = \frac{1}{G(1)}$, $G(1) \neq 0$, $H'(0) = H''(0)$, $H'''(0) < \infty$,
$G'(1) = \frac{1}{2}G(1)$, $G''(1) = \left(\frac{10 - 9B_{M_T}}{8}\right)G(1)$, $G'''(1) < \infty$.

Proof. Let the kth iteration error be $e_{k+1} = x_k - x^*$. If $x{=}x_k$ in the Taylor series expansion of $f(x)$ and $f'(x)$ about x^*, then

$$f\left(x_k\right)=f'\left(x^*\right)\left[e_k+\sum_{n=2}^{4}c_n e_k^n+O\left(e_k^5\right)\right],k{=}0,1,2,\cdots \tag{2. 15}$$

and

$$f'\left(x_k\right)=f'\left(x^*\right)\left[1+\sum_{n=2}^{4}c_n e_k^{n-1}+O\left(e_k^5\right)\right],k{=}0,1,2,\cdots \tag{2. 16}$$

where $c_n=\frac{1}{n!}\frac{f^n(x^*)}{f'(x^*)},\ n\geq 2$.

Using (2.15) and (2.16), the following expressions are obtained.

$$\begin{aligned} s_k=\frac{f(x_k)}{f'(x_k)}=&e_k-c_2e_k^2+(2c_2^2-2c_3)e_k^3+(-4c_2^3+7c_2c_3-c_4)e_k^4\\ &+(8c_2^4-20c_2^2c_3+6c_3^2+10c_2c_4-4c_5)e_k^5+O(e_k^6)\end{aligned} \tag{2. 17}$$

and

$$\begin{aligned} w_k&=x_k-\alpha\frac{f(x_k)}{f'(x_k)}\\ &=x^*+(1-\alpha)e_k+\alpha c_2e_k^2+\alpha(-2c_2^2+2c_3)e_k^3+\alpha(4c_2^3-7c_2c_3+3c_4)e_k^4\\ &\quad+O(e_k^5)\end{aligned} \tag{2. 18}$$

Now by expanding $f'(w_k)$ using Taylor series expansion, we have:

$$\begin{aligned} f'(w_k)=f'(x^*)[&1+2(1-\alpha)c_2e_k+(2\alpha c_2^2+3(1-\alpha)^2c_3)e_k^2+(6(1-\alpha)\alpha c_2c_3\\ &+2\alpha c_2(-2c_2^2+2c_3)+4(1-\alpha)^3c_4)e_k^3+(3c_3(\alpha\alpha^2c_2^2+2(1-\alpha)\alpha(-2c_2^2+2c_3))\\ &+12(1-\alpha)^2\alpha c_2c_4+2\alpha c_2(4c_2^3-7c_2c_3+3c_4)+5(1-\alpha)^4c_5)e_k^4+O(e_k^5)]\end{aligned} \tag{2. 19}$$

The variable v_k of the weight function $G(v_k)$ can be express as

$$\begin{aligned} v_k&=\frac{f'(w_k)}{f'(x_k)}\\ &=1-2\alpha c_2e_k+3\alpha(2c_2^2+(-2+\alpha)c_3)e_k^2-4\alpha(4c_2^3+(-7+3\alpha)\\ &+(3-3\alpha+\alpha^2)c_4)e_k^3+\alpha(40c_2^4+(-100+39\alpha)c_2^2c_3+(30-21\alpha)c_3^2\\ &+2(25-24\alpha+10\alpha^2)c_2c_4+5(-4+6\alpha-4\alpha^2+\alpha^3)c_5)c_k^4+O(c_k^5)\end{aligned} \tag{2. 20}$$

and from Proposition 2.1,

$$\begin{aligned} \Phi_{M_T}\left(s_k,v_k\right)=e_k+\left(\alpha-1\right)c_2e_k^2+\left(B_{M_T}c_2^2+D_{M_T}c_3\right)e_k^3&\\ +\left(E_{M_T}c_2^3+F_{M_T}c_2c_3+J_{M_T}c_4\right)e_k^4+O(e_k^5)&\end{aligned} \tag{2. 21}$$

