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Approximation of fixed points for a pair of certain nonexpansive type mappings with applications

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Abstract. We consider an iterative scheme to approximate common fixed points of a pair of single and multi-valued mappings satisfying the so-called condition (C_α) in Banach spaces. We also compare its rate of convergence with some existing iterative schemes. We show that our iterative scheme converges faster than these schemes. It has been confirmed numerically and depicted graphically. We prove the convergence results for our proposed scheme. We also provide an application of our iterative scheme to solve the Split Common Fixed Point (SCFP) Problem.

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1. Introduction

Fixed point theory has emerged as one of the leading fields of mathematics with applications in diverse fieds. Iterative schemes are key to approximating (common) fixed points. This field has attracted many mathematicians around the globe. In this paper, we consider an iterative scheme to approximate common fixed points of a pair of single and multi-valued mappings satisfying the so-called condition (C_α) in Banach spaces. We also compare its rate of convergence with some existing iterative schemes. We show that our iterative scheme converges faster than these schemes. It has been confirmed numerically and depicted graphically. We prove the convergence results for our proposed scheme. We also provide an application of our iterative scheme to solve the Split Common Fixed Point (SCFP) Problem.

2. Notations and Preliminaries

Let Ω be a normed space, B_s a nonempty subset Ω. A single-valued mapping $\mathcal{R} : \mathrm{B}_s \to \mathrm{B}_s$ is said to be

- contraction if

$$\|\mathcal{R}a - \mathcal{R}b\| \leq \wp\|a - b\|, \ \ \wp \in (0, 1),$$

- nonexpansive if

$$\|\mathcal{R}a - \mathcal{R}b\| \leq \|a - b\|, \ \forall \, a, b \in \mathrm{B}_s.$$

If $\nu^* = \mathcal{R}\nu^*$ then ν^* is a fixed point of $\mathcal{R}$ and the set of fixed point is denoted as $\mathcal{F}_p(\mathcal{R}) = \{\nu^* \in \mathrm{B}_s : \nu^* = \mathcal{R}\nu^*\}$. Several authors, including [2, 10, 26], have investigated the fixed points existence and approximation for nonexpansive mappings. Because of its applications in a variety of practical problems, numerous writers have recently generalized the concept of nonexpansive mappings [29].

Recently, Pant et al.[18] proposed more generalized form of nonexpansive mappings. A mapping $\mathcal{R} : \mathrm{B}_s \to \mathrm{B}_s$ is called generalized α-nonexpansive mapping if

$$\|\mathcal{R}a - \mathcal{R}b\| \leq \alpha\|b - \mathcal{R}a\| + \alpha\|a - \mathcal{R}b\| + (1 - 2\alpha)\|a - b\|, \ \ \forall a, b \in \mathrm{B}_s,$$

holds for $\alpha \in [0, 1)$. Fixed point existence and approximation of such mappings were obtained in different spaces (See [10] and therein).

We can represent the family of nonempty bounded closed subsets of Ω as $\mathcal{F}_{cb}(\mathrm{B}_s)$, and the family of nonempty compact convex subsets of Ω as $\mathcal{K}_c(\mathrm{B}_s)$. Note that the Hausdroff distance on $\mathcal{F}_{cb}(\mathrm{B}_s)$ is given by

$$\mathrm{H}(\mathbb{F}, \mathbb{G}) = \max\{\sup_{a\in\mathbb{F}} d(a, \mathbb{G}) \ , \sup_{b\in\mathbb{G}} d(b, \mathbb{F})\}$$

where $d(a, \mathbb{G}) = \inf\{\|a - b\| : b \in \mathbb{G}\}$.

If $\mathrm{S} : \mathrm{B}_s \to \mathcal{F}_{cb}(\mathrm{B}_s)$ is a multivalued mapping and $\nu^* \in \mathrm{S}\nu^*$ then ν^* is said to be the fixed point of multivalued mapping S and its fixed point set is represented by $\mathcal{F}_p(\mathrm{S})$. It is also known that if B_s is a nonempty bounded closed and convex subset of a uniformly convex Banach space (UCBS) Ω, then $\mathcal{S}$ has a fixed point [15].

A multivalued mapping $\mathrm{S} : \mathrm{B}_s \to \mathcal{F}_{cb}(\mathrm{B}_s)$ is

- contraction if

$$\mathrm{H}(\mathrm{S}a, \mathrm{S}b) \leq \wp\|a - b\|, \quad \forall \, a, b \in \mathrm{B}_s, \wp \in (0, 1),$$

- nonexpansive if

$$\mathrm{H}(\mathrm{S}a, \mathrm{S}b) \leq \|a - b\|, \quad \forall \, a, b \in \mathrm{B}_s.$$

In the recent years, fixed point theory for multivalued mappings has attracted the attention of several researchers. Many single-valued mappings have been generalized and extended to multivalued mapping [14, 30]. Recently, Iqbal et al. [10] introduced the generalized multivalued α-nonexpansive mappings on Banach spaces as follows. A multivalued mapping $\mathrm{S} : \mathrm{B}_s \to \mathcal{F}_{cb}(\mathrm{B}_s)$ is said to be generalized α-nonexpansive mapping if

$$\mathrm{H}(\mathrm{S}a, \mathrm{S}b) \leq \alpha\, d(b, \mathrm{S}a) + \alpha d(a, \mathrm{S}b) + (1-2\alpha)\|a-b\|$$

holds for all $a, b \in \mathrm{B}_s$.
S satisfies the condition (C_α) if

$$\begin{aligned} &\frac{1}{2} d(a, \mathrm{S}a) \leq \|a-b\| \qquad \text{implies} \\ &\mathrm{H}(\mathrm{S}a, \mathrm{S}b) \leq \alpha\, d(b, \mathrm{S}a) + \alpha d(a, \mathrm{S}b) + (1-2\alpha)\|a-b\| \end{aligned} \tag{2. 1}$$

for all $a, b \in \mathrm{B}_s$.

Many iterative schemes have been developed for both single and multi-valued mappings to approximate fixed points. A few of them are given by Mann [16], Ishikawa [11], S-iteration [1], Thakur et al. [25].

Recently, Akkasriworn and Sokhuma [3] considered a pair of single and multi-valued nonexpansive mappings and using a modified S-iterative scheme proved that the sequence thus obtained converges to a common fixed point (CFP) of both mappings. Dhompongsa et al. [8] established a CFP theorem for a pair of nonexpansive commuting mappings. Later, in 2010, Sokhuma et al. [12] applied the Ishikawa iterative scheme to a pair of both single and multi-valued nonexpansive mappings in Banach spaces. They utilized this modified Ishikawa iterative scheme as a method for approximating the CFP of the mappings.

$$\begin{cases} a_1 & \in \mathrm{B}_s \\ b_\ell & = (1-\upsilon_\ell)a_\ell + \upsilon_\ell u_\ell, \;\; \ell \in \mathbb{Z}^+ \\ a_{\ell+1} & = (1-\vartheta_\ell)a_\ell + \vartheta_\ell \mathcal{R} b_\ell, \end{cases} \tag{2. 2}$$

where $u_\ell \in \mathrm{S}a_\ell$ and $0 < r \leq \vartheta_\ell, \upsilon_\ell \leq w < 1$.

In 2016, Akkasriworn and Sokhuma [3] introduced the modified S-iterative scheme, which operates on a single and multi-valued nonexpansive mappings in Banach spaces called it the modified S-iterative scheme:

$$\begin{cases} p_1 & \in \mathrm{B}_s \\ q_\ell & = (1-\upsilon_\ell)p_\ell + \upsilon_\ell z_\ell, \;\; \ell \in \mathbb{Z}^+ \\ p_{\ell+1} & = (1-\vartheta_\ell)z_\ell + \vartheta_\ell \mathcal{R} q_\ell, \end{cases} \tag{2. 3}$$

where $z_\ell \in \mathrm{S}p_\ell$ and $0 < r \leq \vartheta_\ell, \upsilon_\ell < w \leq 1$.

