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Coupled Gerdjikov-Ivanov System and its Exact Solutions through Darboux Transformation

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Abstract.: The coupled Gerdjikov-Ivanov (GI) system is discussed and its elementary Darboux transformation (DT) is constructed. From elementary (DT), we constructed $2N$-fold (DT). The symmetrical properties of the Lax pair and iteration of $2N$-fold (DT) give different types of solutions for both zero seed and non-zero seed. Zero seed solutions include bright-bright (one and two) Soliton solutions. In non-zero seed solutions, we obtained breather, Ma breather, dark-bright Soliton, breather fission, and dark-bright rogue wave solutions are obtained and plot it.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Darboux transformation, Exact solutions, Coupled Gerdjikov-Ivanov system.

1. Introduction

Integrable systems play very dominant role in nonlinear science. Different integrable models describe different nonlinear phenomenon's like water waves, nonlinear optics, molecular biology, condensed matter and cold atom physics, plasma physics etc [15, 9, 8, 27, 7, 2, 25, 10, 23, 26, 18]. To solve integrable systems is much difficult task for both theoretical physicists and mathematicians. Different methods have been developed to solve these systems to obtain its solutions. Darboux transformation, Inverse scattering transformation, Bäcklund transformation and Hirota's bilinear method are famous techniques to solve integrable systems [25, 5, 12, 14, 4, 31, 6, 16, 13, 33]. The nonlinear Schrödinger (NLS) equation play significant role in different fields [31, 3, 24]. With various modifications, researchers obtained third kind of derivative nonlinear Schrödinger equation called Gerdjikov-Ivanov equation for higher-order perturbations [11].

$$ir_t + r_{xx} - i|r|^2 r_x^* + \frac{1}{2} r^3 r^{*2} \tag{1. 1}$$

This integrable system has much important in nonlinear optics. In fiber optics, Soliton has great importance for propagation of signals transmission [30, 1, 17]. Without arrangements, the signals disperse and causes lose of signals. This is due to linear effects (dispersion, attenuation, polarization mode dispersion) and nonlinear effects (Kerr effects, stimulated Raman scattering, stimulated brillouin scattering) in the medium [22]. The balance between linear and nonlinear effects in the medium causes Soliton, which are responsible for long transmission. Many researchers obtained different solutions like breather, rogue wave and Soliton of different integrable systems. The vector equations having two or more than two components give more accuracy during investigation than scalar models. The eigenvalues of Lax pair must be equal to the eigenvalues of its adjoint. In [20, 21], authors discuss the symmetries and reductions for matrix mKdV and multicomponent NLS equations and find Soliton solutions by using binary DT. Therefore, we choose coupled Gerdjikov-Ivanov (GI) system to obtain different types of solutions and investigate it [32, 28].

$$\begin{aligned} ir_{1t} + r_{1xx} - ir_1\left(r_1 r_{1x}^* + r_2 r_{2x}^*\right) + \frac{1}{2}\left(|r_1|^2 + |r_2|^2\right)^2 r_1 &= 0, \\ ir_{2t} + r_{2xx} - ir_2\left(r_1 r_{1x}^* + r_2 r_{2x}^*\right) + \frac{1}{2}\left(|r_1|^2 + |r_2|^2\right)^2 r_2 &= 0. \end{aligned} \tag{1. 2}$$

From coupled (GI) system in eq.(1.2), one can easily obtain (GI) system in eq.(1.1) by applying equality $r_1 = r_2$. Many authors applied Darboux transformation to (NLS) equation and obtained different types of solutions like Soliton, breather and rogue wave [31, 19, 29].

In this article, we apply Darboux transformation (DT) to coupled (GI) system. We first construct elementary (DT) for 3×3 matrix spectral problem in section 2. With the help of elementary (DT), we find 2N-fold (DT) in section 3. By using this method, we obtain different solutions like bright-bright Soliton first-order and second-order for zero seed in sub section 4.1. For non-zero seed, we obtain breather, Ma breather, dark-bright Soliton and dark-bright rogue wave in sub section 4.2.

2. Elementary Darboux transformation

To construct the elementary Darboux transformation of coupled Gerdjikov-Ivanov system, we start with spectral problem

$$\Psi_x = L\Psi, \;\; L = \lambda^2 S + \lambda R + \frac{1}{2}R^2 S \tag{2. 3}$$

$$\begin{aligned} \Psi_t &= M\Psi, \\ M &= 2\lambda^4 S + 2\lambda^3 R + \lambda^2 R^2 S + \lambda R_x S + \frac{1}{2}(R_x R - R R_x) - \frac{1}{4}R^4 S \end{aligned} \tag{2. 4}$$

Where

$$R = \begin{pmatrix} 0 & 0 & r_1 \\ 0 & 0 & r_2 \\ -r_1^* & -r_2^* & 0 \end{pmatrix}, \;\; S = \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$$

Here, r_1, r_2 are potentials and λ is spectral parameter. The Lax pair in equations (2.3) and (2.4) satisfy integrability condition, $L_t - M_x + [L, M] = 0$ which leads coupled (GI) system in eq.(1.2).

We introduce related conjugate equation of eq.(2.3) and (2.4) for constructing (DT) as

$$-\Phi_x = \Phi L, \qquad -\Phi_t = \Phi M \tag{2. 5}$$

Where, Φ represent 3-row vector. The matrices $L^\dagger(\lambda)$ and $M^\dagger(\lambda)$ satisfy the symmetric equations as

$$L^\dagger(\lambda) = -L(\lambda^*), \qquad M^\dagger(\lambda) = -M(\lambda^*) \tag{2. 6}$$

and

$$\delta L(\lambda)\delta = L(-\lambda), \qquad \delta M(\lambda)\delta = M(-\lambda) \tag{2. 7}$$

Where, δ denoted as diagonal matrix with (1,1,-1) diagonal entries.

Now suppose $\Psi_1 = (\eta_1, \eta_2, \eta_3)^T$ is the solution of eq.(2.3) and (2.4) at $\lambda = \lambda_1$ then from eq.(2.6) we can show that $\Psi_1^\dagger$ is special solution of eq.(2.5) at $\lambda = \lambda_1^*$. We can also show that the solution $\delta\Psi_1$ for eq.(2.7) at $\lambda = -\lambda_1$ give the solution Ψ_1 at $\lambda = \lambda_1$.

From the above discussion, the Darboux matrix D for two fold problems is constructing as

$$\begin{aligned} D &= D[2]D[1], \\ &= \left(I + \frac{\lambda_1 - \lambda_1^*}{\lambda - \lambda_1^*}\Omega_1\right)\left(I + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*}\Omega_2\right) \\ &= I + \frac{Y_1}{\lambda - \lambda_1^*} + \frac{Y_2}{\lambda - \lambda_1^*} \end{aligned} \tag{2. 8}$$

Where, I is an identity matrix having order 3×3 and other notations in eq.(2.8) are denoted as

$\Omega_1 = \frac{\Psi_1 \Psi_1^\dagger}{\Psi_1^\dagger \Psi_1}, \quad \Omega_2 = \frac{\Psi_2[1]\Psi_2^\dagger[1]}{\Psi_2^\dagger[1]\Psi_2[1]}$

Here

$$\begin{aligned} \Psi_2[1] &= D[1]|_{\lambda=-\lambda}\delta\Psi_1, \\ &= \left(I + \frac{\lambda_1 - \lambda_1^*}{\lambda - \lambda_1^*}\Omega_1\right)|_{\lambda=-\lambda_1}\delta\Psi_1 \\ \Psi_2^\dagger[1] &= \Psi_1^\dagger \delta D[1]^{-1}|_{\lambda=-\lambda_1^*}, \\ &= \Psi_1^\dagger \delta \left(I + \frac{\lambda_1 - \lambda_1^*}{\lambda - \lambda_1^*}\Omega_1\right)|_{\lambda=-\lambda_1^*} \end{aligned}$$

We can write the new potentials for Darboux matrix D in eq.(2.8) as

$$\widetilde{r}_1 = r_1 + 2i(Y_1 + Y_2)_{13}, \quad \widetilde{r}_1 = r_1 + 2i(Y_1 + Y_2)_{23} \tag{2. 9}$$

where, $Y_1 + Y_2 = (\lambda_1 - \lambda_1^*)\Omega_2 + (\lambda_1 - \lambda_1^*)\Omega_1$ and $(Y_1 + Y_2)_{mn}$ represent mth row and nth column of matrix $(Y_1 + Y_2)$.

