Punjab University Journal of Mathematics (2021),53(6),377-385
https://doi.org/1.52280/pujm.2021.530602

Relations on topologized groups

Muhammad Kashif Maqbool *[1], Muhammad Siddique Bosan[2] , Abdul Rauf Khan [3] and Zaheer Ahmad [4]

[1] Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Punjab, Pakistan

[2,3,4] Department of Mathematics, Khwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, Punjab, Pakistan

Email: kashifmaqbool9@gmail.com*[1], siddiquebosan1966@gmail.com[2], khankts@gmail.com[3], zaheeir@gmail.com[4]

Received: 25 February,2021 / Accepted: 25 May, 2021 / Published online: 25 June, 2021

Abstract.: In our present paper, topological groups are being discussed, where the relations with counter examples built the interest in the generalized structure. Some of these structures have also been converted into the other structures using topological isomorphism. In our work, the identity element plays the important role in lieu of arbitrary element. The role of topology has the more interest in our discipline.

AMS (MOS) Subject Classification Codes: 22A05, 22A30, 54C08, 54H99
Key Words: Semi-continuous mapping, group structure, quasi topological group, irresolute topological group and s-topological group

1. Introduction

A mathematical discipline having compatibility between the topology and group is called the topological group [12]. This discipline has very beneficial applications in all about natural sciences. In our structure operations of the continuity and its general forms where the algebraic operations of multiplicity and inverse will be discussed. This weaker form of continuity with topological groups started in the era of 1990s. Twenty-thirty years ago many interesting results were explored relating to this discipline. In 2014, Moiz et. al. explored the notion of quasi $s-$topological groups and quasi irresolute topological groups [2]. In 1963, Levine introduced the concept of semi-open sets [8]. With this idea researchers surveyed variety of concepts and its generalization by applying semi-open set notions [5, 7, 16]. Levine introduced semi continuity in [8] and Crossley have discussed about semi open sets [6]. In 1965, Bohn introduced notion of semi topological groups, which in our sense, is $s-$topological groups and also explored significant results on the semi topological groups [1].

In our case the interesting work is to study topological groups properties by using the generalized concept of continuous mapping and openness [3, 9, 10, 11, 14]. We have also explored the relations about these generalized discipline and presented counter examples as well [2, 3, 14]. Relations between these classes of groups endowed with a topology are investigated. It is very important to mention that this notion of S-topological groups vary from the concept of semi topological groups already available in the literature in [1]. Siab contributed his concept on generalization of topological groups by applying the irresolute mappings and expanded his work in the form of two classes, irresolute and $Irr-$topological groups, their properties, and clear the variety from topological groups [15]. In 2014, Bosan explored the generalization of openness in the form of quasi irresolute and semi Irr-topological groups [2]. We have also surveyed the generalization of quotient topological spaces by using the Levine semi continuity [13].

In this paper, we intend to prove the relation between s-topological groups and irresolute topological groups.

2. Topologized Groups Properties

Throughout this paper in the definition of topologized groups $(A, *, \tau)$, where $(A, *)$ and (A, τ) are respective a group and a topological space. The mapping

$$\mathrm{s : A \times A \to A}$$

represented as

$$s(a, b) = a * b, \text{ for each } a, b \in A,$$

is a multiplication mapping. Also

$$i : A \to A \text{ represented by } i(a) = a^{-1}, \text{ for each } a \in A,$$

is inverse mapping.

Definition 3. $(A, *, \tau)$ is an *S-topological group* if and only if
the multiplication mapping and the inverse mapping are semi-continuous.

Definition 4. [1] A triple $(A, *, \tau)$ is an s-topological group such that for all $a, b \in A$ and each open set R of $a * b^{-1}$ with the property that

$$\mathrm{S * T^{-1} \subset R},$$

for some semi open sets S of a and T of b

Example, where the importance of the Sierpiński topology, that how the character changed.