Let B_s be a nonempty close convex subset of a UCBS Ω and $\mathcal{R} : \mathrm{B}_s \to \mathrm{B}_s$ a nonexpansive mapping. Thakur et al. [25] introduced the following three step iterative scheme starting from $a_1 \in \mathrm{B}_s$:

$$\begin{cases} a_1 & \in \mathrm{B}_s \\ c_\ell & = (1-\omega_\ell)a_\ell + \omega_\ell \mathcal{R} a_\ell, \\ b_\ell & = (1-\upsilon_\ell)c_\ell + \upsilon_\ell \mathcal{R} c_\ell, \;\; \ell \in \mathbb{Z}^+ \\ a_{\ell+1} & = (1-\vartheta_\ell)\mathcal{R} c_\ell + \vartheta_\ell \mathcal{R} b_\ell, \end{cases} \tag{2. 4}$$

where $\{\vartheta_\ell\}, \{\upsilon_\ell\}$ and $\{\omega_\ell\}$ are real sequences in $(0, 1)$.

Motivated by the works of [3] and [12], In this study, we present a modified version of the iterative scheme (2. 4) tailored for a pair of mappings, one single-valued and the other multivalued, both satisfying the condition (C_α). We consider a Banach space Ω that possesses uniform convexity and let B_s represent a nonempty closed and convex subset of Ω. The mappings $\mathcal{R} : \mathrm{B}_s \to \mathrm{B}_s$ and $\mathrm{S} : \mathrm{B}_s \to \mathcal{F}_{cb}(\mathrm{B}_s)$ correspond to the single and multi-valued mappings, respectively. It is worth noting that both mappings adhere to the condition (C_α). Our modified iterative scheme with initial value a_1 is then defined by

$$\begin{cases} a_1 & \in \mathrm{B}_s \\ c_\ell & = (1 - \omega_\ell)a_\ell + \omega_\ell u_\ell, \\ b_\ell & = (1 - \upsilon_\ell)c_\ell + \upsilon_\ell \mathcal{R}c_\ell, \quad \ell \in \mathbb{Z}^+ \\ a_{\ell+1} & = (1 - \vartheta_\ell)\mathcal{R}c_\ell + \vartheta_\ell \mathcal{R}b_\ell, \end{cases} \tag{2. 5}$$

where $u_\ell \in \mathrm{S}a_\ell$ and $\{\vartheta_\ell\}, \{\upsilon_\ell\}, \{\omega_\ell\}$ are sequences satisfying $0 < \vartheta_\ell, \upsilon_\ell, \omega_\ell < 1$.

We establish some convergence results in a nonempty convex subset of a UCBS. We compare the iterative scheme with some existing iterative schemes. We show that this iterative scheme converges faster than Mann, Iskikawa and S-iterative scheme which has been confirmed numerically and graphically. As an application, we approximate the solution of SCFP problem with the help of our iterative scheme.

Next, we have some definitions and results which are useful in proving the main results.

Lemma 2.1. [18] *Suppose tha Ω is a Banach space and $\mathcal{R}$ a mapping on a subset of Ω. Suppose that $\mathcal{R}$ satisfies condition (C_α). Then $\|a - \mathcal{R}b\| \leq \|a - b\| + \frac{(3+\alpha)}{(1-\alpha)}\|\mathcal{R}a - a\|$ holds for all $a, b \in \mathrm{B}_s$.*

In 1991, Sahu [21] established an important property of UCBS, which can be stated as follows.

Lemma 2.2. [21] *Assume that Ω is a UCBS and $\{z_\ell\}$ be a sequence in $(0, 1)$ for all $\ell \geq 1$. If $\{a_\ell\}$ and $\{b_\ell\}$ are in Ω such that $\limsup_{\ell\to\infty} \|a_\ell\| \leq m$, $\limsup_{\ell\to\infty} \|b_\ell\| \leq m$ and $\lim_{\ell\to\infty} \|(1 - z_\ell)a_\ell + z_\ell b_\ell\| = m$ for some $m \geq 0$. Then $\lim_{\ell\to\infty} \|a_\ell - b_\ell\| = 0$.*

The following will be used in comparing the convergence rate of iterative schemes.

Definition 2.3. [4] *Let $\{x_\ell\} \to x$ and $\{y_\ell\} \to y$ be the sequences of real numbers. If $\lim\limits_{\ell\to\infty} \frac{|x_\ell - x|}{|y_\ell - y|} = 0$, then $\{x_\ell\}$ converges faster than $\{y_\ell\}$.*

Definition 2.4. [4] *Assume that $\{v_\ell\}$ and $\{w_\ell\}$ are the sequences of fixed point iteration which converges to the fixed point p, the error estimates*

$$\|v_\ell - p\| \leq x_\ell \; \forall \; \ell \geq 1$$
$$\|w_\ell - p\| \leq y_\ell \; \forall \; \ell \geq 1$$

are available where the positive sequences $\{x_\ell\}, \{y_\ell\} \to 0$. If $\{x_\ell\}$ converges faster than $\{y_\ell\}$, then $\{v_\ell\}$ converges faster than $\{w_\ell\}$ to p.

Lemma 2.5. [3] *Suppose that B_s is a nonempty closed convex subset of a Banach space Ω. Then*

$$d(b, \mathrm{S}b) \leq \|b - a\| + d(a, \mathrm{S}a) + \mathrm{H}(\mathrm{S}a, \mathrm{S}b),$$

where $a, b \in \mathrm{B}_s$ *and* $\mathrm{S} : \mathrm{B}_s \to \mathcal{F}_{cb}(\mathrm{B}_s)$.

The demiclosedness principle is a fundamental principle that holds significance in the study of both asymptotic and ergodic behavior. In the setting of a Banach space Ω and its subset B_s, $\mathcal{R}$ is considered demiclosed if it satisfies the stated property: for any sequence a_ℓ in B_s such that $a_n \rightharpoonup a$ (weak convergence) and $\mathcal{R}(a_\ell) \to b$, it follows that $\mathcal{R}(a) = b$. In other words, when a sequence in $\mathrm{B}_s \rightharpoonup a$ and the image of corresponding sequence under $\mathcal{R}$ converges to b, the demiclosedness property guarantees that $\mathcal{R}(a)$ is equal to b.

Lemma 2.6. [5] *Assume a nonempty subset* B_s *of a UCBS* Ω *and* $\mathcal{R} : \mathrm{B}_s \to \mathrm{B}_s$ *is nonexpansive. If* $\{a_\ell\} \rightharpoonup \nu^*$ *in* B_s *and* $\{a_\ell - \mathcal{R}a_\ell\} \to 0$ *then,* $\nu^* \in \mathcal{F}_p(\mathcal{R})$.

Lemma 2.7. [28] *Suppose a sequence* $\{x_\ell\} \in \mathbb{R} \cup \{0\}$ *such that*

$$x_{\ell+1} \leq (1 - \wp_\ell)x_\ell + y_\ell + z_\ell$$

with $\{\wp_\ell\} \subset (0, 1), \{y_\ell\}$ *and* $\{z_\ell\} \subset \mathbb{R}$ *such that*

(a) $\sum\limits_{\ell=1}^{\infty} \wp_\ell = \infty$.

(b) $\sum\limits_{\ell=1}^{\infty} |y_\ell| < \infty$ *or* $\limsup\limits_{\ell\to\infty} \frac{y_\ell}{\wp_\ell} \leq 0$.

(c) $\sum\limits_{\ell=1}^{\infty} z_\ell < \infty$.

Then, $x_\ell \to 0$.

Lemma 2.8. [28] *Consider* ι, υ *in* Ω *where* Ω *is a real Banach space.Then, the following holds:*

- $\|\iota + \upsilon\|^2 \leq \|\iota\|^2 + 2\langle \upsilon, j(\iota + \upsilon)\rangle$, *for each* $j(\kappa + \upsilon) \in J(\kappa + \upsilon)$.
- $\|\iota + \upsilon\|^2 \geq \|\iota\|^2 + 2\langle \upsilon, j(\iota)\rangle$.

The value of $\mathcal{R} \in \Omega^*$ *at* $\iota \in \Omega$ *is denoted by* $\langle \iota, \mathcal{R}\rangle$, *where* Ω^* *be a dual space of* Ω. *Then the multivalued mapping* $J : \Omega \to 2^{\Omega^*}$ *defined by*

$$J(x) = \{\mathcal{R} \in \Omega^* : \langle \iota, \mathcal{R}\rangle = \|\iota\|^2 = \|\mathcal{R}\|^2\}$$

for all $\iota \in \Omega$, *is called the normalized duality mapping.*

Consider a space l^∞, which is a Banach space consists of bounded real sequences equipped with the supremum norm. Within this space, there exists a bounded linear functional ϖ which satisfies the following conditions:

a) For $\ell \in \mathbb{N}$ if $\zeta_\ell \in l^\infty$ and $\zeta_\ell \geq 0$, then $\varpi(\{t_\ell\}) \geq 0$;
b) For $\ell \in \mathbb{N}$ if $\zeta_\ell = 1$, then, $\varpi(\{t_\ell\}) = 1$;
c) $\varpi(\{\zeta_\ell\}) = \varpi(\{\zeta_{\ell+1}\})$ for all $\{\zeta_\ell\} \in l^\infty$.