Theorem 2.1. *The new potentials in eq.(2.9) change the Lax pair in eq.(2.3) and (2.4) accordingly*

$$\widetilde{\Psi}_x = \widetilde{L}\widetilde{\Psi}, \qquad \widetilde{\Psi}_t = \widetilde{M}\widetilde{\Psi} \tag{2. 10}$$

Where

$$\begin{aligned} \widetilde{L} &= (D_x + DL)D^{-1}, \\ \widetilde{M} &= (D_t + DM)D^{-1} \end{aligned} \tag{2. 11}$$

We can easily confirm that the $\widetilde{L}$ and $\widetilde{M}$ have same as L and M in eq.(2.3) and (2.4).

3. 2N-FOLD DARBOUX TRANSFORMATION

By using the elementary Darboux transformation method in section 2, now we derive 2N-fold Darboux transformation as follows.

Let Ψ_k is 2N solutions of Lax pair in eq.(2.3) and (2.4) with $\lambda = \lambda_k$ and Φ_κ is 2N solutions of eq.(2.5) with $\lambda = \varsigma_k$ where $k = (1, 2, 3, ...2N)$. Then the Darboux matrix D for 2N-fold can written as

$$\begin{aligned} D &= D[2N]D[2N-1].....D[1], \\ &= I + \sum_{k=1}^{2N} \frac{Y_k}{\lambda - \varsigma_k} \end{aligned} \tag{3. 12}$$

Where, $Y_k = |v_k\rangle \langle\chi_k|$ and $|v_k\rangle$ is column vector and $\langle\chi_k|$ is row vector.

Moreover, inverse Darboux matrix D is

$$\begin{aligned} D^{-1} &= D[1]^{-1}D[2]^{-1}.....D[2N]^{-1}, \\ &= I + \sum_{k=1}^{2N} \frac{Z_k}{\lambda - \lambda_k} \end{aligned} \tag{3. 13}$$

Where $Z_k = |x_k\rangle \langle\omega_k|$ and $|x_k\rangle$ is column vector and $\langle\omega_k|$ is row vector.

Moreover, the Darboux matrix D satisfy the condition as

$$DD^{-1} = D^{-1}D = I \tag{3. 14}$$

given by

$$\begin{aligned} \left(I + \sum_{l=1}^{2N} \frac{Y_l}{\lambda - \varsigma_l}\right)|_{\lambda=\lambda_k} Z_k &= 0, \\ Y_k \left(I + \sum_{l=1}^{2N} \frac{Z_l}{\lambda - \lambda_l}\right)|_{\lambda=\varsigma_k} &= 0, \\ \left(I + \sum_{l=1}^{2N} \frac{Z_l}{\lambda - \lambda_l}\right)|_{\lambda=\varsigma_k} Y_k &= 0, \\ Z_k \left(I + \sum_{l=1}^{2N} \frac{Y_l}{\lambda - \varsigma_l}\right)|_{\lambda=\lambda_k} &= 0. \end{aligned} \tag{3. 15}$$

Therefore, from above it is easy to obtain that

$$D|_{\lambda=\lambda_k}|x_k\rangle = 0, \qquad \langle\chi_k|D^{-1}|_{\lambda=\varsigma_k} = 0 \tag{3. 16}$$

$$D^{-1}|_{\lambda=\varsigma_k}|v_k\rangle = 0, \qquad \langle\omega_k|D|_{\lambda=\lambda_k} = 0 \tag{3. 17}$$

With the help of

$$D|_{\lambda=\lambda_k}\Psi_k = 0, \quad \Phi_k D^{-1}|_{\lambda=\varsigma_k} = 0 \tag{3. 18}$$

we can get

$$|x_k\rangle = \Psi_k, \quad \langle\chi_k| = \Phi_k \tag{3. 19}$$

So, we can write the eq.(3.16) and (3.17) as

$$|x_k\rangle = -\sum_{l=1}^{2N} |v_l\rangle\Lambda_{lk}, \quad \langle\chi_k| = \sum_{l=1}^{2N} \Lambda_{lk}\langle\omega_l| \tag{3. 20}$$

$$|v_k\rangle = -\sum_{l=1}^{2N} |x_l\rangle \widetilde{\Lambda}_{lk}, \quad \langle \omega_k| = \sum_{l=1}^{2N} \widetilde{\Lambda}_{lk} \langle \chi_l| \tag{3. 21}$$

Here

$$\Lambda_{lk} = \frac{\langle \chi_l | x_k \rangle}{\lambda_k - \varsigma_l}, \qquad \widetilde{\Lambda}_{lk} = \frac{\langle \omega_l | v_k \rangle}{\varsigma_k - \lambda_l} \tag{3. 22}$$

We can show that $\Lambda_{lk}\widetilde{\Lambda}_{lk} = I$. From eq.(3.20) and (3.21) we obtain the follows

$$\begin{aligned}(Y_1 + Y_2 + ... + Y_{2N})_{mn} &= |v_1\rangle^m \langle \chi_1|^n + |v_2\rangle^m \langle \chi_2|^n + ... + |v_{2N}\rangle^m \langle \chi_{2N}|^n \\ &= (|v_1\rangle^m, |v_2\rangle^m, ..., |v_{2N}\rangle^m)(\langle \chi_1|^n, \langle \chi_2|^n, ..., \langle \chi_{2N}|^n)^T \\ &= -(|x_1\rangle^m, |x_2\rangle^m, ..., |x_{2N}\rangle^m)\widetilde{\Lambda}_{lk}(\langle \chi_1|^n, \langle \chi_2|^n \\ , ...\langle \chi_{2N}|^n)^T \end{aligned} \tag{3. 23}$$

where $|x_k\rangle^m$ represent mth component and $\langle \chi_k|^n$ denoted nth component with $(m, n = 1, 2, 3)$ and $(k = 1, 2, 3, ..., 2N)$.

Theorem 3.1. *Let*

$$|x_{2k-1}\rangle = \Psi_k, \quad |x_{2k}\rangle = \delta\Psi_k, \tag{3. 24}$$

$$|\chi_{2k-1}\rangle = \Psi_k^\dagger, \quad |\chi_{2k-1}\rangle = \delta\Psi_k^\dagger \tag{3. 25}$$

Then the Darboux transformation of 2N-fold for the Lax pair in eq.(2.3) and (2.4) is obtain as

$$\widetilde{r}_1 = r_1 + 2i(Y_1 + Y_2 + ...Y_{2N})_{13} \tag{3. 26}$$

$$\widetilde{r}_2 = r_2 + 2i(Y_1 + Y_2 + ...Y_{2N})_{23} \tag{3. 27}$$

One can write the new potentials $\widetilde{r}_1$ and $\widetilde{r}_2$ in compact determinant form as

$$\widetilde{r}_1 = r_1 - 2i\sum_{k=1}^{2N} \frac{\det \Lambda_k}{\det \Lambda} \tag{3. 28}$$

$$\widetilde{r}_2 = r_2 - 2i\sum_{k=1}^{2N} \frac{\det S_k}{\det \Lambda} \tag{3. 29}$$