The Taylor expansion of the weight functions $H\left(s_k\right)$ and $G\left(v_k\right)$ about $s_k=0$ and $v_k=1$ respectively, are

$$H\left(s_k\right) = H(0) + sH'(0) + \frac{1}{2}s^2H''(0) + \frac{1}{6}s^3H'''(0) + \cdots \tag{2. 22}$$

and

$$G\left(v_k\right) = G(1) + (v-1)G'(1) + \frac{1}{2}(v-1)^2G''(1) + \frac{1}{6}(v-1)^3G'''(1) + \cdots \tag{2. 23}$$

Now using (2.21), (2.22), and (2.23) in the second step of (2.12), we get

$$\begin{aligned} x_{k+1} =& (1 - G(1)H(0))e_k + (c_2(G(1) - \alpha G(1) + 2\alpha G'(1))H(1) - G(1)H'(0))e_k^2 \\ &+ (-((B_{M_T}c_2^2 + c_3D_{M_T})G(1)H(0)) - \alpha(3(\alpha-2)c_3G'(1)) + 2c_3^2(3G'(1) \\ &+ \alpha G''(1))H(0) + c_2G(1)H'(0) + 2\alpha c_2G'(1)H'(0) \\ &+ (\alpha+1)c_2(2\alpha c_2G'(1)H(0) - G(0)H'(0)) - \frac{G(1)H'(0)}{2})e_k^3 \\ &+ (4\alpha^3c_4G'(1)H(0) - \frac{1}{3}(c_2^3(3E_{M_T}G(0) + 2\alpha((9\alpha \\ &- 3(11 + B_{M_T}))G'(1) + \alpha(3(\alpha-7)G''(1) - 2\alpha G'''(0))))H(0) \\ &+ c_2^2((-3 + \alpha - B)G(1) + 2\alpha((\alpha-5)G'(1) - \alpha G''(1)))H'(0) \\ &+ 6\alpha G'(1)(2c_4H(0) + c_3H'(0)) - 3\alpha^2G'(1)(4c_4H(0) + c_3H'(0))\cdots \\ &- \frac{1}{6}G(1)(6c_3(D_{M_T} - 2)H'(0) + H'''(0) + 6c_4H(0)H(0)J_{M_T})))e_k^4 \\ &+ O(e_k^5) \end{aligned} \tag{2. 24}$$

The method (2.12) converges to x^* if the first, second, and third term of (2.24) vanish. This is achieved when the following set of system of equations holds.

$$\begin{aligned} &G(1)H'(0) = 0 \\ &1 - G(1)H(0) = 0 \\ &(G(1) - \alpha G(1) + 2\alpha G'(1))H(0) = 0 \\ &\frac{1}{2}G(1)H'(0) = 0 \\ &2G'(1)\left(2\alpha - 1\right) - B_{M_T}G(1) = 0 \\ &3\alpha G'(1)\left(2 - \alpha\right) - D_{M_T}G(1) = 0 \\ &2\alpha\left(3G'(1) - \alpha G''(1)\right) = 0 \end{aligned} \tag{2. 25}$$

The system of equations (2.25) is satisfied if

$$\begin{aligned} &\alpha = \frac{2}{3}, \;\; D_{M_T} = -\alpha, \;\; H(0) = \frac{1}{G(1)}, \quad G(1) \neq 0, \;\; H'(0) = H''(0) = 0, \\ &H'''(0) \leq \infty, \quad G'(1) = -\frac{1}{4}G(1), \\ &G''(1) = \left(\frac{10 - 9B_{M_T}}{8}\right)G(1), \;\; G'''(1) < \infty. \end{aligned} \tag{2. 26}$$

By substituting the weights conditions (2.26) in (2.22) and (2.23), the following general error equation of any method constructed from (2.12) is obtained as:

$$\begin{aligned} e_{k+1} =& \alpha + \frac{1}{162G(1)}(54c_2c_3(12B_{M_T} - 3F_{M_T} - 2)G(1) - 2c_2^3((540B_{M_T} + 81E_{M_T} \\ &- 327)G(0) - 32G'''(0)) - 3G(1)(9G(1)H'''(0) + c_4(52 + 54J_{M_T})))e_k^4 \\ &+ O\left(e_k^5\right) \end{aligned} \tag{2. 27}$$

The error equation in (2.27) establishes the claim that for any concrete method derived from (2.12), will have CO four.