The aforementioned functional ϖ is commonly referred to as a Banach limit. When evaluating ϖ at a sequence $\zeta_\ell \in l^\infty$, the resulting value is denoted as $\varpi_\ell\zeta_\ell$.

Proposition 2.9. [22] *Let* $x \in \mathbb{R}$ *and suppose* $(x_1, x_2, \ldots\ldots) \in l^\infty$ *such that* $\varpi_n(x_n) \leq x$ *for* ϖ *where* ϖ *is the Banach limit and* $\limsup\limits_{\ell\to\infty}(x_{\ell+1} - x_\ell) \leq 0$. *Then* $\limsup\limits_{\ell\to\infty} x_\ell \leq x$.

Lemma 2.10. [9] *Suppose that Ω is a Banach space with uniformly Gâteaux differentiable norm and B_s a nonempty closed convex subset of Ω. Let $\{x_\ell\}$ be a bounded sequence of Ω and $z \in K$ then*

$$\varpi_\ell \|x_\ell - z\|^2 = \min_{y \in K} \varpi_\ell \|x_\ell - y\|^2, \;\Leftrightarrow\; \varpi_\ell \big\langle x - z, j(x_\ell - z)\big\rangle \leq 0.$$

Lemma 2.11. [19] *Consider $\mathrm{B}_s \neq \emptyset$ a closed convex subset of UCBS Ω, and a bounded sequence $\{a_\ell\} \in \Omega$. The set B_{s_1} given as follows:*

$$\mathrm{B}_{s_1} = \big\{\varphi \in \mathrm{B}_s : \varpi_\ell \|a_\ell - \varphi\|^2 = \min_{\psi \in K} \varpi_\ell \|a_\ell - \psi\|^2\big\}$$

The set under consideration consists of precisely one point.

Lemma 2.12. [27] *Suppose Ω is a Banach space that is 2-uniformly smooth, with the best smoothness constant $\zeta > 0$. The following holds:*

$$\|\iota + \upsilon\|^2 \leq \|\iota\|^2 + 2\big\langle \upsilon, j(\iota)\big\rangle + 2\|\zeta\iota\|^2 \forall\, \iota, \upsilon, \in\; \Omega.$$

3. Main Results

First, we show that the iterative scheme (2. 5) converges faster than Modified S-iterative scheme (2. 3). Due to the faster rate of convergence exhibited by (2. 3) compared to existing iterative schemes such as Mann, Ishikawa, and others [3], it follows that our scheme is correspondingly faster than all of them.

Theorem 3.1. *Consider a nonempty closed convex subset B_s within a normed space Ω. Let $\mathcal{R}$ and S be single-valued and multivalued contractions with contraction constant $\wp \in (0, 1)$ having CFP ν^*. Let $\{a_\ell\}$ be defined by the iterative scheme (2. 5) and $\{p_\ell\}$ by (2. 3) where $\{\vartheta_\ell\}, \{\upsilon_\ell\}$ and $\{\omega_\ell\}$ are in $(0, 1)$ for all $\ell \in \mathbb{N}$. Then $\{a_\ell\}$ converges faster than $\{p_\ell\}$.*

Proof. As from the iterative scheme (2. 3),

$$\begin{aligned}
\|q_\ell - \nu^*\| =& \|(1 - \upsilon_\ell)p_\ell + \upsilon_\ell z_\ell - \nu^*\| \\
\leq& (1 - \upsilon_\ell)\|p_\ell - \nu^*\| + \upsilon_\ell \|z_\ell - \nu^*\| \\
\leq& (1 - \upsilon_\ell)\|p_\ell - \nu^*\| + \upsilon_\ell \mathrm{H}(\mathrm{S}p_\ell, \mathrm{S}\nu^*) \\
\leq& (1 - \upsilon_\ell)\|p_\ell - \nu^*\| + \upsilon_\ell \wp \|p_\ell - \nu^*\| \\
=& (1 - (1 - \wp)\upsilon_\ell)\|p_\ell - \nu^*\|,
\end{aligned}$$

and

$$\begin{aligned}
\|p_{\ell+1} - \nu^*\| =& \|(1 - \vartheta_\ell)z_\ell + \vartheta_\ell \mathcal{R}q_\ell - \nu^*\| \\
\leq& (1 - \vartheta_\ell)\|z_\ell - \nu^*\| + \vartheta_\ell \|\mathcal{R}q_\ell - \nu^*\| \\
\leq& (1 - \vartheta_\ell)\wp\|p_\ell - \nu^*\| + \vartheta_\ell \wp \|q_\ell - \nu^*\| \\
\leq& \wp(1 - \vartheta_\ell)\|p_\ell - \nu^*\| + \wp\vartheta_\ell(1 - (1 - \wp)\upsilon_\ell)\|p_\ell - \nu^*\| \\
=& \wp[1 - (1 - \wp)\vartheta_\ell \upsilon_\ell]\|p_\ell - \nu^*\|.
\end{aligned}$$

As, ϑ_ℓ and υ_ℓ are sequences in $(0,1)$, we can find a constant $\vartheta, \upsilon \in \mathbb{R}$ such that $\vartheta_\ell < \vartheta < 1$ and $\upsilon_\ell < \upsilon < 1$ for all $\ell \in \mathbb{N}$. So,

$$\begin{aligned}\|p_{\ell+1} - \nu^*\| \leq& \wp[1 - (1-\wp)\vartheta\upsilon]\|p_\ell - \nu^*\| \\ &. \\ &. \\ &. \\ \leq& \wp^\ell[1 - (1-\wp)\vartheta\upsilon]^\ell\|p_1 - \nu^*\|.\end{aligned}$$

which implies

$$\|p_{\ell+1} - \nu^*\| \leq \wp^\ell[1 - (1-\wp)\vartheta\upsilon]^\ell\|p_1 - \nu^*\| \quad \text{for all } \ell \in \mathbb{N}.$$

Let

$$y_\ell = \wp^\ell[1 - (1-\wp)\vartheta\upsilon]^\ell\|p_0 - \nu^*\|.$$

Now, from iterative scheme (2. 5), we have

$$\begin{aligned}\|c_\ell - \nu^*\| =& \|(1-\omega_\ell)a_\ell + \omega_\ell u_\ell - \nu^*\| \\ \leq& (1-\omega_\ell)\|a_\ell - \nu^*\| + \omega_\ell\|u_\ell - \nu^*\| \\ \leq& (1-\omega_\ell)\|a_\ell - \nu^*\| + \omega_\ell \mathrm{H}(\mathrm{S}a_\ell, \mathrm{S}\nu^*) \\ \leq& (1-\omega_\ell)\|a_\ell - \nu^*\| + \omega_\ell\wp\|a_\ell - \nu^*\| \\ =& (1 - (1-\wp)\omega_\ell)\|a_\ell - \nu^*\|,\end{aligned}$$

