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Classes of Ordinary Differential Equations of Length Biased Exponential Distribution and their Solutions

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Abstract. The purpose of the work is to generate the ordinary differential equations and their solutions for the probability density function, quantile rate function, survival rate function, inverse survival rate function, haz-ard rate function and reversed hazard rate function of the Length Biased Exponential Distribution. The ordinary differential equations and their so-lutions are obtained using the Math of differentiation and integration as a tool together with their boundary conditions. The boundaries and para-meters that portrayed the distribution unavoidably decide the nature, pres-ence, uniqueness, arrangement and the various possible solutions of these ordinary differential equations are better approaches to understand these characteristics. The work will be helpful to analyze the lifetime growth or risk and is of great significance in the field of ecological studies. The method can be very useful for the other probability distributions and can serve as a substitute for the approximation study.

Key Words: Probability Density Function, Quantile Rate Function, Survival Rate Function, Inverse Survival Rate Function, Hazard Rate Function, Reversed Hazard Rate Function.

1. INTRODUCTION

Lifetime processes have subsequently gained prominence through the modelling of their distributions. The length-biased distributions have attracted a lot of attention in the literature because naturally, it is not conceivable to gather information in groups. In numerous handy circumstances, old style distributions accordingly offer deficient modification for genuine information. The normal populace of human, natural life, bugs, plants, fishes and so on, often do not follow the characterized examining structures and thus recorded perceptions of individuals from these populace are biased, except if each perception is given an equivalent possibility of being recorded. The weighted distributions give a brought together way to deal with this type of biased information. When the probability of picking an individual in a general population is comparative with its size, it is called size biased testing. Suppose X$\geq$0 is a random variable with probability density function f(x). Let w(x)$\geq$0 be a weight function, then the probability density function of the weighted random variable X_w is defined as

$$f_w(x) = \frac{w(x)f(x)}{E[w(x)]}, \qquad x > 0,$$

where $E[w(x)]$=$\int w(x)f(x)dx$. when $w(x) = x^b$, the resulting distribution is termed weighted distribution and for $b = 1$, the distribution is known as length biased. Fisher [4] gave the idea of weighted distributions to show the ascertainment tendency. Later Rao [23] developed this thought in a more general way, where the standard distributions were not appropriate to record the observations with equal probabilities. In this way weighted models were itemized in such exceptional conditions to record the observations according to weighted limits. The possibility of biased investigation was first introduced by Cox [1] and Zelen [25]. Patil and Rao [22] surveyed the length biased study with application to natural life and human families. These later led to weighted movement. Relationships among the weighted and length biased distributions were first presented by Khatree [5]. These included security results and various characteristics of biased testing. Over the years, many researchers have analyzed classes of ordinary differential equations of different distributions. These include: exponential and truncated exponential distributions [6], Harris extended exponential distribution [7], exponentiated generalized exponential distribution [8], Gumbel distribution [9], Cauchy, standard cauchy and log-cauchy distributions [10], Frechet Distribution [11], Gompertz and Gamma Gompertz distributions [12], Burr XII and pareto distributions [13], linear failure rate and generalized linear failure rate distributions [14], 3-parameter Weibull distribution [15], exponentiated Frchet distribution [16], half-normal distribution [17], exponentiated pareto distribution [18], Weibull Distribution and their application in ecology [19], convoluted distributions [20], Kumaraswamy distribution [21].

2. MATERIALS AND METHODS

The Exponential Distribution' is extensively used to model the passage of time (between occurrence, or between spans of time). Because of the flexibility of the distribution and constant hazard rate, the distribution has seen many modifications in form of compounding, exponentiation, transmuted and so on. Dara and Ahmad [3], developed the new lifetime distribution, known as Length Biased Exponential Distribution'. The failure rate and the advantages of moment distributions were elaborated by them. Dar et al. [2], generalized it using another parameter. They presented Transmuted Weighted Exponential Distribution (TWED) and researched its diverse trademarks just as auxiliary properties. Two kinds of informational indexes were considered to make correlation between extraordinary instances of TWED, regarding fitting of Length Biased Exponential Distribution. Haq et al. [24], presented Marshall-Olkin Length-Biased Exponential Distribution. They studied its various statistical properties by examining it with the real-life data, which proved a better fit than many other similar types of models. Differential math is used to generate the ordinary differential equations of Length Biased Exponential Distribution, whose solutions are the corresponding functions produced with certain initial conditions.