Where

$$\Lambda = \begin{pmatrix} \frac{\Psi_1^\dagger \Psi_1}{\lambda_1 - \lambda_1^*} & \frac{\Psi_1^\dagger \delta\Psi_1}{-\lambda_1 - \lambda_1^*} & . & . & . & \frac{\Psi_1^\dagger \Psi_N}{\lambda_N - \lambda_1^*} & \frac{\Psi_1^\dagger \delta\Psi_N}{-\lambda_N - \lambda_1^*} \\ \frac{\Psi_1^\dagger \delta\Psi_1}{\lambda_1 + \lambda_1^*} & \frac{\Psi_1^\dagger \Psi_1}{-\lambda_1 - \lambda_1^*} & . & . & . & \frac{\Psi_1^\dagger \delta\Psi_N}{\lambda_N + \lambda_1^*} & \frac{\Psi_1^\dagger \Psi_N}{-\lambda_N - \lambda_1^*} \\ . & . & & & & . & . \\ . & . & . & . & . & . & . \\ . & . & & & & . & . \\ \frac{\Psi_N^\dagger \Psi_1}{\lambda_1 - \lambda_N^*} & \frac{\Psi_N^\dagger \delta\Psi_1}{-\lambda_1 - \lambda_N^*} & . & . & . & \frac{\Psi_N^\dagger \Psi_N}{\lambda_N - \lambda_N^*} & \frac{\Psi_N^\dagger \delta\Psi_N}{-\lambda_N - \lambda_N^*} \\ \frac{\Psi_N^\dagger \delta\Psi_1}{\lambda_1 + \lambda_N^*} & \frac{\Psi_N^\dagger \Psi_1}{-\lambda_1 - \lambda_N^*} & . & . & . & \frac{\Psi_N^\dagger \delta\Psi_N}{\lambda_N + \lambda_N^*} & \frac{\Psi_N^\dagger \Psi_N}{-\lambda_N - \lambda_N^*} \end{pmatrix} \tag{3. 30}$$

By inserting $(2k-1)th$ row with $(\eta_1^{(1)}\eta_3^{(k)*}, \eta_1^{(1)}\eta_3^{(k)*}, ..., \eta_1^{(N)}\eta_3^{(k)*})$ in Λ to obtain Λ_{2k-1}.

By inserting $2kth$ row with $(-\eta_1^{(1)}\eta_3^{(k)*}, -\eta_1^{(1)}\eta_3^{(k)*}, ..., -\eta_1^{(N)}\eta_3^{(k)*})$ in Λ to obtain Λ_{2k}.

By inserting $(2k-1)th$ row with $(\eta_2^{(1)}\eta_3^{(k)*}, \eta_2^{(1)}\eta_3^{(k)*}, ..., \eta_2^{(N)}\eta_3^{(k)*})$ in Λ to obtain S_{2k-1}.

By inserting $2kth$ *row with* $(-\eta_2^{(1)}\eta_3^{(k)*}, -\eta_2^{(1)}\eta_3^{(k)*}, ..., -\eta_2^{(N)}\eta_3^{(k)*})$ *in* Λ *to obtain* S_{2k}.

From eq.(3.28) and (3.29) for $N = 1$ one can obtain the basic solution $(\eta_1, \eta_2, \eta_3)^T$ for Lax pair in eq.(2.3) and (2.4) as

$$\widetilde{r}_1 = r_1 + 8\xi\zeta \frac{(\xi E + i\zeta F)\eta_1\eta_3^*}{\xi^2E^2 + \zeta^2F^2} \tag{3. 31}$$

$$\widetilde{r}_2 = r_2 + 8\xi\zeta \frac{(\xi E + i\zeta F)\eta_2\eta_3^*}{\xi^2E^2 + \zeta^2F^2} \tag{3. 32}$$

Where $E = |\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2$, $F = |\eta_1|^2 + |\eta_2|^2 - |\eta_3|^2$ and $\lambda = \lambda_1 = \xi + i\zeta$.

In addition, the transformation in eq.(3.31) and (3.32) is consistent in eq.(2.9).

4. Exact solutions for the coupled GI system

Now we apply the above mention transformation to obtain both zero seed and non-zero seed solutions.

4.1. **Zero seed solutions.** We take $r_1 = 0, r_2 = 0$ for the system in eq.(1.2), the Lax pair in eq.(2.3) and (2.4) give the basic solution as

$$\Psi(\lambda) = \begin{pmatrix} \Upsilon_1 e^{-i\lambda^2 x - 2i\lambda^4 t} \\ \Upsilon_2 e^{-i\lambda^2 x - 2i\lambda^4 t} \\ \Upsilon_3 e^{i\lambda^2 x + 2i\lambda^4 t} \end{pmatrix} \tag{4. 33}$$

Here Υ_j $(j = 1, 2, 3)$ are real constants and $\Upsilon_1 \neq \Upsilon_2$.

4.1.1. *One Soliton solution.* By using Darboux transformation in eq.(3.31) and (3.32) with the help of the basic solution in eq.(4.33) with $\lambda = \xi + i\zeta$ and also $\xi\zeta \neq 0$ we can obtain one Soliton solution as

$$\widetilde{r}_1 = r_1 + 8\xi\zeta \frac{(\xi E_1 + i\zeta F_1)\Upsilon_1\Upsilon_3 e^{2i\varrho_2}}{\xi^2E_1^2 + \zeta^2F_1^2} \tag{4. 34}$$

$$\widetilde{r}_2 = r_2 + 8\xi\zeta \frac{(\xi E_1 + i\zeta F_1)\Upsilon_2\Upsilon_3 e^{2i\varrho_2}}{\xi^2E_1^2 + \zeta^2F_1^2} \tag{4. 35}$$

Here

$E_1 = 2\sqrt{\Upsilon_1^2 + \Upsilon_2^2}\Upsilon_3 \cosh(2\varrho_1 + \vartheta)$, $F_1 = 2\sqrt{\Upsilon_1^2 + \Upsilon_2^2}\Upsilon_3 \sinh(2\varrho_1 + \vartheta)$, $\varrho_1 = 2\xi\zeta[x + 4(\xi^2 - \zeta^2)t]$, $\varrho_2 = -(\xi^2 - \zeta^2)x - 2(\xi^4 + \zeta^4 - 6\xi^2\zeta^2)t$, and $\vartheta = \ln \frac{\sqrt{\Upsilon_1^2+\Upsilon_2^2}}{\Upsilon_3}$.

By inserting these values in eq.(4.34) and (4.35) we can write the solution in the following form as

$$|\widetilde{r}_1|^2 = \frac{16\xi^2\zeta^2\Upsilon_1^2}{(\Upsilon_1^2 + \Upsilon_2^2)[\xi^2\cosh^2(2\varrho_1 + \vartheta) + \zeta^2\sinh^2(2\varrho_1 + \vartheta)]} \tag{4. 36}$$

$$|\widetilde{r}_2|^2 = \frac{16\xi^2\zeta^2\Upsilon_2^2}{(\Upsilon_1^2 + \Upsilon_2^2)[\xi^2\cosh^2(2\varrho_1 + \vartheta) + \zeta^2\sinh^2(2\varrho_1 + \vartheta)]} \tag{4. 37}$$

We can plot this Soliton solution by choosing suitable parameters. Both potentials $|\widetilde{r}_1|^2$ and $|\widetilde{r}_2|^2$ give bright-bright Soliton of eq.(1.2) shown in Figure 1 as