so that

$$\begin{aligned}\|b_\ell - \nu^*\| =& \|(1-\upsilon_\ell)c_\ell + \upsilon_\ell\mathcal{R}c_\ell - \nu^*\| \\ \leq& (1-\upsilon_\ell)\|c_\ell - \nu^*\| + \upsilon_\ell\|\mathcal{R}c_\ell - \nu^*\| \\ \leq& (1-\upsilon_\ell)\|c_\ell - \nu^*\| + \upsilon_\ell\wp\|c_\ell - \nu^*\| \\ =& (1 - (1-\wp)\upsilon_\ell)\|c_\ell - \nu^*\| \\ \leq& (1 - (1-\wp)\upsilon_\ell)(1 - (1-\wp)\omega_\ell)\|a_\ell - \nu^*\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|a_{\ell+1} - \nu^*\| =& \|(1-\vartheta_\ell)\mathcal{R}c_\ell + \vartheta_\ell\mathcal{R}b_\ell - \nu^*\| \\ \leq& (1-\vartheta_\ell)\|\mathcal{R}c_\ell - \nu^*\| + \vartheta_\ell\|\mathcal{R}b_\ell - \nu^*\| \\ \leq& (1-\vartheta_\ell)\wp\|c_\ell - \nu^*\| + \vartheta_\ell\wp\|b_\ell - \nu^*\| \\ \leq& \wp(1-\vartheta_\ell)(1 - (1-\wp)\omega_\ell)\|a_\ell - \nu^*\| + \\ & \wp\vartheta_\ell(1 - \upsilon_\ell(1-\wp)(1 - (1-\wp)\omega_\ell)\|a_\ell - \nu^*\| \\ =& \wp(1 - (1-\wp)\omega_\ell)[1 - \vartheta_\ell + \vartheta_\ell(1 - (1-\wp)\upsilon_\ell)]\|a_\ell - \nu^*\| \\ =& \wp(1 - (1-\wp)\omega_\ell)[1 - (1-\wp)\vartheta_\ell\upsilon_\ell)]\|a_\ell - \nu^*\| \\ =& \wp[1 - (1-\wp)\omega_\ell - (1-\wp)\vartheta_\ell\upsilon_\ell + (1-\wp)^2\vartheta_\ell\upsilon_\ell\omega_\ell]\|a_\ell - \nu^*\| \\ <& \wp[1 - (1-\wp)\omega_\ell - (1-\wp)\vartheta_\ell\upsilon_\ell\omega_\ell + (1-\wp)\vartheta_\ell\upsilon_\ell\omega_\ell]\|a_\ell - \nu^*\| \\ \leq& \wp(1 - (1-\wp)\omega_\ell)\|a_\ell - \nu^*\|\end{aligned}$$

As, ω_ℓ is a sequences in $(0,1)$, we can find a constant $\omega \in \mathbb{R}$ such that $\omega_\ell < \omega < 1$ and for all $\ell \in \mathbb{N}$. So,

$$\begin{aligned}\|a_{\ell+1} - \nu^*\| \leq & \wp(1-(1-\wp)\omega)\|a_\ell - \nu^*\| \\ & . \\ & . \\ & . \\ \leq & \wp^\ell[1-(1-\wp)\omega]^\ell\|a_1 - \nu^*\|.\end{aligned}$$

which implies

$$\|a_{\ell+1}\| \leq \wp^\ell[1-(1-\wp)\omega]^\ell\|a_1 - \nu^*\| \quad \text{for all } \ell \in \mathbb{N}.$$

Let

$$x_\ell = \wp^\ell[1-(1-\wp)\omega]^\ell\|a_1 - \nu^*\|.$$

Then

$$\begin{aligned}\frac{x_\ell}{y_\ell} = & \frac{\wp^\ell[1-(1-\wp)\omega]^\ell\|a_1 - \nu^*\|}{\wp^\ell[1-(1-\wp)\vartheta\upsilon]^\ell\|p_1 - \nu^*\|} \\ = & \frac{[1-(1-\wp)\omega]^\ell\|a_1 - \nu^*\|}{[1-(1-\wp)\vartheta\upsilon]^\ell\|p_1 - \nu^*\|} \\ & \to 0 \text{ as } \ell \to \infty.\end{aligned}$$

Consequently, by definition (2.3) and (2.4), $\{a_\ell\}$ converges faster than $\{p_\ell\}$. □

Lemma 3.2. *Suppose a UCBS Ω, and B_s is a nonempty compact convex subset of Ω. Consider a single-valued mappings $\mathcal{R} : \mathrm{B}_s \to \mathrm{B}_s$ and a multivalued mapping $\mathrm{S} : \mathrm{B}_s \to \mathcal{F}_{cb}(\mathrm{B}_s)$. These mappings satisfy the condition (C_α). Let $\nu^* \in \mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S})$, and let $\{a_\ell\}$ be the sequence defined by equation (2. 5). Then, $\lim_{\ell\to\infty} \|a_\ell - \nu^*\|$ exists, where $\nu^* \in \mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S})$.*

Proof. Let $a_1 \in \mathrm{B}_s$ and $\nu^* \in \mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S})$. Consider

$$\begin{aligned}\|a_{\ell+1} - \nu^*\| = & \|(1-\vartheta_\ell)\mathcal{R}c_\ell + \alpha_\ell \mathcal{R}b_\ell - \nu^*\| \\ \leq & (1-\vartheta_\ell)\|\mathcal{R}c_\ell - \nu^*\| + \vartheta_\ell\|\mathcal{R}b_\ell - \nu^*\|. \end{aligned} \tag{3. 6}$$

Now $\frac{1}{2}\|\nu^* - \mathcal{R}\nu^*\| = 0 \leq \|a_\ell - \nu^*\|$, therefore

$$\begin{aligned}\|\mathcal{R}c_\ell - \mathcal{R}\nu^*\| \leq & \alpha\|\nu^* - \mathcal{R}c_\ell\| + \alpha\|c_\ell - \mathcal{R}\nu^*\| + (1-2\alpha)\|c_\ell - \nu^*\| \\ \leq & \alpha(\|\nu^* - \mathcal{R}\nu^*\| + \|\mathcal{R}c_\ell - \mathcal{R}\nu^*\|) + \alpha(\|c_\ell - \nu^*\| + \|\nu^* - \mathcal{R}\nu^*\|) \\ & + (1-2\alpha)\|c_\ell - \nu^*\| \\ \leq & \alpha\|\mathcal{R}c_\ell - \mathcal{R}\nu^*\| + \alpha\|c_\ell - \nu^*\| + (1-2\alpha)\|c_\ell - \nu^*\|\end{aligned}$$

implies

$$\|\mathcal{R}c_\ell - \nu^*\| \leq \|c_\ell - \nu^*\|. \tag{3. 7}$$

Next,

$$\begin{aligned}\|c_\ell-\nu^*\| =&\|(1-\omega_\ell)a_\ell+\gamma_\ell u_\ell-\nu^*\|\\ \le&(1-\omega_\ell)\|a_\ell-\nu^*\|+\omega_\ell\|u_\ell-\nu^*\|\\ \le&(1-\omega_\ell)\|a_\ell-\nu^*\|+\gamma_\ell \mathrm{H}(\mathrm{S}a_\ell,\mathrm{S}\nu^*). \end{aligned} \tag{3. 8}$$

As $\frac{1}{2}d(\nu^*,\mathrm{S}\nu^*)=0\le\|a_\ell-\nu^*\|$, so

$$\begin{aligned}\mathrm{H}(\mathrm{S}a_\ell,\mathrm{S}\nu^*) \le&\alpha d(\nu^*,\mathrm{S}a_\ell)+\alpha d(a_\ell,\mathrm{S}\nu^*)+(1-2\alpha)\|a_\ell-\nu^*\|\\ \le&\alpha\{d(\nu^*,\mathrm{S}\nu^*)+d(\mathrm{S}a_\ell,\mathrm{S}\nu^*)\}+\alpha\{d(a_\ell,\nu^*)+d(\nu^*,\mathrm{S}\nu^*)\}\\ &+(1-2\alpha)\|a_\ell-\nu^*\|\\ \le&\alpha d(\mathrm{S}a_\ell,\mathrm{S}\nu^*)+\alpha d(a_\ell,\nu^*)+(1-2\alpha)\|a_\ell-\nu^*\|\\ \le&\alpha \mathrm{H}(\mathrm{S}a_\ell,\mathrm{S}\nu^*)+\alpha\|a_\ell-\nu^*\|+(1-2\alpha)\|a_\ell-\nu^*\|\end{aligned}$$

yields

$$\mathrm{H}(\mathrm{S}a_\ell,\mathrm{S}\nu^*)\le\|a_\ell-\nu^*\|.$$

Putting it in (3. 8), we get

$$\|c_\ell-\nu^*\|\le\|a_\ell-\nu^*\|.$$

Thus by (3. 7), we obtain

$$\|\mathcal{R}c_\ell-\nu^*\|\le\|a_\ell-\nu^*\|. \tag{3. 9}$$

Also,

$$\begin{aligned}\|\mathcal{R}b_\ell-\mathcal{R}\nu^*\| \le&\alpha\|\nu^*-\mathcal{R}b_\ell\|+\alpha\|b_\ell-\mathcal{R}\nu^*\|+(1-2\alpha)\|b_\ell-\nu^*\|\\ \le&\alpha(\|\nu^*-\mathcal{R}\nu^*\|+\|\mathcal{R}b_\ell-\mathcal{R}\nu^*\|)+\alpha(\|b_\ell-\nu^*\|+\|\nu^*-\mathcal{R}\nu^*\|)\\ &+(1-2\alpha)\|b_\ell-\nu^*\|\\ \le&\alpha\|\mathcal{R}b_\ell-\mathcal{R}\nu^*\|+\alpha\|b_\ell-\nu^*\|+(1-2\alpha)\|b_\ell-\nu^*\|\end{aligned}$$