3. RESULTS

The probability density function of Length Biased Exponential Distribution is

$$f(x) = \theta^2 x e^{-\theta x}; \quad \theta > 0, \quad x > 0 \tag{3. 1}$$

It can also be defined by the mean of Gamma distribution

$$f(x, \alpha, \beta) = \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0,$$

with parametric values $\alpha = 2$ and $\beta = \frac{1}{\theta}$.
The cumulative distribution function is defined as

$$F(x) = 1 - (1 + \theta x)e^{-\theta x} \tag{3. 2}$$

a) Probability Density Function
Differentiating the equation (3.1)

$$f'(x) = \theta^2 \left[e^{-\theta x} - \theta x e^{-\theta x}\right],$$

Using equation (3.1)

$$f'(x) = \big(\frac{1}{x} - \theta\big) f(x),$$

The expression exists under the necessary condition $\theta > 0, x > 0$ and can be simplified as

$$xf'(x) + (\theta x - 1)f(x) = 0 \tag{3. 3}$$

which is the required first order differential equation. Simplifying and taking the integral, we have

$$\int \frac{f'(x)}{f(x)} dx = \int \big(\frac{1}{x} - \theta\big) dx,$$

$$lnf(x) = lnx - \theta x + lnA,$$

$$f(x) = Axe^{-\theta x} \quad (3.\ 4)$$

For $f(1) = \theta^2 e^{-\theta}$, or for $f(\frac{1}{\theta}) = \frac{\theta}{e} = (0.3678)\theta$, we have

$$A = \theta^2,$$

Putting it in (3.4) yields the equation (3.1).

b) Quantile Rate Function

We proceed in equation (3.2) as

$$y = 1 - (\theta Q(y) + 1)e^{-\theta Q(y)} \quad (3.\ 5)$$

$$-\theta Q(y) + ln(\theta Q(y) + 1) = ln(1 - y),$$

By differentiating, we get

$$-\theta Q'(y) + \frac{\theta Q'(y)}{\theta Q(y) + 1} = -\frac{1}{1 - y},$$

The expression exists for $\theta > 0$, $0 \leq y < 1$. Now the equation becomes

$$-\theta Q'(y)\left(\frac{\theta Q(y) + 1 - 1}{\theta Q(y) + 1}\right) = -\frac{1}{1 - y},$$

$$-\theta Q'(y)(\theta Q(y)) = -\frac{\theta Q(y) + 1}{1 - y},$$

$$\theta^2(1 - y)Q(y)Q'(y) + \theta Q(y) + 1 = 0 \quad (3.\ 6)$$

which is the ordinary differential equation of order one. On simplifying and taking the integral, we have

$$-\int \frac{1}{1 - y}dy = -\theta^2 \int \frac{Q(y)Q'(y)}{\theta Q(y) + 1}dy,$$

$$-\int \frac{1}{1 - y}dy = -\int \left(\theta Q'(y) - \frac{\theta Q'(y)}{\theta Q(y) + 1}\right)dy,$$

$$ln(1 - y) = -\theta Q(y) + ln(\theta Q(y)) + lnB,$$

$$1 - y = B(\theta Q(y) + 1)e^{-\theta Q(y)} \quad (3.\ 7)$$

For $Q(0) = 0$, we have

$$B = 1,$$

Putting these values in (3.7), yields the equation (3.5).

c) Survival Rate Function

By using the definition, we have

$$S(x) = (1 + \theta x)e^{-\theta x} \quad (3.\ 8)$$

One time differentiation will lead us to

$$S'(x) = \theta e^{-\theta x} - \theta(1 + \theta x)e^{-\theta x},$$

Using equation (3.8)

$$S'(x) = \theta\left[\frac{1}{\theta x+1} - 1\right]S(x),$$

The necessary condition for existence is $\theta > 0, x \geq 0$.

$$(\theta x+1)S'(x) + \theta^2 x S(x) = 0 \tag{3. 9}$$

is the required first order differential equation. Simplifying and taking the integral, we have

$$\int \frac{S'(x)}{S(x)}dx = -\theta^2 \int \frac{x}{\theta x+1}dx,$$

$$\int \frac{S'(x)}{S(x)}dx = -\int \left(\theta - \frac{\theta}{\theta x+1}\right)dx,$$

$$lnS(x) = -\theta x + ln(\theta x+1) + lnC,$$

$$S(x) = C(\theta x+1)e^{-\theta x}, \tag{3. 10}$$

For $S(0) = 1$, we have

$$C = 1,$$

Which by putting in (3.10), yields the equation (3.8).

d) Inverse Survival Rate Function

By the substitution in (3.8), we get

$$y = (\theta Q(y)+1)e^{-\theta Q(y)} \tag{3. 11}$$

$$-\theta Q(y) + ln(\theta Q(y)+1) = lny,$$

By differentiating, we get

$$-\theta Q'(y) + \frac{\theta Q'(y)}{\theta Q(y)+1} = \frac{1}{y},$$

The expression exists for $\theta > 0, 0 < y \leq 1$. Now the equation becomes

$$-\theta Q'(y)\left(\frac{\theta Q(y)+1-1}{\theta Q(y)+1}\right) = \frac{1}{y},$$

$$-\theta Q'(y)(\theta Q(y)) = \frac{\theta Q(y)+1}{y},$$

$$\theta^2 y Q(y)Q'(y) + \theta Q(y) + 1 = 0 \tag{3. 12}$$

which is the first order differential equation. On simplifying and taking the integral, we have

$$\int \frac{1}{y}dy = -\theta^2 \int \frac{Q(y)Q'(y)}{\theta Q(y)+1}dy,$$

$$\int \frac{1}{y}dy = -\int \left(\theta Q'(y) - \frac{\theta Q'(y)}{\theta Q(y)+1}\right)dy,$$

$$lny = -\theta Q(y) + ln(\theta Q(y)+1) + lnD,$$

$$y = D(\theta Q(y) + 1)e^{-\theta Q(y)}, \tag{3. 13}$$

For $Q(1) = 0$, we have

$$D = 1,$$

By substituting in (3.13), we will obtain the equation (3.11).