Example 5. $A = \{0, 1\}$ is a group under the usual addition modulo 2 and the topology

$$\tau = \{\emptyset, \{1\}, A\}.$$

$$SO(A \times A) = \{\emptyset, \{(1,1)\}, \{(0,0),(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\},$$

$$\{(0,0),(0,1),(1,1)\}, \{(0,0),(1,0),(1,1)\}, \{(0,1),(1,0),(1,1)\}, \{(0,0),(0,1),(1,0),(1,1)\}\}.$$

We observe that the multiplication mapping s is continuous at points $(0, 0)$ and $(1, 1)$ but neither continuous nor semi continuous at points $(0, 1)$ and $(1, 0)$. If we endow it with the Sierpiński topology

$$\tau = \{\emptyset, \{0\}, \mathrm{A}\},$$

where

$$\text{SO}(A \times A) = \{\emptyset, \{(0,0)\}, \{(0,0),(0,1)\}, \{(0,0),(1,0)\}, \{(0,0),(0,1),(1,0)\},$$

$$\{(0,0),(0,1),(1,0),(1,1)\}, \{(0,0),(1,1)\}, \{(0,0),(0,1),(1,1)\}, \{(0,0),(1,0),(1,1)\}\}.$$

Note that the multiplication mapping s is continuous at each point of the domain except for $(1,1)$, where at this point semi continuity exists. For the open set $\{0\}$ in the range set A containing $m(1,1)$, there exists a semi open set

$$\{(0,0),(1,1)\}$$

in the domain of multiplication mapping s with the property that $m(\{(0,0),(1,1)\}) \subset \{0\}$.

The inverse mapping

$$\mathrm{i : A \to A}$$

is continuous and hence semi-continuous. Therefore, $(A, +_2, \tau)$ is an S-topological group but not a topological group. Cao et. al. [4] proved that $(A, +_2, \tau)$ is not a semi topological group.

Definition 6. $(A, *, \tau)$ is a quasi irresolute topological group if and only if
(a) the left and right translations are irresolute
(b) the inverse mapping is irresolute.

The following example plays an important role of the topology in our work.

Example 7. The set $A = \{1,3,5,7\}$ is a group under the usual multiplication modulo 8 and the topology

$$\tau = \{\emptyset, A, \{7\}, \{3,5,7\}\}.$$

We have

$$SO(A) = \{\emptyset, A, \{7\}, \{1,7\}, \{3,7\}, \{5,7\{1,3,7\}, \{1,5,7\}, \{3,5,7\}\}.$$

$$\tau\times\tau = \{\emptyset, \{(7,7)\}, \{(3,7),(5,7),(7,7)\}, \{(7,3),(7,5),(7,7)\}, \{(1,7),(3,7),(5,7),(7,7)\},$$
$$\{(7,1),(7,3),(7,5),(7,7)\}, \{(3,7),(7,3),(5,7),(7,5),(7,7)\}, \{(1,7),(3,7),(7,3),(5,7),$$
$$(7,5),(7,7)\}, \{(7,1),(3,7),(7,3),(5,7),(7,5),(7,7)\}, \{(1,7),(7,1),(3,7),(7,3),(5,7),$$
$$(7,5),(7,7)\}, \{(3,3),(3,5),(3,7),(5,3),(5,5),(5,7),(7,3),(7,5),(7,7)\}, \{(7,1),(3,3),$$
$$(3,5),(5,3),(3,7),(7,3),(5,5),(5,7),(7,5),(7,7)\}, \{(1,7),(3,3),(3,5),(5,3),(3,7),(7,3),$$
$$(5,5),(5,7),(7,5),(7,7)\}, \{(3,1),(5,1),(7,1),(3,3),(3,5),(3,7),(5,3),(5,5),(5,7),(7,3),$$
$$(7,5),(7,7)\}, \{(1,3),(1,5),(1,7),(3,3),(3,5),(3,7),(5,3),(5,5),(5,7),(7,3),(7,5),(7,7)\},$$
$$\{(1,7),(3,1),(5,1),(7,1),(3,3),(3,5),(3,7),(5,3),(5,5),(5,7),(7,3),(7,5),(7,7)\},$$
$$\{(1,3),(1,5),(7,1),(1,7),(3,3),(3,5),(3,7),(5,3),(5,5),(5,7),(7,3),(7,5),(7,7)\},$$
$$\{(1,3),(1,5),(1,7),(3,1),(5,1),(7,1),(3,3),(3,5),(3,7),(5,3),(5,5),(5,7),(7,3),(7,5),$$
$$(7,7)\}, A \times A\}$$

It is not very hard to obtain $SO(A\times A)$ and observe that A has no favour of any topologized group in our discipline and when we change the topology, we have a favour as in the following example.

$$\tau = \{\emptyset, A, \{1\}, \{1,3,7\}\}.$$

We have

$$SO(A) = \{\emptyset, A, \{1\}, \{1,3\}, \{1,5\}, \{1,7\}, \{1,3,5\}, \{1,3,7\}, \{1,5,7\}\}.$$

Now it is not difficult to check that A is an S-topological group and not a topological group. It is very interesting that this family of $SO(A)$ is a topology again on A. However, the group A with this topology $SO(A)$ is not a topological group. For the open set $\{1\}$ containing

$$3 \odot_8 3^{-1},$$

there does not exist any open/semi-open set V containing 3 with the property of satisfying continuity inclusion.