gives

$$\|\mathcal{R}b_\ell-\nu^*\|\le\|b_\ell-\nu^*\|.$$

But

$$\begin{aligned}\|\mathcal{R}b_\ell-\nu^*\|\le&\|b_\ell-\nu^*\|\\ =&\|(1-\upsilon_\ell)c_\ell+\upsilon_\ell\mathcal{R}c_\ell-\nu^*\|\\ \le&(1-\upsilon_\ell)\|c_\ell-\nu^*\|+\upsilon_\ell\|\mathcal{R}c_\ell-\nu^*\|\\ \le&(1-\upsilon_\ell)\|a_\ell-\nu^*\|+\upsilon_\ell\|a_\ell-\nu^*\|\end{aligned}$$

implies

$$\|\mathcal{R}b_\ell-\nu^*\|\le\|a_\ell-\nu^*\|. \tag{3. 10}$$

Hence using (3. 10) and (3. 9) in (3. 6), we get

$$\|a_{\ell+1}-\nu^*\|\le\|a_\ell-\nu^*\|$$

which means that $\{\|a_\ell-\nu^*\|\}$ is a nonincreasing sequence of reals and hence $\lim_{\ell\to\infty}\|a_\ell-\nu^*\|$ exists. □

Lemma 3.3. *Suppose that* B_s, Ω, $\mathcal{R}$ *and* S *be as in Lemma 3.2. Suppose* $\{c_\ell\}$ *is defined in* (2. 5). *If* $0 < r \leq \vartheta_\ell, \upsilon_\ell, \omega_\ell \leq w < 1$ *then* $\lim_{\ell\to\infty} \|c_\ell - \mathcal{R}c_\ell\| = 0$.

Proof. Let $\nu^* \in \mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S})$. From Lemma 3.2, $\lim_{\ell\to\infty} \|a_\ell - \nu^*\|$ exists.
Suppose for some $h \geq 0$,

$$\lim_{\ell\to\infty} \|a_\ell - \nu^*\| = h \tag{3. 11}$$

From the inequalities in the previous lemma, we can have by taking lim sup as $\ell \to \infty$:

$$\limsup_{\ell\to\infty} \|b_\ell - \nu^*\| \leq \limsup_{\ell\to\infty} \|a_\ell - \nu^*\| = h \tag{3. 12}$$

$$\limsup_{\ell\to\infty} \|\mathcal{R}c_\ell - \nu^*\| \leq \limsup_{\ell\to\infty} \|a_\ell - \nu^*\| = h \tag{3. 13}$$

$$\limsup_{\ell\to\infty} \|c_\ell - \nu^*\| \leq \limsup_{\ell\to\infty} \|a_\ell - \nu^*\| = h. \tag{3. 14}$$

$$\limsup_{\ell\to\infty} \|u_\ell - \nu^*\| \leq \limsup_{\ell\to\infty} \|a_\ell - \nu^*\| = h. \tag{3. 15}$$

Further

$$\begin{aligned}\|a_{\ell+1} - \nu^*\| =& \|(1-\vartheta_\ell)\mathcal{R}c_\ell + \vartheta_\ell \mathcal{R}b_\ell - \nu^*\| \\ \leq& (1-\vartheta_\ell)\|\mathcal{R}c_\ell - \nu^*\| + \vartheta_\ell\|\mathcal{R}b_\ell - \nu^*\|.\end{aligned}$$

Thus from the inequalities $\|\mathcal{R}c_\ell - \nu^*\| \leq \|a_\ell - \nu^*\|$ and $\|\mathcal{R}b_\ell - \nu^*\| \leq \|b_\ell - \nu^*\|$ of the previous lemma, we can write

$$\|a_{\ell+1} - \nu^*\| \leq (1-\vartheta_\ell)\|a_\ell - \nu^*\| + \vartheta_\ell\|b_\ell - \nu^*\|. \tag{3. 16}$$

By adding and subtracting $\vartheta_\ell\|a_{\ell+1} - \nu^*\|$ on right side of (3. 16) and simplifying, we obtain

$$\begin{aligned}\vartheta_\ell\|a_{\ell+1} - \nu^*\| \leq& (1-\vartheta_\ell)\|a_\ell - \nu^*\| - (1-\vartheta_\ell)\|a_{\ell+1} - \nu^*\| + \vartheta_\ell\|b_\ell - \nu^*\| \\ \|a_{\ell+1} - \nu^*\| \leq& \frac{(1-\vartheta_\ell)}{\vartheta_\ell}\{\|a_\ell - \nu^*\| - \|a_{\ell+1} - \nu^*\|\} + \|b_\ell - \nu^*\| \\ \|a_{\ell+1} - \nu^*\| \leq& \frac{(1-r)}{r}\{\|a_\ell - \nu^*\| - \|a_{\ell+1} - \nu^*\|\} + \|b_\ell - \nu^*\|.\end{aligned}$$

Now taking lim inf as $\ell \to \infty$ of both sides, we reach at

$$\liminf_{\ell\to\infty} \|a_{\ell+1} - \nu^*\| \leq \liminf_{\ell\to\infty} \|b_\ell - \nu^*\|.$$

As $\lim_{\ell\to\infty} \|a_{\ell+1} - \nu^*\| = h$, so

$$h \leq \liminf_{\ell\to\infty} \|b_\ell - \nu^*\|. \tag{3. 17}$$

And then (3. 17) and (3. 12) give us

$$\begin{aligned}h =& \lim_{\ell\to} \|b_\ell - \nu^*\| \\ =& \lim_{\ell\to} \|(1-\upsilon_\ell)c_\ell + \upsilon_\ell\mathcal{R}c_\ell - \nu^*\| \\ =& \lim_{\ell\to} \|(1-\upsilon_\ell)c_\ell + \upsilon_\ell\mathcal{R}c_\ell - (1-\upsilon_\ell)\nu^* - \upsilon_\ell\nu^*\| \\ =& \lim_{\ell\to} \|(1-\upsilon_\ell)(c_\ell - \nu^*) + \upsilon_\ell(\mathcal{R}c_\ell - \nu^*)\|.\end{aligned} \tag{3. 18}$$

From (3. 18), (3. 14), (3. 13) and by applying Lemma (2.2), we get

$$\lim_{\ell\to\infty} \|c_\ell - \mathcal{R}c_\ell\| = 0.$$

□

Lemma 3.4. *Suppose that* B_s, Ω, $\mathcal{R}$ *and* S *be as in Lemma 3.2. Suppose* $\{c_\ell\}$ *is defined in* (2. 5)*. Suppose* $\{a_\ell\}$ *is a sequence defined in* (2. 5)*. If* $0 < r \leq \vartheta_\ell, \upsilon_\ell, \omega_\ell \leq w < 1$*, then* $\lim\limits_{\ell\to\infty} \|a_\ell - u_\ell\| = 0$*, where* $u_\ell \in \mathrm{S}a_\ell$*.*

Proof. Let $\nu^* \in \mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S})$. From (3. 11), we have $\lim_{\ell\to\infty} \|a_\ell - \nu^*\| = h$. Also, from (3. 15), we have

$$\limsup_{\ell\to\infty} \|u_\ell - \nu^*\| \leq \limsup_{\ell\to\infty} \|a_\ell - \nu^*\| = h.$$

Further,

$$\begin{aligned} h &= \lim_{\ell\to\infty} \|a_{\ell+1} - \nu^*\| \\ &= \lim_{\ell\to\infty} \|(1-\vartheta_\ell)a_\ell + \vartheta_\ell u_\ell - \nu^*\| \\ &= \lim_{\ell\to\infty} \|(1-\vartheta_\ell)a_\ell + \vartheta_\ell u_\ell - (1-\vartheta_\ell)\nu^* - \vartheta_\ell \nu^*\| \\ &= \lim_{\ell\to\infty} \|(1-\vartheta_\ell)(a_\ell - \nu^*) + \vartheta_\ell(u_\ell - \nu^*)\|. \end{aligned}$$