Remark 2.1: For any two weight functions $H(s)$ and $G(v)$ jointly satisfying the conditions in (2.26), a family of CO four means-based IM for solving (1.1) can be put forward. For instance:

$H(s) = \left(\frac{1}{\delta} + \theta s^3\right)$ and $G(v) = \delta - \frac{1}{4}\delta(v-1) + \frac{\eta}{2}\delta(v-1)^2 + \tau(v-1)^3$

where $\delta \neq 0$, $\theta < \infty$ and $\tau < \infty$ are free constant. These weight functions when substituted in (2.12) yield a family of methods given as:

$$\begin{aligned} y_k &= x_k - \frac{2}{3}s_k; \\ x_{k+1} &= x_k - \Phi_{M_T}\left(s_k, v_k\right)\left(\frac{1}{\delta} + \theta s_k^3\right)\left(\delta - \frac{1}{4}\delta\sigma + \frac{\eta}{2}\delta\sigma^2 + \tau\sigma^3\right), \\ k &= 0, 1, 2, \cdots \end{aligned} \tag{2. 28}$$

where $\sigma = v_k - 1$ and $\eta = \left(\frac{10-9B_{M_T}}{8}\right)$.
We state here that the family of order four methods developed in [13,14] is a consequential member of the method in (2.28) with $\theta = \tau = 0$ (More details on this can be seen in Remark 2.3). For this reason method (2.28) is referred to as family of families of CO four methods for solving (1.1).

Remark 2.2: It is important to note that the variable η (depending on B_{M_T}) is responsible for the differences in IM developed from (2.28) using different types of means in M_T. Using Proposition 2.2, the values of η when different types of means are used in M_T are presented in Table 1.

TABLE 1. Values of η for different methods.

M_T	B_{M_T}	η
Arithmetic mean	$2-4\alpha+\alpha^2$	$\frac{3}{2}$
Harmonic mean	$2(\alpha-1)^2$	1
Geometric mean	$\frac{1}{2}(\alpha^2-8\alpha+4)$	$\frac{5}{4}$
Hearonian mean	$\frac{7}{6}\alpha^2-4\alpha+2$	$\frac{17}{12}$
Quadratic mean	$\frac{1}{2}(2-4\alpha+\alpha^2)$	$\frac{7}{4}$

Remark 2.3: To construct concrete form of the method in (2.28) for a chosen M_T (corresponding to η), arbitrary values are assigned to the free parameters δ, θ and τ. For instance, if $\eta = \frac{3}{2}, \delta = 1, \theta = \tau = 0$ in (2.28), then we have the method put forward in Lofti (2014)[13] and Chicharro et al.,(2019) [4] denoted as M_1 and given as:

$$x_{k+1} = x_k - \frac{2s_k}{1+v_k}\left(2 - \frac{7}{4}v_k + \frac{3}{4}v_k^2\right). \tag{2. 29}$$

Some other typical methods are presented in Table 2.