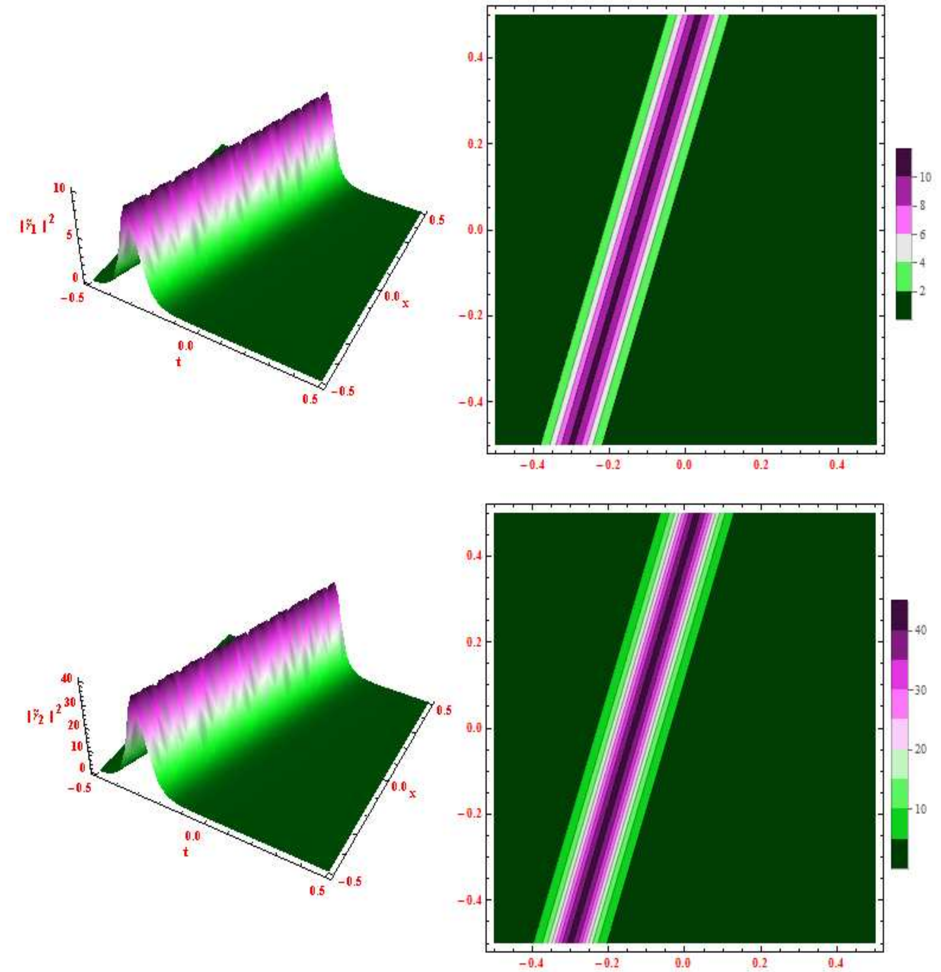


Figure:1 Bright-bright one Soliton solution with contour plot having parameters $\xi = \frac{1}{2}$, $\zeta = -1$ and $\Upsilon_1 = \Upsilon_3 = 1$, $\Upsilon_2 = 2$

4.1.2. *Two Soliton solution.* If $N = 2$ then the solution in eq.(3.28) and (3.29) with basic solution in eq.(4.33) having suitable parameters $\lambda_1 = \frac{1}{4} + \frac{1}{2}i$, $\lambda_2 = \frac{1}{2} + \frac{1}{4}i$, $\Upsilon_1 = \Upsilon_3 = 1$ and $\Upsilon_2 = 2$ give two Soliton solution as

$$|\widetilde{r}_1| = \frac{6e^{(-\frac{3}{64}+\frac{i}{16})(8ix+(4+3i)t)}\left((-2-i)e^{\frac{3t}{4}} + (1+2i)e^{\frac{3ix}{4}} + (10-5i)e^{x} - (5-10i)e^{\frac{3t}{4}+\left(1+\frac{3i}{4}\right)x}\right)}{-80i\cos\left(\frac{3x}{4}\right) + 100i\cosh\left(\frac{3t}{4}\right) + 108\cosh(x) - 75\sinh\left(\frac{3t}{4}\right) + 117\sinh(x)} \quad (4.\ 38)$$

$$|\widetilde{r}_2| = \frac{12e^{(-\frac{3}{64}+\frac{i}{16})(8ix+(4+3i)t)}\left((-2-i)e^{\frac{3t}{4}} + (1+2i)e^{\frac{3ix}{4}} + (10-5i)e^{x} - (5-10i)e^{\frac{3t}{4}+\left(1+\frac{3i}{4}\right)x}\right)}{-80i\cos\left(\frac{3x}{4}\right) + 100i\cosh\left(\frac{3t}{4}\right) + 108\cosh(x) - 75\sinh\left(\frac{3t}{4}\right) + 117\sinh(x)} \quad (4.\ 39)$$

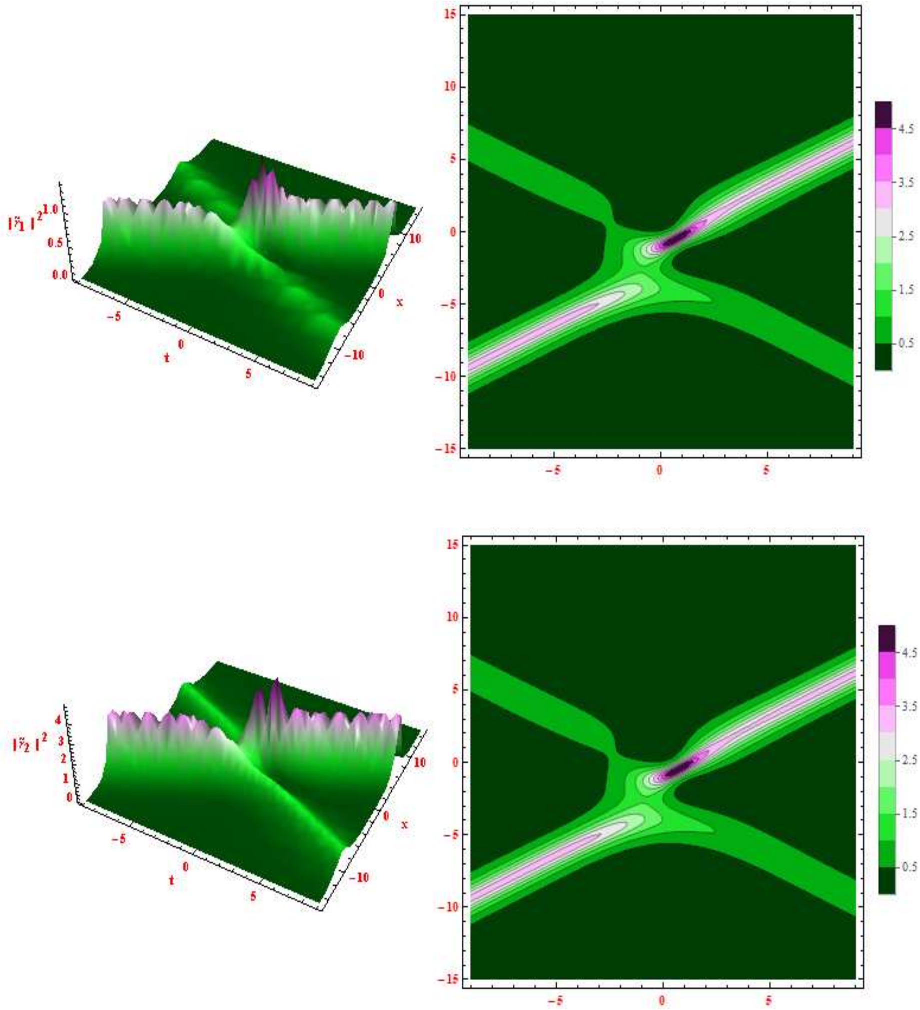


Figure:2 Two Soliton solution and contour plot having parameters $\lambda_1 = \frac{1}{4} + \frac{1}{2}i$, $\lambda_2 = \frac{1}{2} + \frac{1}{4}i$, $\Upsilon_1 = \Upsilon_3 = 1$ and $\Upsilon_2 = 2$

Moreover, the parameters $\lambda_1 = \frac{1}{4} + \frac{1}{2}i$, $\lambda_2 = \frac{1}{2} + \frac{1}{4}i$ have interchangeable real and imaginary parts. If we take other values then we could see oscillation in amplitudes of Soliton.