So by applying Lemma (2.2), we get

$$\lim_{\ell\to\infty} \|a_\ell - u_\ell\| = 0.$$

□

Lemma 3.5. *Suppose that* B_s, Ω, $\mathcal{R}$ *and* S *be as in Lemma 3.2. Suppose* $\{c_\ell\}$ *is defined in* (2. 5)*. Suppose* $\{a_\ell\}$ *is a sequence defined in* (2. 5)*. If* $0 < r \leq \vartheta_\ell, \upsilon_\ell, \omega_\ell \leq w < 1$*, then* $\lim\limits_{\ell\to\infty} \|\mathcal{R}a_\ell - a_\ell\| = 0$*.*

Proof. By Lemma 2.1, we have

$$\begin{aligned} \|\mathcal{R}a_\ell - a_\ell\| &\leq \|\mathcal{R}a_\ell - c_\ell\| + \|c_\ell - a_\ell\| \\ &\leq \frac{(3+\alpha)}{(1-\alpha)}\|c_\ell - \mathcal{R}c_\ell\| + \|c_\ell - a_\ell\| + \|c_\ell - a_\ell\| \\ &= \frac{(3+\alpha)}{(1-\alpha)}\|c_\ell - \mathcal{R}c_\ell\| + 2\|(1-\omega_\ell)a_\ell + \omega_\ell u_\ell - a_\ell\| \\ &= \frac{(3+\alpha)}{(1-\alpha)}\|c_\ell - \mathcal{R}c_\ell\| + 2\|\omega_\ell a_\ell - \omega_\ell u_\ell\| \\ &= \frac{(3+\alpha)}{(1-\alpha)}\|c_\ell - \mathcal{R}c_\ell\| + 2\omega_\ell\|a_\ell - u_\ell\| \end{aligned}$$

Taking limit as $\ell \to \infty$ on both sides,

$$\lim_{\ell\to\infty} \|\mathcal{R}a_\ell - a_\ell\| \leq \frac{(3+\alpha)}{(1-\alpha)} \lim_{\ell\to\infty} \|c_\ell - \mathcal{R}c_\ell\| + 2 \lim_{\ell\to\infty} \|a_\ell - u_\ell\|.$$

Hence Lemma 3.3 and Lemma 3.4 give us

$$\lim_{\ell\to\infty}\|\mathcal{R}a_\ell - a_\ell\| = 0.$$

□

Here, we present a definition which is useful in proving our next result.

Definition 3.6. *Let $\mathcal{A}$ be a subset of Ω. Then $\mathcal{A}$ is said to be asymptotically closed if for $a \in \Omega$ and every $\epsilon > 0$,*

$$d(a,\mathcal{A}) < \epsilon \text{ implies } a \in \mathcal{A}.$$

Remark 3.7. *Every closed set is asymptotically closed but converse is not true.*

Example 3.8. *Let $\mathcal{A} = [0,1) \subset [0,1] = \Omega$. Take $a = 0$ then $d(0,[0,1)) = 0 < \epsilon$. Obviously, $0 \in [0,1)$. Thus $\mathcal{A}$ is not closed but asymptotically closed.*

Let $\mathcal{A} = [0,1] \cap \mathbb{Q} \subset \mathbb{R} = \Omega$. Then, it is not closed but asymptotically closed.

In the forthcoming discussion, we provide various sufficient conditions that ensure the existence of CFP for a pair of mappings, where one mapping is single-valued and the other is multivalued. These mappings satisfy the Condition (C_α), which allows us to establish the existence of CFP.

Theorem 3.9. *Consider a UCBS Ω, and B_s be a nonempty compact convex subset of Ω. Suppose we have a pair of mappings: a single-valued mapping $\mathcal{R} : \mathrm{B}_s \to \mathrm{B}_s$ and a multivalued mapping $\mathrm{S} : \mathrm{B}_s \to \mathcal{F}_{cb}(\mathrm{B}_s)$, both satisfying the condition (C_α). Suppose $\mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S}) \neq \emptyset$. Let $\{a_\ell\}$ be a sequence defined as in (2. 5), where $0 < r \leq \vartheta_\ell, \upsilon_\ell, \omega_\ell \leq w < 1$. If $\{a_{\ell_i}\}$ a subsequence of $\{a_\ell\}$ such that $\{a_{\ell_i}\} \to p$, then it follows that $\mathrm{z} \in \mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S})$.*

Proof. From Lemma 3.5, we have

$$0 = \lim_{\ell\to\infty}\|\mathcal{R}a_{\ell_i} - a_{\ell_i}\| = \lim_{\ell\to\infty}\|(I-\mathcal{R})(a_{\ell_i})\|.$$

Since $(I-\mathcal{R})$ is demiclosed at 0, we have $(I-\mathcal{R})\mathrm{z} = 0$ and hence $\mathrm{z} = \mathcal{R}\mathrm{z}$ i.e., $\mathrm{z} \in \mathcal{F}_p(\mathcal{R})$. Note that $u_\ell \in \mathrm{S}a_\ell$ and $a_\ell \to \mathrm{z}$ as $\ell \to \infty$, so that by Lemma 3.4, we have $d(a_\ell, u_\ell) \to 0$ as $\ell \to \infty$.

Now, our claim is that $u_\ell \to \mathrm{z}$ as $\ell \to \infty$.

$$d(u_\ell,\mathrm{z}) \leq d(u_\ell, a_\ell) + d(a_\ell,\mathrm{z})$$

as $\ell \to \infty$ implies $d(u_\ell,\mathrm{z}) \to 0$ and we are home.

Next, our claim is that $\mathrm{z} \in \mathrm{S}a_\ell$ for sufficiently large ℓ.

For this

$$d(\mathrm{z},\mathrm{S}a_\ell) \leq d(\mathrm{z}, u_\ell) + d(u_\ell, a_\ell).$$

On taking $\ell \to \infty$, we have

$$\lim_{\ell\to\infty} d(\mathrm{z},\mathrm{S}a_\ell) = 0.$$

Thus, for every $\epsilon > 0$ there exists ℓ_0 such that for all $\ell \geq \ell_0$, we have

$$d(\mathrm{z},\mathrm{S}a_\ell) < \epsilon.$$

This implies that $\mathrm{z} \in \mathrm{S}a_\ell$ for sufficiently large ℓ.
Now, $\frac{1}{2}d(a_\ell, \mathrm{S}a_\ell) \leq \|a_\ell - \mathrm{z}\|$, for sufficiently large value of ℓ, hence

$$\begin{aligned}
\mathrm{H}(\mathrm{S}a_\ell, \mathrm{Sz}) \leq& \alpha d(\mathrm{z}, \mathrm{S}a_\ell) + \alpha d(a_\ell, \mathrm{Sz}) + (1-2\alpha)\|a_\ell - \mathrm{z}\| \\
\leq& \alpha\{d(\mathrm{z}, \mathrm{Sz}) + \mathrm{H}(\mathrm{S}a_\ell, \mathrm{Sz})\} + \alpha\{\|a_\ell - \mathrm{z}\| + d(\mathrm{z}, \mathrm{Sz})\} \\
&+ (1-2\alpha)\|a_\ell - \mathrm{z}\| \\
\leq& \alpha \mathrm{H}(\mathrm{S}a_\ell, \mathrm{Sz}) + (1-\alpha)\|a_\ell - \mathrm{z}\| + 2\alpha d(\mathrm{z}, \mathrm{Sz}) \\
(1-\alpha)\mathrm{H}(\mathrm{S}a_\ell, \mathrm{Sz}) \leq& (1-\alpha)\|a_\ell - \mathrm{z}\| + 2\alpha d(\mathrm{z}, \mathrm{Sz}) \\
\mathrm{H}(\mathrm{S}a_\ell, \mathrm{Sz}) \leq& \|a_\ell - \mathrm{z}\| + \big(\frac{2\alpha}{1-\alpha}\big) d(\mathrm{z}, \mathrm{Sz}).
\end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned}
d(\mathrm{z}, \mathrm{Sz}) \leq& \|\mathrm{z} - a_{\ell_i}\| + d(a_{\ell_i}, \mathrm{S}a_{\ell_i}) + \mathrm{H}(\mathrm{S}a_{\ell_i}, \mathrm{Sz}) \\
\leq& |\mathrm{z} - a_{\ell_i}\| + \|a_{\ell_i} - u_{\ell_i}\| + \|a_{\ell_i} - \mathrm{z}\| + \big(\frac{2\alpha}{1-\alpha}\big) d(\mathrm{z}, \mathrm{Sz}) \\
d(\mathrm{z}, \mathrm{Sz}) - \big(\frac{2\alpha}{1-\alpha}\big) d(\mathrm{z}, \mathrm{Sz}) \leq& |\mathrm{z} - a_{\ell_i}\| + \|a_{\ell_i} - u_{\ell_i}\| + \|a_{\ell_i} - \mathrm{z}\| \\
(\frac{1-3\alpha}{1-\alpha}) d(\mathrm{z}, \mathrm{Sz}) \leq& |\mathrm{z} - a_{\ell_i}\| + \|a_{\ell_i} - u_{\ell_i}\| + \|a_{\ell_i} - \mathrm{z}\| \to 0 \; as \; i \; \to \infty.
\end{aligned}$$

It shows that $\mathrm{z} \in \mathcal{F}_p(\mathrm{S})$.
Hence $\mathrm{z} \in \mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S})$. □

We conclude this section by proving a strong convergence theorem using our iterative scheme (2. 5) as follows.