TABLE 2. Concrete forms of the family of methods and their error equations

$Methods$	η	δ	θ	τ	e_{k+1}	$Remark$
M_1	3/2	1	0	0	$\frac{1}{9}(33c_2^3 - c_2c_3 + c_4)e_k^4$	Method in [13,4]
M_2	3/2	1	1	1	$(\frac{1}{9}c_4 - c_2c_3 + \frac{329}{81}c_2^3 - \frac{1}{6})e_k^4$	New Method
M_3	1	1	0	0	$(\frac{75}{27}c_2^3 - c_2c_3 + \frac{c_4}{9})e_k^4$	Method in [4]
M_4	1	2	2	6	$\frac{1}{9}(37c_2^3 - 9c_2c_3 + c_4 - 6)e_k^4$	New Method
M_5	$\frac{5}{4}$	1	0	0	$(\frac{89}{27}c_2^3 - c_2c_3 + \frac{c_4}{9})e_k^4$	Method in [4]
M_6	$\frac{5}{4}$	$\frac{1}{2}$	3	0	$(\frac{89}{27}c_2^3 - c_2c_3 + \frac{c_4}{9} - \frac{1}{4})e_k^4$	New Method
M_7	$\frac{17}{12}$	1	0	0	$(\frac{287}{81}c_2^3 - c_2c_3 + \frac{c_4}{9})e_k^4$	Method in [4]
M_8	$\frac{17}{12}$	3	1	1	$\left(\frac{883}{243}c_2^3 - c_2c_3 + \frac{c_4}{9} - \frac{1}{2}\right)e_k^4$	New Method
M_9	$\frac{7}{4}$	1	0	0	$(\frac{109}{27}c_2^3 - c_2c_3 + \frac{c_4}{9})e_k^4$	Method in [4]
M_{10}	$\frac{7}{4}$	1	1	1	$(\frac{3599}{81}c_2^3 - c_2c_3 + \frac{c_4}{9} - \frac{1}{6})e_k^4$	New Method

3. NUMERICAL IMPLEMENTATION

To demonstrate the applicability of the developed iterative methods, some standard nonlinear problems in literature are solved. The performance of some concrete methods of the developed families of methods presented in Table 2 are compared with some existing fourth order method in Sharma *et al.* [21] (SM)

$$x_{k+1} = x_k - \frac{4f(x_k)}{f'(x_k)+3f'(w_k)}\left(1+s_k^3\right) - \frac{9}{16}\left(\frac{\phi}{f'(x_k)}\right)^2 s_k^3, \tag{3. 30}$$

where $\phi = \frac{f'(x_k)-f'(w_k)}{s_k}$.

and Chun *et al.* [6] (CM)

$$x_{k+1} = x_k - \frac{f'(x_k)+3f'(w_k)}{2f'(x_k)-f'(w_k)}\frac{f(x_k)}{f'(x_k)}. \quad (3.\ 31)$$

All computer programs written for computations were implemented in MAPLE 2017 version environment, with 1000 digits of mantissa on Intel Celeron(R) CPU 1.6 GHz with 2 GB of RAM processor. The stopping criteria used for all program is $|f\,(x_{k+1})| \leq 10^{-100}$. The metrics used for comparison includes Number of Iterations $(Iter)$, Absolute value of function of last iteration value $|f\,(x_{k+1})|$ and Computational local convergence order due to Petkovic ρ_{coc} [8] given as,

$$\rho_{coc} = \frac{log\,|f(x_{k+1})|}{log\,|f(x_k)|}. \quad (3.\ 32)$$

The developed methods $(M_{2j}, j = 1,2,3,4,5)$ and the compared methods SM and CM were used to approximate the solution of the nonlinear problems under same computation conditions and computation results obtained are presented in Table 3. The test functions $f_1(x) = sin(x) - x^2 + 1$ and $f_2(x) = (x-1)^3 - 1$ taken from [5] were adopted for numerical test:

TABLE 3. Methods results comparison for the test functions

$f_i(x)$	$Methods$	x_0	$iter$	$\|f\,(x_{k+1})\|$	ρ_{coc}
	SM		5	$2.9346e-397$	4.0182
	CM		4	$1.2067e-154$	4.0538
	M_1		4	$4.8394e-121$	4.0493
$f_1(x)$	M_2	2	4	$3.4228e-107$	4.0953
	M_4		5	$9.9384e-168$	4.0000
	M_6		5	$2.7762e-286$	3.9722
	M_8		5	$1.8910e-165$	4.0244
	M_{10}		4	$3.9356e-104$	4.0856
	SM		4	$1.6738e-108$	4.0000
	CM		4	$1.2469e-122$	3.9355
	M_1		5	$2.5977e-337$	4.0119
	M_2		5	$8.2931e-364$	4.0000
$f_2(x)$	M_4	2.5	5	$8.6029e-381$	4.0105
	M_6		4	$1.6872e-204$	4.0000
	M_8		4	$2.1119e-101$	4.0400
	M_{10}		5	$2.3967e-351$	3.9889

3.1. **Results discussion.** From Table 3, the numerical experience on some concrete forms $(M_{2j}, j = 1,2,3,4,5)$ of the developed family of families of methods (3.28) agrees with theoretical order of convergence obtained. For instance, it can be seen from the last column of Table 3 that the computational order of convergence ρ_{coc} of $(M_{2j}, j = 1,2,3,4,5)$ is four . It is also worthy of note that the developed method solved all the test problems with error margins that are competitive with methods compared.