By the following example we obtain an $S-$topological group in which inverse mapping is not continuous.

Example 8. Consider a group under addition modulo 3

$$A = \{2,1,0\},$$

where topology is

$$\tau = \{\phi, A, \{0\}, \{2,0\}\}.$$

and family of semi open sets is $SO(A) = \{\emptyset, A, \{0\}, \{1,0\}, \{2,0\}\}$. Note that the inverse mapping i is continuous at all points of the domain except at point 1, where it is semi continuous because for the point 1, we take an open set $V = \{2,0\}$ containing $i(1) = 2$ and there is only one open set $U = A$ in the domain topology containing 1 but $i(U) \nsubseteq V$. However, there exists a semi open set $\{0,1\}$ with the property that $i(\{0,1\}) \subset V$.

Definition 9. A mapping between two topological spaces is $S-$homeomorphism if it is bijective, semi continuous and pre semi open.

The following fundamental result is very important.

Theorem 10. *Each left/right translation in A ($s-$topological group) is an $S-$homeomorphism.*

Proof. Suppose that for any $a \in A$. Suppose b is a general element in A and let R be an open set of

$$\ell_a(b) = a * (b^{-1})^{-1}.$$

By using the Definition 4, we get

$$S * T^{-1} \subset R.$$

As in the special case,

$$a * T^{-1} \subset R.$$

Using Theorem 5 [1], the set T^{-1} is a semi open set of b. Thus, ℓ_a is semi continuous on A. Suppose $\ell_a(p_1) = \ell_a(p_2)$. This implies that $a * p_1 = a * p_2$. This implies by cancelation law that $p_1 = p_2$. This implies that ℓ_a is one-one. Now for every $p \in A$, we get $\ell_a(a^{-1} * p) = p$, for some $a^{-1} * p \in$ which implies ℓ_a is onto. By lemma [15] for every semi open set P in the domain A, $\ell_a(P) = a * P$ is semi open in the range A. This implies that ℓ_a is pre semi open. Hence ℓ_a is $S-$homeomorphism. □

Definition 11. [15] A triple $(A, *, \tau)$ is an Irr-topological group if the multiplication mapping and the inverse mapping defined in the S-topological group above are irresolute.

Definition 12. [15] A triple $(A, *, \tau)$ is an irresolute topological group such that for all $a, b \in A$ and each semi open set R of $a * b^{-1}$ with the property that

$$\mathrm{S} * \mathrm{T}^{-1} \subset \mathrm{R},$$

for some semi open sets S of a and T of b

Example 13.

$$A = \{c, b, a, e\}$$

is group and

$$\tau = \{\phi, \{e\}, \{e, a, c\}, A\}$$

is a topology and the set $SO(A) = \{\phi, \{e\}, \{c, e\}, \{b, e\}, \{c, e\}, \{e, a, b\}, \{e, b, c\}, \{e, a, c\}, A\}$ is the collection of all semi open sets. Here we can check semi continuity at every point of the domain proving that $(A, *, \tau)$ is just $Irr-$topological group but not with weaker condition because

$$a = a, b = c,$$

we get

$$P_a P_{c^{-1}} = P_a P_c = \{a, e\}\{c, e\} = A \subsetneq Q_{ac^{-1}} = Q_{ac} = Q_b = \{e, b\}.$$

Theorem 14. *An irresolute topological group $(A, *, \tau)$ implies an $s-$topological group.*

Proof. Suppose that

$$a, b \in A, and$$

$$R \subset A$$

is an open set of

$$a * b^{-1}.$$

Then by using the Definition 12, there exits semi open sets S of a and T of b with the property that

$$S * T^{-1} \subset R.$$

which proves the requirement. □

Theorem 15. *An Irr-topological group $(A, *, \tau)$ implies an $S-$topological group.*