e) Hazard Rate Function

By definition, we have

$$h(x) = \frac{\theta^2 x}{\theta x + 1}, \tag{3. 14}$$

Differentiating the equation gives

$$h'(x) = \theta^2 \left[\frac{1}{\theta x + 1} - \frac{\theta x}{(\theta x + 1)^2}\right],$$

Using equation (3.14)

$$h'(x) = \left[\frac{1}{x} - \frac{\theta}{\theta x + 1}\right] h(x),$$

$$h'(x) = \left[\frac{1}{x} - \frac{h(x)}{\theta x}\right] h(x),$$

The necessary condition for existence is $\theta > 0, x > 0$. Simplification will yield the first order differential equation as

$$\theta x h'(x) + h^2(x) - \theta h(x) = 0, \tag{3. 15}$$

For the solution of the differential equation, we proceed as

$$1 = \theta\left(- x\frac{h'(x)}{h^2(x)} + \frac{1}{h(x)}\right),$$

$$\int dx = \theta \int d\Big(\frac{x}{h(x)}\Big),$$

$$x + E = \frac{\theta x}{h(x)},$$

$$h(x) = \frac{\theta x}{x + E}, \tag{3. 16}$$

For $h(1) = \frac{\theta^2}{\theta+1}$, for $h(\frac{1}{\theta}) = \frac{\theta}{2}$, we have

$$E = \frac{1}{\theta},$$

By the substitution in (3.16), we get (3.14).

f) Reverse Hazard Rate Function

Using definition, we get

$$j(x) = \frac{\theta^2 x e^{-\theta x}}{1 - (\theta x + 1)e^{-\theta x}}, \tag{3. 17}$$

One time differentiation implies

$$j'(x) = \frac{\theta^2(1-\theta x)e^{-\theta x}}{1-(\theta x+1)e^{-\theta x}} - \frac{\theta^2 x e^{-\theta x}[0-\big(\theta-\theta(\theta x+1)\big)e^{-\theta x}]}{(1-(\theta x+1)e^{-\theta x})^2},$$

Using equation (3.17) gives

$$j'(x) = \left[\frac{1-\theta x}{x} + \frac{(\theta-\theta^2 x-\theta)e^{-\theta x}}{1-(\theta x+1)e^{-\theta x}}\right] j(x),$$

$$j'(x) = \left[\frac{1-\theta x}{x} - j(x)\right] j(x),$$

On simplification, we get the required first order differential equation

$$xj'(x) + xj^2(x) + (\theta x - 1)j(x) = 0, \tag{3. 18}$$

The equation holds for $\theta > 0, x > 0$. For the solution, we proceed as

$$1 = -\frac{j'(x)}{j^2(x)} + \big(\frac{1}{x} - \theta\big)\frac{1}{j(x)},$$

The integrating factor of the differential equation is $xe^{-\theta x}$. Multiplying and taking the integral to get

$$\int xe^{-\theta x}dx = \int d\left(\frac{xe^{-\theta x}}{j(x)}\right),$$

$$-x\frac{e^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta^2} + F = \frac{1}{j(x)}xe^{-\theta x},$$

$$j(x) = \frac{\theta^2 x e^{-\theta x}}{F\theta^2 - (\theta x+1)e^{-\theta x}}, \tag{3. 19}$$

For $j(1) = \frac{\theta^2 e^{-\theta}}{1-(\theta+1)e^{-\theta}}$, for $j(\frac{1}{\theta}) = \frac{\theta}{e-2} = (0.3922)\theta$, we have

$$F = \frac{1}{\theta^2},$$

These values give the equation (3.17).

4. Conclusion

Ordinary differential equations have been derived for the likelihood elements of Length Biased Exponential Distribution. The differential and integral analytics with proficient mathematical improvements were utilized to determine the different classes of these differential equations and their respective solutions. The outcomes have indicated that the ordinary differential equations are not restricted to the probability density function however other likelihood functions. The procedure cannot be applied to the distributions, if the probability density function and cumulative distribution function are not differentiable. There is need to explore and urgent utilization of these ordinary differential equations as their convenience in the field of environment and information management is yet under considered. The exploration is vital in biological inspections, as the investigation of behavior of the distribution can be stretched out to the conduct of the ordinary differential equations that characterize them.

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