4.2. **Nonzero seed solutions.** We take $r_1 = \Upsilon e^{i(ax+bt)}$, $r_2 = 0$ for the system in eq.(1.2), the Lax pair in eq.(2.3) and (2.4) give the basic solution as

$$\Psi(\lambda)=\begin{pmatrix} e^{\frac{\sqrt{h}(x+2\lambda^2t-at)+i(ax+bt)}{2}}+e^{\frac{-\sqrt{h}(x+2\lambda^2t-at)+i(ax+bt)}{2}} \\ e^{\left(-i\lambda^2x-2i\lambda^4t\right)} \\ \frac{\sqrt{h}+ia+2i\lambda^2-i\Upsilon^2}{2\lambda\Upsilon}e^{\frac{\sqrt{h}(x+2\lambda^2t-at)-i(ax+bt)}{2}}+\frac{-\sqrt{h}+ia+2i\lambda^2-i\Upsilon^2}{2\lambda\Upsilon}e^{\frac{-\sqrt{h}(x+2\lambda^2t-at)-i(ax+bt)}{2}} \end{pmatrix} \tag{4. 40}$$

Here
$b=-\left(a^2+a\Upsilon^2-\frac{1}{2}\Upsilon^4\right)$ and $h=-a^2-4\lambda^4-4a\lambda^2-\Upsilon^4+2a\Upsilon^2$.

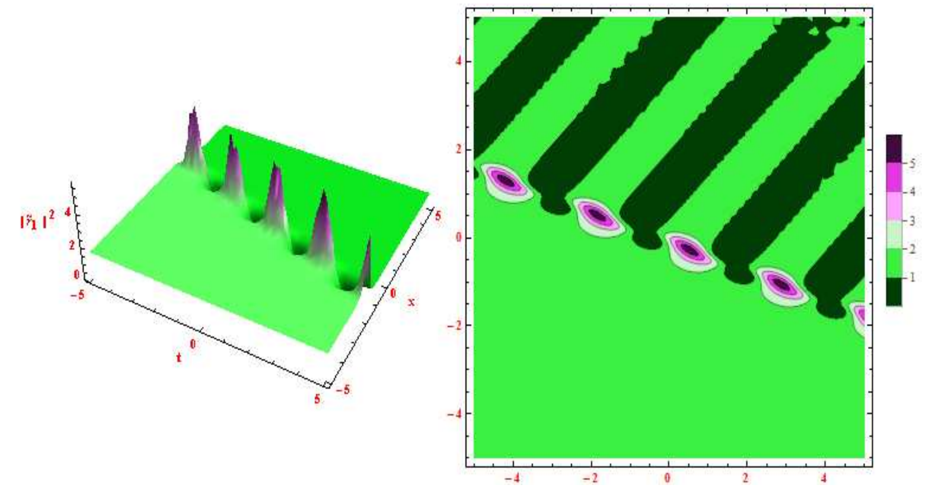
4.2.1. *Breather solution.* If we take $a=\frac{\Upsilon^2}{2}$ then $b=-\frac{\Upsilon^4}{4}$ and $h=-\left(\frac{\Upsilon^2}{2}+2\lambda^2\right)^2$ becomes. By taking $\lambda=\xi_1+i\zeta_1$ and inserting these values into eq.(4.40) and then from eq.(3.31) and (3.32) for $N=1$ one can obtain first order breather solution as

$$\widetilde{r}_1 = \Upsilon e^{i\left(\frac{\Upsilon^2x}{2}-\frac{\Upsilon^2t}{4}\right)}+8\xi_1\zeta_1\frac{\left(\xi_1E_2+i\zeta_1F_2\right)\Omega_1\Omega_2^*}{\xi_1^2E_2^2+\zeta_1^2F_1^2}, \tag{4. 41}$$

$$\widetilde{r}_2 = 8\xi_1\zeta_1\frac{\left(\xi_1E_2+i\zeta_1F_2\right)\Omega_2^*e^{i\tau_2+\tau_1}}{\xi_1^2E_2^2+\zeta_1^2F_1^2} \tag{4. 42}$$

Here $E_2=\Omega_1\Omega_1^*+e^{2\tau_1}+\Omega_2\Omega_2^*$, $F_2=\Omega_1\Omega_1^*+e^{2\tau_1}-\Omega_2\Omega_2^*$, $\Omega_1=e^{i(-\tau_2+\tau_3)-\tau_1}+e^{i\tau_2+\tau_1}$, $\Omega_2=\frac{-2\zeta_1+2i\xi_1}{\Upsilon}e^{-i\tau_2-\tau_1}+\frac{\Upsilon\zeta_1-i\xi_1\Upsilon}{2\left(\xi_1^2+\zeta_1^2\right)}e^{i(\tau_2-\tau_3)+\tau_1}$, $\tau_1=2\xi_1\zeta_1x+8\xi_1\zeta_1\left(\xi_1^2-\zeta_1^2\right)t$, $\tau_2=\left(\zeta_1^2-\xi_1^2\right)x-2\left(\xi_1^4+\zeta_1^4-6\xi_1^2\zeta_1^2\right)t$ and $\tau_3=\frac{\Upsilon^2x}{2}-\frac{\Upsilon^2t}{4}$.

By choosing $\Upsilon=1$, $\xi_1=1$ and $\zeta_1=-\frac{1}{2}$ we plot first order breather solution in Figure 3 as



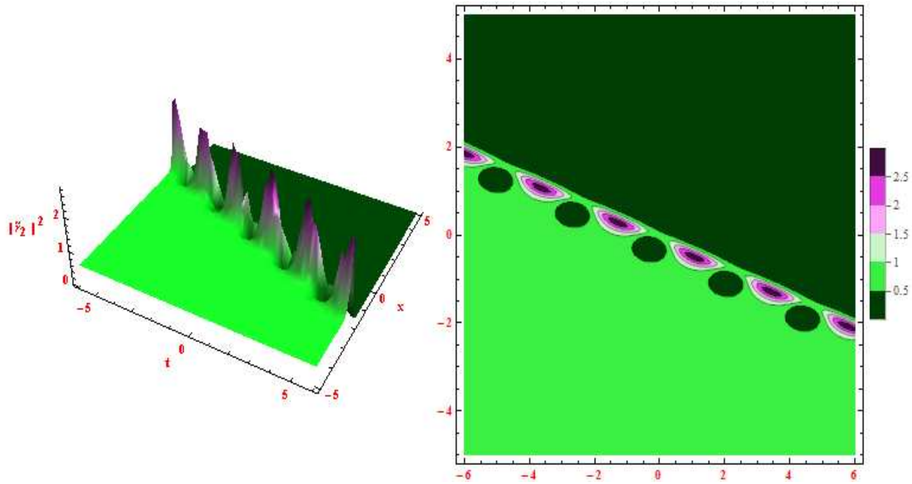


Figure:3 Breather solution and contour plot having parameters $\xi_1 = 1,\ \zeta_1 = -\frac{1}{2}$ and $\Upsilon = 1$

Applying similar fashion for $N = 2$ we can obtain and plot second order breather solution.