Proof. The proof is obvious because irresoluteness implies a semi continuous mapping. □

Theorem 16. *An irresolute topological group $(A, *, \tau)$ implies an $S-$topological group.*

Proof. The proof is very clear because an irresolute topological group implies, by Theorem 14 an $s-$topological group which is an $S-$ topological group. □

Theorem 17. *A topological group $(A, *, \tau)$ implies an $s-$topological group.*

Proof. The proof is obvious because continuous mapping implies a semi continuous mapping. □

Theorem 18. *An s-topological group $(A, *, \tau)$ implies an $S-$topological group.*

Proof. The proof is obvious by the definitions of $s-$ topological group and $S-$ topological group. □

The following example is the counter example of Theorem 17 and Theorem 18.

Example 19. Suppose that

$$A = \{7, 5, 3, 1\}$$

is a group under multiplication modulo 8 and

$$\tau = \{\phi, \{7\}, \{7, 3, 5\}, A\}$$

is topology.

Here we can check semi continuity at each point of domain proving that A is an $S-$topological group but not weaker of irresolute since for,

$$a = 3, b = 7$$

with semi open $\{7, 5\}$ of

$$a \odot_8 b^{-7},$$

there do not exist respective semi opens S and T of a and b with the property that

$$S \odot_8 T^{-7} \subset \{7, 5\}.$$

Here, A due to missing of continuity of multiplication mapping s at the point

$$(3, 3) \text{ of } A \times A.$$

is not a topological group.

Example 20. The set $A = \{\omega^2, \omega, 1\}$ is a group under multiplication and topology on A is

$$\tau = \{\phi, A, \omega\}\}.$$

Then

$$SO(A) = \{\phi, A, \{\omega\}, \{\omega, 1\}, \{\omega^2, \omega\}\}.$$

$$\tau \times \tau = \{\phi, A \times A, \{(\omega, \omega)\}, \{(\omega, 1), (\omega, \omega), (\omega, \omega^2)\}, \{(1, \omega), (\omega, \omega), (\omega^2, \omega)\}, \{(\omega, 1), (1, \omega), (\omega, \omega), (\omega, \omega^2), (\omega^2, \omega)\}.$$

Note that A does not obey any structure in our case. If we endow this group with the topology like Bosan et. al. [3],

$$\tau = \{\phi, A, \{1\}\}.$$

Then

$$SO(A) = \{\phi, A, \{1\}, \{\omega, 1\}, \{\omega^2, 1\}\}.$$

multiplication and inverse mappings obey only semi continuous here.

Since

$$a = \omega, b = \omega^2,$$

we get,

$$P_\omega * P_{(\omega^2)^{-1}} = \{\omega, 1\} * \{\omega, 1\} = A \nsubseteq Q_{\omega*(\omega^2)^{-1}} = Q_{\omega*\omega} = Q_{\omega^2} = \{\omega^2, 1\}.$$

Definition 21. A mapping between two topological spaces is semi homeomorphism if it is bijective pre semi open and irresolute.

Lemma 22. [15] *Left and right translations are semi homeomorphism in irresolute topological group A.*

Theorem 23. *Let*

$$\zeta : (A, *, \tau_A) \to (B, *, \tau_B)$$

be a semi homeomorphism, where A and B are respective quasi irresolute and quasi $s-$ topological groups. Then semi continuity at e_A implies the semi continuity on A.

Proof. Let $a \in A$. Suppose R is an open set of $\zeta(a)$. By the semi continuity of the left translation, there exists a semi open set Q of e_B with the property that

$$l_b(Q) = b * Q \subset R.$$

Since ζ is semi continuous at e_A,

$$\zeta(P) \subset Q$$

for some semi open set P of e_A in A. Again, due to pre semi openness of

$$l_a : A \to A$$

we get,

$$\zeta(a * P) = \zeta(a) * zeta(P) = b * zeta(P) \subset b * Q \subset R.$$

So ζ is semi continuous at $a \in A$, and therefore ζ is semi continuous on A. □

Theorem 24. *Suppose*

$$\zeta : \mathrm{A} \to \mathrm{B}$$

is topologically isomorphism with

$$\zeta(\mathrm{a}^{-1}) = (\zeta(\mathrm{a}))^{-1},$$

where A and B are respective topological and $s-$topological groups. Then prove that B has the domain character of topological group.