4. Some Applications to Real Life Models

In this section, some concrete forms of the developed methods $(M_{2j}, j = 1, 2, 3, 4, 5)$ are applied to solve some real life problems that have been expressed in nonlinear equations. Their numerical results were compared with the results by the methods SM and CM.

Application 1: (Colebrook-White Equation [7]). Given a flow rate in a pipeline, one important means of determining pressure drop is the friction factor f, [22]. A relationship between the friction factor, Raynolds number R, Pipe roughness ϵ and the pipe inner diameter D is described by the Colebrook-White equation given as:

$$\sqrt{\frac{1}{f}} = -2log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{R\sqrt{f}}\right), \tag{4. 33}$$

where $R > 4000$. The expression in (4.33) can be written in the form of (1.1) as:

$$f(f^*) = \sqrt{\frac{1}{f^*}} + 2log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{R\sqrt{f^*}}\right), \tag{4. 34}$$

where $\epsilon/D = 10^{-4}$ and $R = 10^5$. Table 4 shows the computation results when the developed methods $(M_{2i}, i = 1, 2, 3, 4, 5)$ and the compared methods (SM and CM) are applied to solve the problem in (4.34) with an initial guess $x_0 = 0.01$. All the methods located the solution $f^* = 0.73164090877 \cdots$ of (4.34) with few number of iterations and similar accuracy.

TABLE 4. Comparison of methods results for Application 1.

$Methods$	x_0	$iter$	$\lvert f(x_{k+1})\rvert$	ρ_{coc}
SM		6	4.7897e-296	4.0000
CM		4	9.0271e-109	4.0370
M_1		5	3.5381e-355	3.9888
M_2		6	3.7252e-210	3.9623
M_4	0.01	6	4.1820e-106	3.9259
M_6		6	2.4273e-344	4.0000
M_8		6	4.4259e-279	3.9857
M_{10}		6	3.03401e-192	4.0000

Application 2: (Population growth [2]) The expression governing the population growth is given as

$$\frac{dN(t)}{dt} = \mu N(t) + \sigma, \tag{4. 35}$$

where $N(t)$ is population at time t, μ is constant rate of birth and σ is constant rate of immigration. The solution of the differential equation can be obtained as:

$$N(t) = N_0 e^{\mu t} + \frac{\sigma}{\mu}\left(e^{\mu t} - 1\right), \tag{4. 36}$$

where N_0 is population at initial point (t=0). Suppose the population of certain town is 1000000 at initial point, and 43.5 percent of its population immigrate into the town in the first year, and that at the end of the year 58.6 percent of the initial population were added to the town population. It may be of interest to determine the birth rate of the town population, then we must determine μ. Substitute the data in (4.36) yield

$$f(\mu) = 1586000 - \frac{435000}{\mu}\left(e^{\mu} - 1\right) - 1000000e^{\mu} = 0. \tag{4. 37}$$

The computation results obtained by applying the developed methods and the compared methods are presented in Table 5.

TABLE 5. Comparison of methods results for Application 2.