Theorem 3.10. *Consider the setting described in Theorem* 3.9. *Suppose* $\{a_\ell\}$ *is a sequence as defined in* (2. 5). *Assuming that* $0 < r \leq \vartheta_\ell, \upsilon_\ell, \omega_\ell \leq w < 1$, *it can be concluded that* $\{a_\ell\} \to \mathrm{z}$ *where* z *is CFP of* $\mathcal{R}$ *&* S.

Proof. As $\{a_\ell\}$ is a sequence in the compact set B_s, there exists a subsequence $\{a_{\ell_i}\}$ of $\{a_\ell\}$ that converges strongly to a point $\mathrm{z} \in \mathrm{B}_s$, i.e.,

$$\lim_{i\to\infty} \|a_{\ell_i} - \mathrm{z}\| = 0.$$

By Theorem 3.9, we have $\mathrm{z} \in \mathcal{F}_p(\mathcal{R}) \cap \mathcal{F}_p(\mathrm{S})$ and by Lemma 3.2, $\lim_{\ell\to\infty} \|a_\ell - \mathrm{z}\|$ exists. It must be the case that

$$\lim_{\ell\to\infty} \|a_\ell - \mathrm{z}\| = \lim_{i\to\infty} \|a_{\ell_i} - \mathrm{z}\| = 0.$$

Therefore $\{a_\ell\} \to \mathrm{z}$ which is a CFP of both $\mathcal{R}$ & S. □

4. Numerical Ananlysis

Now, we give an example which holds for both single and multi-valued mappings satisfying Condition (C_α) in usual Banach spaces. Let $\mathrm{B}_s = [0, 4]$ and $\mathcal{R} : \mathrm{B}_s \to \mathrm{B}_s$, $\mathrm{S} : \mathrm{B}_s \to \mathcal{F}_{cb}(\mathrm{B}_s)$ be the mappings defined as

$$\mathcal{R}(a) = \begin{cases} \frac{4a+2}{5}, & \text{if } 0 \leq a \leq 2 \\ 2, & \text{if } 2 < a \leq 4 \end{cases}$$

and

$$\mathrm{S}(a) = \begin{cases} [0, \frac{3a+4}{5}], & \text{if } 0 \leq a \leq 2 \\ \{2\}, & \text{if } 2 < a \leq 4. \end{cases}$$

To show that $\mathcal{R}$ is single-valued generalized α-nonexpansive mapping, we consider the following three cases:
Case:1 Let $a \in [0, 2]$, $b \in (2, 4]$ and $\alpha = \frac{1}{3}$.

$$\begin{aligned} \alpha|b - \mathcal{R}(a)| + \alpha|a - \mathcal{R}(b)| + (1-2\alpha)|a-b| =& \frac{1}{3}|b - (\frac{4a+2}{5})| + \frac{1}{3}|a-2| + \frac{1}{3}|a-b| \\ \geq& \frac{1}{3}|(b - (\frac{4a+2}{5})) - (a-2)| + \frac{1}{3}|a-b| \\ \geq& \frac{1}{3}|\frac{9a}{5} - b + \frac{8}{5}| + \frac{1}{3}|a-b| \\ \geq& \frac{1}{3}|\frac{9a}{5} - a - b + b + \frac{8}{5}| \\ \geq& \frac{1}{3}|\frac{9a-5a}{5} + \frac{8}{5}| \\ \geq& \frac{1}{3}|\frac{4a+8}{5}| \\ =& \frac{4}{15}|a+2| \\ \geq& \frac{4}{5}|a-2| = |\mathcal{R}a - \mathcal{R}b| \end{aligned}$$

Case:2 Let $a, b \in [0, 2]$, and $\alpha = \frac{1}{3}$.

$$\begin{aligned} \alpha|b - \mathcal{R}(a)| + \alpha|a - \mathcal{R}(b)| + (1-2\alpha)|a-b| =& \frac{1}{3}|b - (\frac{4a+2}{5})| + \frac{1}{3}|a - (\frac{4b+2}{5})| + \frac{1}{3}|a-b| \\ \geq& \frac{1}{3}|(b - (\frac{4a+2}{5})) - ((a - (\frac{4b+2}{5})))| + \frac{1}{3}|a-b| \\ \geq& \frac{1}{3}|\frac{4a}{5} - a - \frac{4b}{5} - b| + \frac{1}{3}|a-b| \\ \geq& \frac{1}{3}|\frac{9a}{5} - \frac{9b}{5}| + \frac{1}{3}|a-b| \\ \geq& \frac{3}{5}|a-b| + \frac{1}{3}|a-b| \\ =& \frac{14}{15}|a-b| \\ \geq& \frac{4}{5}|a-b| = |\mathcal{R}a - \mathcal{R}b| \end{aligned}$$

Case:3 The case $a, b \in (2, 4]$ is trivial.

Hence $\mathcal{R}$ is generalized $\frac{1}{3}$-nonexpansive mapping.

Next, we have to show that S is multivalued generalized α-nonexpansive mapping. Here also we have considered the following cases:

Case:1 Let $a \in [0,2], b \in (2,4]$ and $\alpha = \frac{1}{3}$.

$$\begin{aligned}\alpha d(b,\mathrm{S}(a)) + \alpha d(a,\mathrm{S}(b)) + (1-2\alpha)d(a,b) =& \frac{1}{3}|b-(\frac{3a+4}{5})| + \frac{1}{3}|a-2| + \frac{1}{3}|a-b| \\ \geq& \frac{1}{3}|(b-(\frac{3a+4}{5})) - (a-2) - (a-b)| \\ \geq& \frac{1}{3}|(b-\frac{3a}{5}-\frac{4}{5}-a+2-a+b| \\ =& \frac{1}{3}|\frac{-3a}{5}-2a+2b-\frac{4}{5}+2| \\ \geq& \frac{1}{3}|\frac{13a}{5}-2b-\frac{6}{5}| \\ \geq& \frac{3}{5}|a-2| = \mathrm{H}(\mathrm{S}a,\mathrm{S}b)\end{aligned}$$

Case:2 Let $a, b \in [0,2]$ and $\alpha = \frac{1}{3}$.

$$\begin{aligned}\alpha d(b,\mathrm{S}(a)) + \alpha d(a,\mathrm{S}(b)) + (1-2\alpha)d(a,b) =& \frac{1}{3}|b-(\frac{3a+4}{5})| + \frac{1}{3}|a-(\frac{3b+4}{5})| + \frac{1}{3}|a-b| \\ \geq& \frac{1}{3}|-\frac{3a}{5}-2a+\frac{3b}{5}+2b| \\ \geq& \frac{13}{15}|a-b| \\ \geq& \frac{3}{5}|a-b| = \mathrm{H}(\mathrm{S}a,\mathrm{S}(b))\end{aligned}$$

Case:3 The case $a, b \in (2,4]$ is trivial.

Thus S is multivalued $\frac{1}{3}$-nonexpansive mapping.

It is clear that $\nu^* = 2$ is a CFP of $\mathcal{R}$ and S. Suppose $\{a_\ell\}$ is a sequence as defined in (2. 5).We not only prove that $\{a_\ell\}$ converges strongly to the CFP $\nu^* = 2$ of $\mathcal{R}$ and S but also faster than Mann, Ishikawa and S-iterative schemes.