Proof. Take $t_1, t_2 \in B$ and say $P_{t_1 * t_2^{-1}}$ is an open set of

$$t_1 * t_2^{-1}.$$

Then by continuity,

$$\zeta^{\leftarrow}(P_{t_1 * t_2^{-1}})$$

is an open set in A. Since ζ is bijective, there exist $p_1, p_2 \in A$ with the property that

$$\zeta(p_1) = t_1,\ \zeta(p_2) = t_2,$$

that is

$$p_1 = \zeta^{\leftarrow}(t_1),\ p_2 = \zeta^{\leftarrow}(t_2).$$

Now A is a topological group, so there are open sets

$$Q_{p_1} \text{ and } Q_{p_2}$$

such that

$$Q_{p_1} * Q_{p_2^{-1}} \subset \zeta^{\leftarrow}(P_{t_1 * t_2^{-1}}).$$

This implies that

$$\zeta(Q_{p_1} * Q_{p_2^{-1}}) \subset P_{t_1 * t_2^{-1}}.$$

or

$$\zeta(Q_{p_1}) * \zeta(Q_{p_2^{-1}}) \subset P_{t_1 * t_2^{-1}}.$$

This implies that

$$\zeta(Q_{p_1}) * (\zeta(Q_{p_2}))^{-1} \subset P_{t_1 * t_2^{-1}}.$$

Since ζ is open, so

$$\zeta(Q_{p_1}) \text{ and } \zeta(Q_{p_2})$$

are open in B. Assume that,

$$\zeta(Q_{p_1}) = R_{t_1} \text{ and } \zeta(Q_{p_2}) = R_{t_2}.$$

Then

$$R_{t_1} * (R_{t_2})^{-1} \subset P_{t_1 * t_2^{-1}}.$$

Hence required. □

REFERENCES

1. E. Bohn, *Semi-topological groups,* The American Mathematical Monthly, **72,** No. 9 (1965) 996-998.
2. M. S. Bosan et al., *On quasi irresolute and semi-irr-topological groups,* Afindad, **80,** No. 574 (2014) 1241-1252.
3. M. S.Bosan, M. Dn. Khanand L. DR Koscinac, *On s-topological groups,* Mathematica Moravica, **18,** No. 2 (2014) 35-44.
4. J. Cao, R. Drozdowski, and Z. Piotrowski, *Weak continuity properties of topologized groups*, Czechoslovak mathematical journal, **60,** No.1 (2010) 133-148.
5. P. Das, *Note on some applications of semi-open sets,* Prog. math., **7** (1973) 33-44.
6. S. G. Crossley and SK. Hildebrand, *Semi-topological properties,* Fundamenta Mathematicae, **74,** No.3(1972) 233-254.
7. T.Husain, *Semi-topological groups and linear spaces,* Mathematische Annalen, **160,** No.2 (1965) 146-160.
8. N. Levine, *Semi-open sets and semi-continuity in topological spaces,* The American mathematical monthly, **70,** No. 1 (1963) 36-41
9. M.K. Maqbool, C. Ozel, M. A Yousaf, M. Dilbar, Z. Ahmad, and M. S. Bosan, *A-hurewiczness and normality in semi topological groups*, submitted.
10. M. K. Maqbool, A. Yousaf, M. S. Bosan, and S. Jafari, *A-paracompactness and strongly a-screenability in topological groups*, European Journal of Pure and Applied Mathematics, **13,** No. 2 (2020) 280-286.
11. M. K. Maqbool, A. Yousaf, M. Dilbar, M. S. Bosan, M. Younis, and S. Sadique, A paradigmatic approach to quasi topological loops: Classification and characterization, Punjab Univ. j. math. **52,** No. 6 (2020) 107-117.
12. J. R. Munkres, *Topology,* Prentice Hall Upper Saddle River, NJ, 2000.
13. R. Noreen, M.S. Bosan, et al., *Semi-quotient mappings and spaces,* Open Mathematics, **14,** No. 1 (2016) 1014-1022.
14. R. Noreen, M.S. Bosan, and M.D.Khan, *Semi connectedness in irresolute topological groups,* Science International, **27,** No. 6 (2015).
15. A. Siab and L. DR Koscinac, *Irresolute-topological groups,* Mathematica Moravica, **19,** No.1 (2015) 73-80.
16. M. Tkachenko and AV. Arkhangelskii, *Topological groups and related structures*, Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2008).