$Methods$	x_0	$iter$	$\lvert f(x_{k+1})\rvert$	ρ_{coc}
SM		5	4.9598e-265	4.2742
CM		4	8.9603e-138	4.6000
M_1		5	1.1961e-234	4.3333
M_2		5	2.1856e-303	4.2676
M_4	1	5	5.4625e-323	4.2500
M_6		5	1.1112e-156	4.4571
M_8		6	2.4989e-125	4.6296
M_{10}		5	9.1122e-316	4.2133

Application 3: (Radiation [12,20]) Consider the Plank's radiation law which calculate the energy density within an isothermal blackbody given as

$$\varphi(\lambda) = \frac{8\pi ch\lambda^{-5}}{e^{ch/\lambda kT} - 1}, \tag{4. 38}$$

where λ, t, h and c represents the radiation wavelength, blackbody absolute temperature, Boltzmann's constant, Planck's constant and c speed of light. A typical radiation problem might involve calculating the wavelenth λ which agree with maximum energy density $\varphi(\lambda)$. Differentiating Equation (4.38) with respect to λ, yield

$$\varphi'(\lambda) = \frac{8\pi ch\lambda^{-6}}{e^{ch/\lambda kT} - 1}\left(\frac{\left(\frac{ch}{\lambda kT}\right)}{e^{ch/\lambda kT} - 1} - 5\right). \tag{4. 39}$$

From Equation (4.39), the function $\varphi(\lambda)$ attain its maxima when

$$\frac{\left(\frac{ch}{\lambda kT}\right)}{e^{ch/\lambda kT} - 1} - 5 = 0. \tag{4. 40}$$

Now, set $x = \frac{ch}{\lambda kT}$ in Equation (4.40), we get

$$x = 5 - 5e^{-x}. \tag{4. 41}$$

The expression in (4.41) can be written in the form of (1.1). Hence,

$$f(x) = 5 - 5e^{-x} - x = 0. \tag{4. 42}$$

Obviously, $x = 0$ is a trivial solution. It was discussed in [12] that a solutions of (4.42) occur near $x = 5$. Here, we approximate the solution $x^* = 4.965114231744276303698$ of (4.10) with an intial approximation $x = 5$ using the methods developed here in and compared with results of some methods in Table 6. Consequently, the energy density is maximum when the approximated wavelength of radiation is $\lambda = \frac{ch}{4.965114231744276303698(kT)}$.

TABLE 6. Comparison of methods results for Application 3.

$Methods$	x_0	$iter$	$\lvert f(x_{k+1})\rvert$	ρ_{coc}
SM		4	6.9245e-374	3.9787
CM		3	9.3221e-169	4.0238
M_1		3	2.8092e-164	4.0000
M_2		4	6.9180e-374	3.9787
M_4	5	4	1.0551e-322	3.9753
M_6		4	6.4843e-359	3.9889
M_8		4	2.5287e-333	3.9643
M_{10}		4	6.9168e-374	3.9787

5. CONCLUSIONS

In this paper, the method of Weerankoon and Fernando [25] have been modified with the main objective of scaling its order of convergence from three to four. This was made possible by replacing the arithmetic mean of $f'(x_k)$ and $f'(w_k)$ used in its second step with other means expressions and two attached weight functions $H(s)$ and $G(v)$. The analysis of convergence of the developed method reveals the conditions for which many other CO four methods can be constructed from it. In fact, a recent developed family of CO four methods in Lofti (2014) [13] and Chicharro et al., (2019a) [4] is one of its consequential member. The developed methods, were applied to approximate the solution of some nonlinear problems and real life problems expressed in nonlinear equation that recently appeared in literature. In all the problems solved, the computation experience from the developed methods shows promising competence in solving the problems as compared with the results of some contemporary methods. It may be of interest to extended the methods put forward in this manuscript to determining multiple roots of (1.1). Also the methods can be generalized in Banach spaces.

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REFERENCES

[1] R, Behl, C. Chun, A. S. Alshormani, and S. S. Motsa, *A General Way to Construct a New Optimal Scheme with Eighth-Order Convergence for Nonlinear Equations,* International J. of Computational Methods, **17,** No.1 (2020) 1-15.

[2] R. L. Burden, and J. D. Faires, *Numerical Analysis*, Brooks/Cole Publishing Company, California, 2005.

[3] P. B. Chand, F. I. Chicharro and P. Jain, *On the design and analysis of high order Weerakoon-Fernando method based on weight functions,* Computational and Mathematical Methods, **2,** No.5 (2020) doi: 10.1002/cmm4.1114.