4.2.2. *Ma-breather first order-solution.* If we take $a = 0$ then nonzero seed solution becomes $r_1 = \Upsilon e^{\frac{i\Upsilon^4 t}{2}}$, $r_2 = 0$. Inserting these into Lax pair of eq.(2.3) and (2.4) then the basic solution in eq.(4.40) takes the form as

$$\Psi(\lambda) = \begin{pmatrix} e^{\frac{\sqrt{h}(x+2\lambda^2 t)+ibt}{2}} + e^{\frac{-\sqrt{h}(x+2\lambda^2 t)+ibt}{2}} \\ e^{(-i\lambda^2 x - 2i\lambda^4 t)} \\ \frac{\sqrt{h}+2i\lambda^2-i\Upsilon^2}{2\lambda\Upsilon} e^{\frac{\sqrt{h}(x+2\lambda^2 t)-ibt}{2}} + \frac{-\sqrt{h}+2i\lambda^2-i\Upsilon^2}{2\lambda\Upsilon} e^{\frac{-\sqrt{h}(x+2\lambda^2 t)-ibt}{2}} \end{pmatrix} \tag{4. 43}$$

Inserting this basic solution in eq.(3.31) and (3.32) and setting $E = E_3$, $F = F_3$ also $\lambda = \xi_2 + i\zeta_2$ for $|\xi_2| = |\zeta_2|$. we take imaginary part for $h = 0$ and $\xi_2 = \zeta_2 = \varkappa$ we obtain

$$\widetilde{r}_1 = \Upsilon e^{\frac{i\Upsilon^4 t}{2}} + 8\varkappa^2 \frac{(\varkappa E_3 + i\varkappa F_3)\, g_1}{\varkappa^2 E_3^2 + \varkappa^2 F_3^2}, \tag{4. 44}$$

$$\widetilde{r}_2 = 8\varkappa^2 e^{2\varkappa^2 x} \frac{(\varkappa E_3 + i\varkappa F_3)\, g_2[\cos(8\varkappa^4 t) + i\sin(8\varkappa^4 t)]}{\varkappa^2 E_3^2 + \varkappa^2 F_3^2} \tag{4. 45}$$

Here

$$\begin{aligned} E_3 &= 2\cos(2\tau_5) + 2\cosh(2\tau_4) + e^{4\varkappa^2 x} + \frac{1}{2\Upsilon^2\varkappa^2}[\Upsilon^4\cos(2\tau_5) + 16\varkappa^4\cosh(2\tau_4) \\ &\quad -\Upsilon^2\sqrt{16\varkappa^4 - \Upsilon^4}\sin(2\tau_5) - 4\varkappa^2\sqrt{16\varkappa^4 - \Upsilon^4}\sinh(2\tau_4)], \\ F_3 &= 2\cos(2\tau_5) + 2\cosh(2\tau_4) + e^{4\varkappa^2 x} - \frac{1}{2\Upsilon^2\varkappa^2}[\Upsilon^4\cos(2\tau_5) + 16\varkappa^4\cosh(2\tau_4) \\ &\quad -\Upsilon^2\sqrt{16\varkappa^4 - \Upsilon^4}\sin(2\tau_5) - 4\varkappa^2\sqrt{16\varkappa^4 - \Upsilon^4}\sinh(2\tau_4)]. \end{aligned}$$

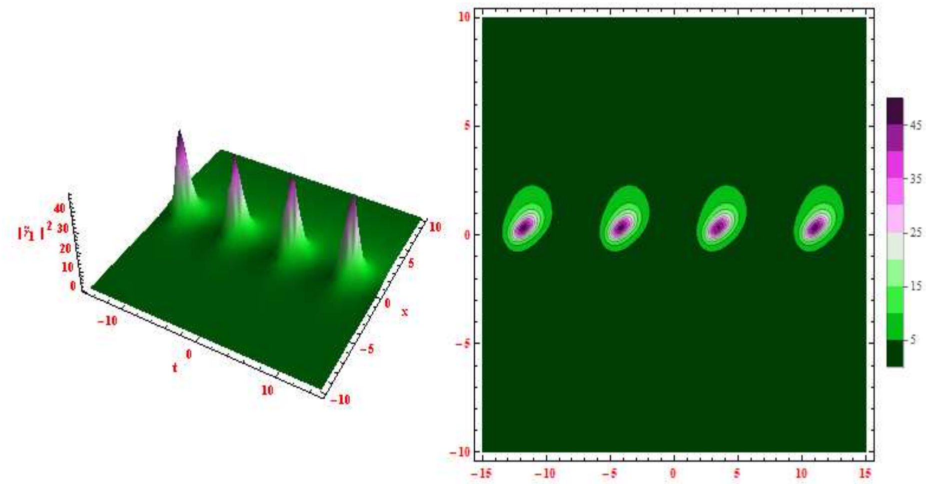
and

$$\begin{aligned} g_1 &= \frac{1+i}{2\Upsilon\varkappa}e^{\frac{i\Upsilon^4 t}{2}}[\left(i\Upsilon^2-4\varkappa^2\right)\left(\cos\left(2\tau_5\right)+\cosh\left(2\tau_4\right)\right) \\ &\quad +\sqrt{16\varkappa^4-\Upsilon^4}\left(-i\sin\left(2\tau_5\right)+\sinh\left(2\tau_4\right)\right)], \\ g_2 &= e^{\frac{i\Upsilon^4 t}{4}}[\frac{\left(\varkappa+i\varkappa\right)\sqrt{16\varkappa^4-\Upsilon^4}}{4\Upsilon\varkappa^2}\left(e^{\tau_4-i\tau_5}-e^{-\tau_4+i\tau_5}\right) \\ &\quad -\frac{\left(\varkappa+i\varkappa\right)\left(4\varkappa^2-i\Upsilon^2\right)}{4\Upsilon\varkappa^2}\left(e^{\tau_4-i\tau_5}+e^{-\tau_4+i\tau_5}\right)] \end{aligned}$$

also

$$\tau_4=\frac{\sqrt{16\varkappa^4-\Upsilon^4}}{2}x, \qquad \tau_5=2\varkappa^2 t\sqrt{16\varkappa^4-\Upsilon^4}$$

Now there are many different situations, which are gives as follows.
Case-1. When we take $16\varkappa^4-\Upsilon^4>0$ by setting condition $\Upsilon<2\varkappa$ then we obtain time periodic breather (Ma breather) as plot in figure 4 as



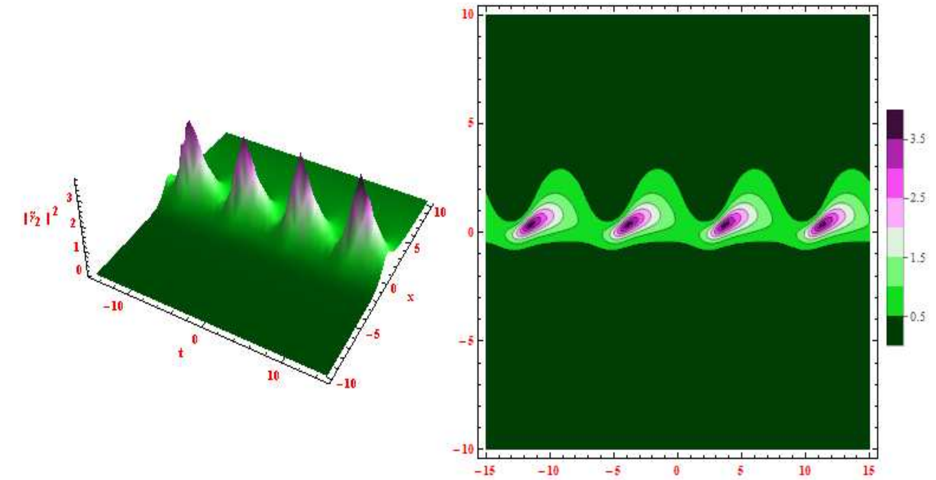
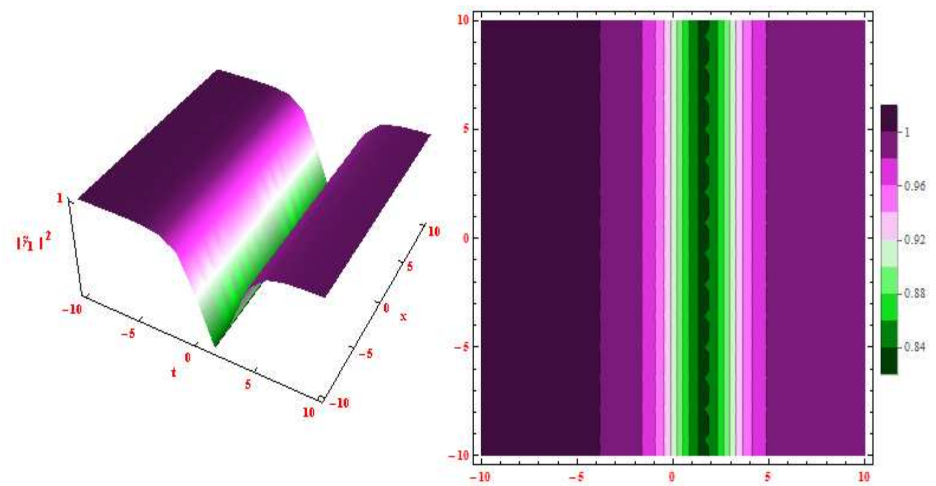


Figure:4 Ma breather solution and contour plot having parameters $\varkappa = \frac{1}{2}$ and $\Upsilon = \frac{3}{4}$.
Case-2. When we take $\Upsilon = 2\varkappa$ then $16\varkappa^4 - \Upsilon^4 = 0$ and the solution in eq.(4.44) give dark Soliton at $|\widetilde{r}_1|^2 \to 1$ and eq.(4.45) give dark bright Soliton at $|\widetilde{r}_2|^2 \to 0$. The dark-bright Soliton solution is also obtain in nonlinear Schrödinger equations.



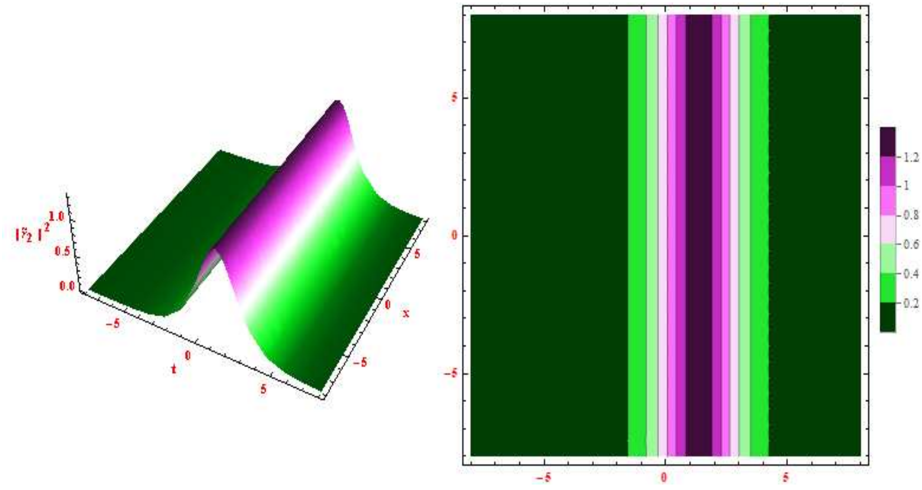
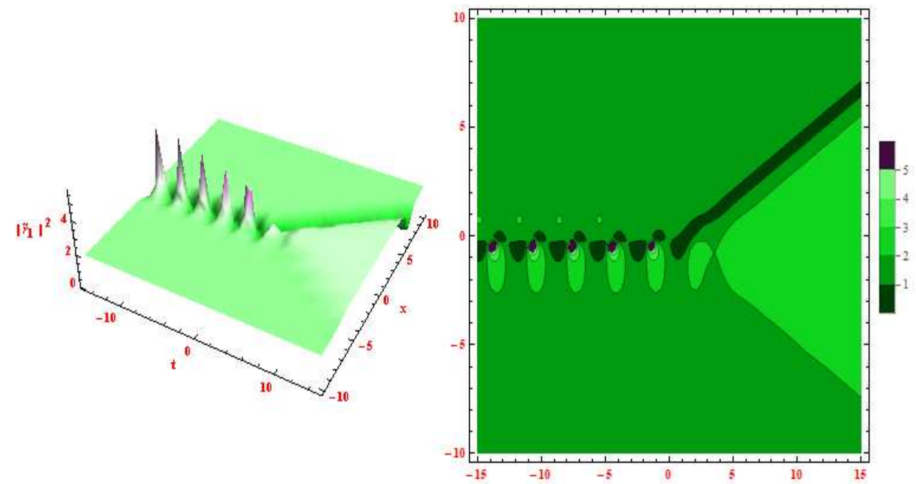


Figure:5 Dark-bright Soliton and contour plot having parameters $\varkappa = \frac{1}{2}$ and $\Upsilon = 1$.
Case-3. If we take $\Upsilon > 2\varkappa$ then $16\varkappa^4 - \Upsilon^4 < 0$ and the solution in eq.(4.44) and (4.45) give breather fission solution. The energy of potentials is not conserves independently. We can derive conservation law from eq.(1.2) as

$$\left(|r_1|^2 + |r_2|^2\right)_t = \left[\frac{1}{2}\left(|r_1|^2 + |r_2|^2\right)^2 - i\left(r_1 r_{1x}^* - r_{1x} r_{1x}^*\right) - i\left(r_1 r_{1x}^* - r_{2x} r_{2x}^*\right)\right]_x \quad (4.46)$$

Now the total energy of potentials $|r_1|^2$ and $|r_2|^2$ is constant energy with the help of $\int_{-\infty}^{+\infty} \left(|r_1|^2 + |r_2|^2\right) \tau x$ =constant.



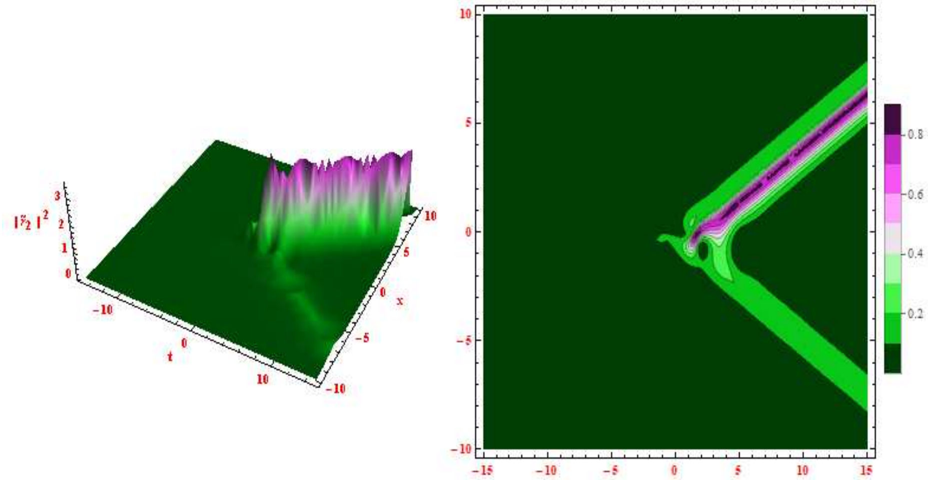


Figure:6 Breather fission solution and contour plot having parameters $\varkappa = \frac{1}{2}$ and $\Upsilon = \frac{3}{2}$.

Similarly if we set $\lambda = \xi_2 - i\zeta_2$ then the condition $\xi_2 = -\zeta_2 = \varkappa$ becomes and we can obtain Soliton fusion solution. In addition, the total energy of system is conserves during fusion.