[4] F. I. Chicharro, A. Cordero, T. H. Martinez and J. R. Torregrossa, *CMMSE-2019 maen-based iterative methods for solving nonlinear chemistry problems*, Journal of Mathematical Chemistry, (2019a).https://doi:10.1007/s10910-019-01085-2

[5] F. I. Chicharro, A. Cordero, N. Garrido and J. R. Torregrossa, *Wide stability in a new family of optimal fourth-order iterative methods*,Comp. and Math Methods, (2019b)**1**: *Wiley* https://doi.org/10.1002/cmm4.1023

[6] C. Chun, M. Y. Lee, B. Neta, and J. Dzunic, *On optimal fourth-order iterative methods free from second derivative an their dynamics,* Appl. Math. Comput.**218,**No. 11 (2012) 6427-6438.

[7] C. F. Colebrook, and C. M. White, *Experiments with fluid friction in roughened pipes,* Proc. R. Soc. Lond., **161** (1937) 367-381.

[8] M. Grau-Sanchez, M. Noguera, A. Grau and J. R. Herrero, *A study on new computational local orders of convergence,* (2012) *arXiv:1202.4236[math.NA].*

[9] D. Herceg, D. Herceg. *Third-order modifications of Newtons method based on Stolarsky and Gini means,* J. of Computational and Appl. Mathematics, **245** (2013) 53-61.

[10] D. Herceg, D. Herceg, *Means based modifications of Newtons method for solving nonlinear equations,* J. of Computational and Appl. Mathematics, **219** (2013) 6126-6133.

[11] D. Herceg, D. Herceg, *Sixth-order modifications of Newtons method based on Stolarsky and Gini means*,J. of Computational and Appl. Mathematics, **267** (2014) 244253.

[12] D. Jain, *Families of Newton-like methods with fourth-order onvergence,* Inter. J. of Computer Mathematics, **90,** No.5 (2013) 1072-1082.

[13] T. Lofti, *A new optimal method of fourth-order convergence for solving non-linear equations*, Int. J. Industrial Mathematics, **6,**(No.2 (2014) 1-4.

[14] T. Lukic and N.Ralevic, *Geometric mean Newton's method for simple and multiple roots,*Appl. Math. Lett. **21** (2008) 30-36.

[15] O. Ogbereyivwe and V. Ojo-Orobosa, *High Order Quadrature Based Iterative Method for Approximating the Solution of Nonlinear Equations, Casp. J. of Math. Sc.,* **9,** No.2 (2020) 243-255.

[16] O. Ogbereyivwe and V. Ojo-Orobosa, Family of optimal two-step fourth order iteration method and its extension for solving Nonlinear Equations, J. of Interdisciplinary Mathematics, **24,** No.5 (2021) 1347-1365.

[17] O. Ogbereyivwe and K. O. Muka, *Multistep Quadrature based methods for nonlinear system of equations with singular jacobian,* J. of Applied Mathematics and Physics, **7,** No.3 (2019) 702-725.

[18] A. M. Ostrowski, *Solution of equations and systems of equations,* Academic Press, New York, 1966.

[19] A. Ozban, *Some new variants of Newton's method*,Appl. Math. Lett, **17** (2004) 677-682.

[20] S. Parimala and J. Jayakumar, *Some new higher order weighted Newton methods for solving nonlinear equation with applications,* Math. and Comput. Appl. **24** (2019) 59-69.

[21] E, Sharma, S. Pandy and M. Dwivedi, *New optimal fourth order iterative method for solving nonlinear equations*,Int. J. on Emerging Technol, **11,** No.3 (2020) 755-758.

[22] E, Shashi, *Transmission pipeline calculations and Simulations Manual, Fluid flow in pipes,* (Elsevier, London), (2015) 149-234.

[23] J. F. Traub, *Iterative Methods for the Solution of Equations,* Prentice-Hall, New Jersey, 1964.

[24] S, Weerakoon and T. G. I. Fernando, *A variant of Newton's method with third-order convergence,* App. Math. Lett, **13,** No. 1 (2000) 87-93.

[25] Z. Xiaojain. *A class of Newton's method with third-order convergence*, Appl. Math. Lett, **20** (2007) 1026-1030.