4.2.3. *Dark-bright rogue wave solution.* If we take $a = \frac{\Upsilon^2 - \varepsilon^2}{2}$, $b = -\frac{\Upsilon^4}{4} - \frac{\varepsilon^4}{4} + (\Upsilon\varepsilon)^2$ then we construct basic solution $\Psi(\lambda)$ for rogue wave solution. The basic solution appears as both polynomial and exponential function. For this discriminant is zero when we choose $\lambda = \frac{\varepsilon}{2} \pm \frac{\Upsilon}{2} i$, where ε is nonzero constant.

$$a^2 + 4\left(\lambda^4 + a\lambda^2 + \frac{\Upsilon^4}{4} - \frac{a\Upsilon^2}{2}\right) = 0 \tag{4. 47}$$

Then the basic solution for Lax pair in eq.(2.3) and (2.4) takes the form as

$$\Psi(\lambda) = \begin{pmatrix} \left[k_1 + k_2\left(2\lambda^2 - a\right)t + k_2 x\right] e^{\left(\frac{i(ax+bt)}{2}\right)} \\ k_3 e^{\left(-i\lambda^2 x - 2i\lambda^4 t\right)} \\ \frac{1}{\lambda\Upsilon}\left[\left(k_1 + k_2\left(2\lambda^2 - a\right)t\right)\left(\frac{i\left(a-\Upsilon^2\right)}{2} + i\lambda^2\right) + k_2\left(1 + \frac{i\left(a-\Upsilon^2\right)}{2}x + i\lambda^2 x\right)\right] e^{\left(\frac{-i(ax+bt)}{2}\right)} \end{pmatrix} \tag{4. 48}$$

Where k_i represent real constant as $(i = 1, 2, 3)$.

We can obtain first order rogue wave solution by inserting eq.(4.48) into eq.(3.31) and (3.32) by choosing $E = E_4$, $F = F_4$ and $\lambda = \frac{\varepsilon}{2} + \frac{\Upsilon}{2} i$ as

$$\widetilde{r}_1 = \Upsilon e^{i(ax+bt)} + \frac{4\varepsilon\Upsilon\left(\varepsilon E_4 + i\Upsilon F_4\right)\Omega_3 \Omega_4^* e^{(iax+ibt)}}{\varepsilon^2 E_4^2 + \Upsilon^2 F_4^2}, \tag{4. 49}$$

$$\widetilde{r}_2 = \frac{4\varepsilon\Upsilon\left(\varepsilon E_4 + i\Upsilon F_4\right) k_3 \Omega_4^* e^{\left(\tau_6 + \frac{iax}{2} + \frac{ibt}{2}\right)}}{\varepsilon^2 E_4^2 + \Upsilon^2 F_4^2} \tag{4. 50}$$

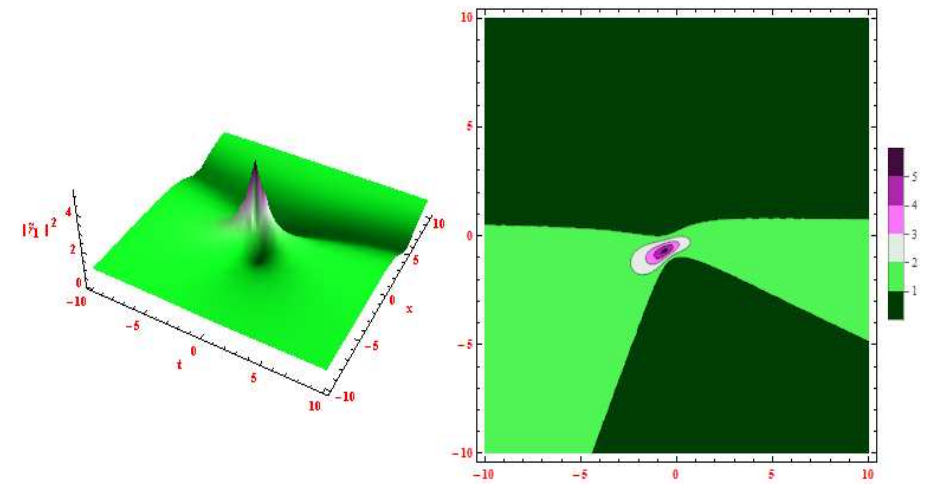
Where

$$\begin{aligned} E_4 &= \Omega_3\Omega_3^* + k_3^2 e^{\Upsilon\varepsilon\left(-\Upsilon^2 t + x + t\varepsilon^2\right)} + \Omega_4\Omega_4^*, \\ F_4 &= \Omega_3\Omega_3^* + k_3^2 e^{\Upsilon\varepsilon\left(-\Upsilon^2 t + x + t\varepsilon^2\right)} - \Omega_4\Omega_4^* \end{aligned}$$

and

$$\begin{aligned} \tau_6 &= \frac{i}{8}\left[-2x + t\left(\Upsilon - i\varepsilon\right)^2\right]\left(\Upsilon - i\varepsilon\right)^2, \\ \Omega_3 &= k_1 + xk_2 - \frac{1}{2}t\left[2a + \left(\Upsilon - i\varepsilon\right)^2\right]k_2, \\ \Omega_4 &= \frac{1}{4\Upsilon\left(\Upsilon - i\varepsilon\right)}\left[-8ik_2 - \left(2a - 3\Upsilon^2 + 2i\Upsilon\varepsilon + \varepsilon^2\right)\left(-2k_1 + \left(2at - 2x + t\left(\Upsilon - i\varepsilon\right)^2\right)k_2\right)\right] \end{aligned}$$

We plot dark-bright rogue wave solution of eq.(4.49) and (4.50). The solution $|\widetilde{r}_1|^2 \to 1$, its maximum and minimum values is 4.6 and 0 respectively. The solution $|\widetilde{r}_2|^2 \to 0$, its maximum and minimum values is 2.6 and 0 respectively.



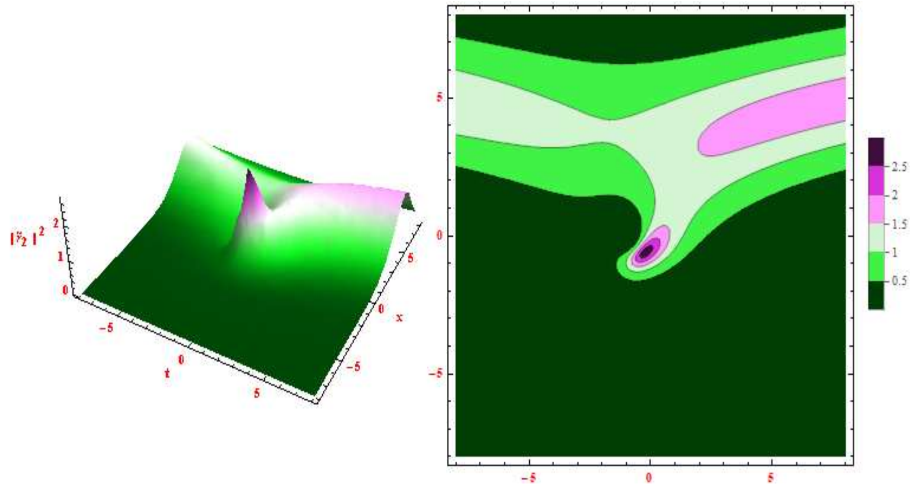


Figure:7 Dark-bright rogue wave solution and contour plot having parameters $\Upsilon = 1$, $\varepsilon = 1$, and $k_1 = k_2 = k_3 = 1$

By choosing $\lambda = \frac{\varepsilon}{2} - \frac{\Upsilon}{2}i$ one can obtain Bright-dark rogue wave solution.

5. Conclusion

In this article, we constructed elementary Darboux transformation for coupled (GI) equation. From the symmetries of the Lax pair and elementary (DT), we constructed 2N-fold (DT). Different types of solutions are obtained for both zero seed and non-zero seed. In zero seed, we obtained bright-bright Soliton solution, which has great importance in optical fibers for long transmission of signals without lose of energy. In non-zero seed, we obtained breather, Ma breather, dark-bright Soliton, breather fission and rogue wave solutions. The air causes small disturbances on the surface of water. These disturbances are first appears in the form of breathers of small amplitude. The oscillations in breathers causes Soliton and rogue waves, which have large amplitudes and causes destruction for ships and shores. With the help of these solutions, we can make arrangements to control different parameters like amplitude, frequency, energy and wavelength etc. By controlling these parameters, we can control the surface of water near seaports and shores, which is some time causes huge destruction. In similar fashion, we can also control transmission of signals in fiber optics. Therefore, the coupled (GI) equation and its solutions have different applications in fluid dynamics, fiber optics and plasma physics.

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