Now, we perform the experiments to determine and compare the convergence behaviour of the iterative scheme (2. 5) with others. In the Table (1), we test the convergence analysis of different iterative schemes with control sequences $\vartheta_\ell = \sqrt{\frac{\ell}{3\ell+3}}$, $\upsilon_\ell = \sqrt{\frac{\ell+2}{\ell^2+5}}$, $\omega_\ell = \sqrt{\frac{\ell+1}{\ell+5}}$ with initial guess $a_1 = 3$. Figure (1) shows graphical convergence of the iterative schemes. We see that our fixed point iterative scheme (2. 5) converges to the fixed point $\nu^* = 2$ faster than Mann, Ishikawa and S-iterative schemes.

5. APPLICATIONS TO SPLIT COMMON FIXED POINT PROBLEM

In their notable work published in 1994 [6], Censor and Elfving introduced the concept of the split feasibility problem (SPF). This problem can be formulated in the following manner:

Consider $\mathcal{P}$ and $\mathcal{Q}$ which is nonempty closed convex subset of two finite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Additionally, suppose that $\Upsilon : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. The aims of SPF to find an element

$$\nu^* \in \mathcal{P} \text{ such that } \Upsilon(\nu^*) \in \mathcal{Q}. \tag{5. 19}$$

Steps	our process	S-Iteration	Ishikawa	Mann
a_1	3.000000	3.000000	3.000000	3.000000
a_2	2.506392	2.650545	2.830353	2.779807
a_3	2.256433	2.423209	2.689485	2.608099
a_4	2.129856	2.275317	2.572516	2.474200
a_5	2.065758	2.179106	2.475390	2.369784
...	....	....	....	....
...	....	....	....	....
...	....	....	....	....
a_{10}	2.002190	2.020869	2.187656	2.106631
...	....	....	....	....
...	....	....	....	....
...	....	....	....	....
a_{15}	2.000073	2.002432	2.074076	2.030748
...	....	....	....	....
...	....	....	....	....
...	....	....	....	....
a_{21}	2.000001	2.000184	2.024280	2.006914
a_{22}	2.000000	2.000120	2.020161	2.005392

TABLE 1. Convergence behavior of Iterative Schemes

The solution set of (5. 19) is as

$$\Omega = \nu^* \in \mathcal{P} : \Upsilon(\nu^*) \in \mathcal{Q} = \mathcal{P} \cap \Upsilon^{-1}(\mathcal{Q}). \tag{5. 20}$$

The split feasibility problem (SPF) has found numerous applications and has been extended from finite to infinite-dimensional Hilbert spaces (cf. [13] and references therein). Censor & Segal [7] introduced the concept of the split common fixed point problem (SCFP) as a generalization of the SPF and the convex feasibility problem (CFP). They presented different schemes to solve the SCFP problem. Zhao and He [31] developed the viscosity approximation algorithms tailored for the SCFP, focusing on quasi-nonexpansive mappings $\mathcal{R}$ with the property that $I - \mathcal{R}$ is demiclosed at 0. Moudafi [17] presented a modified algorithm and provided an alternative, simpler proof for the results obtained by Zhao. Schöpfer et al. [20], Takahashi & Yao [23] and Tang et al. [24] solve SPF via different algorithms in the setting of Banach spaces.

Suppose that Ω_1, Ω_2 be the real Banach space and $\Upsilon : \Omega_1 \to \Omega_2$, be a bounded linear operator along with its adjoint $\Upsilon^* : \Omega_2 \to \Omega_1$. Additionally, we have a multivalued mapping $\mathrm{S} : \Omega_1 \to \mathcal{F}_{cb}(\Omega_1)$ satisfying condition (2. 1), and a single-valued mapping $\mathcal{R} : \Omega_2 \to \Omega_2$ satisfying the (C_α) condition.

We will address the SCFP and present a iterative scheme to solve it. The SCFP can be formulated as follows:

$$\text{Find } \nu^* \in \mathcal{F}_p(\mathrm{S}) \text{ such that } \Upsilon(\nu^*) \in \mathcal{F}_p(\mathcal{R}). \tag{5. 21}$$

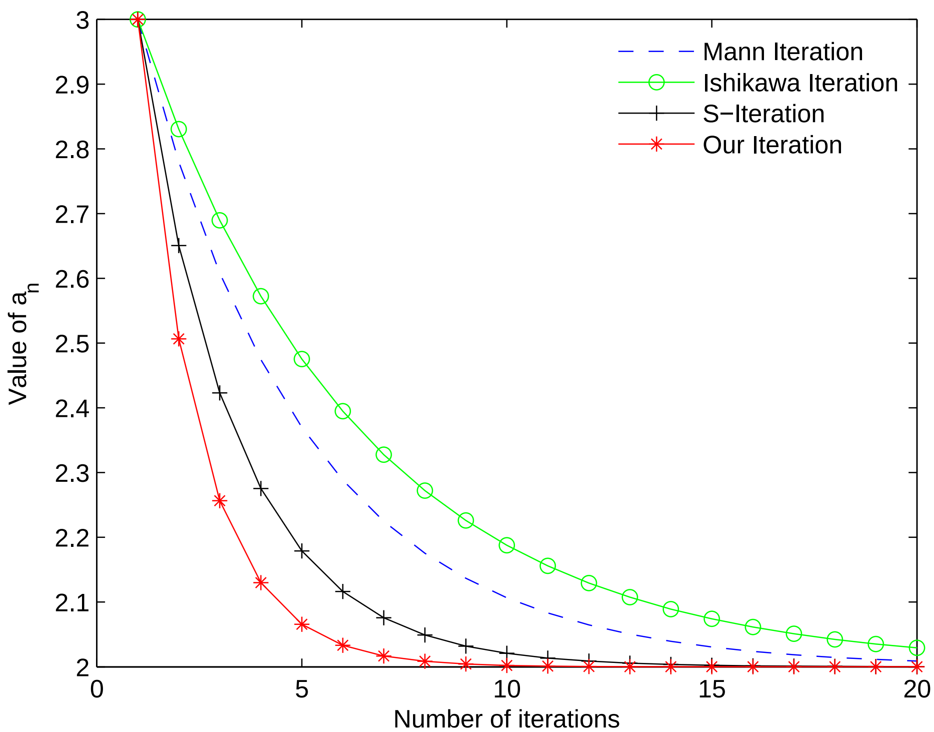

FIGURE 1. Convergence behaviour of our iteration scheme with S, Ishikawa and Mann Iterative schemes

Throughout the rest of the paper, we will denote the set of solutions of the SCFP (5. 21) as Γ, which can be defined as follows:

$$\Gamma = \{a \in \mathcal{F}_p(\mathrm{S}) : \Upsilon(a) \in \mathcal{F}_p(\mathcal{R})\} = \mathcal{F}_p(\mathrm{S}) \cap \Upsilon^{-1}(\mathcal{F}_p(\mathcal{R})).$$

Theorem 5.1. *Consider the previously defined Ω_1, Ω_2, Υ, Υ^*, S, and $\mathcal{R}$. Let $\breve{\rho}$ is a contraction on Ω_1. For any $a_0 \in \Omega_1$, the sequence $\{a_\ell\}$ is defined as:*

$$\begin{cases} d_\ell = & a_\ell + \sigma_\ell \Upsilon^*(\mathcal{R} - I)\Upsilon(a_\ell) \\ c_\ell & = (1-\omega_\ell) d_\ell + \omega_\ell u_\ell \\ b_\ell & = (1-\upsilon_\ell) c_\ell + \upsilon_\ell \breve{\rho} c_\ell \\ a_{\ell+1} & = (1-\vartheta_\ell)\breve{\rho} c_\ell + \vartheta_\ell \breve{\rho} b_\ell, \end{cases} \tag{5. 22}$$

for all $\ell \in \mathbb{N}$, where $\vartheta_\ell, \upsilon_\ell, \omega_\ell \subset (0,1)$ & $\sigma_\ell > 0$ satisfying the assumptions given below:

A_1. *$\vartheta_\ell \to 0$ as $\ell \to \infty$,*

A_2. $\sum\limits_{\ell=0}^{\infty} \vartheta_\ell = \infty$,

A_3. *either* $\sum_{\ell=0}^{\infty} |\vartheta_{\ell+1} - \vartheta_\ell| < \infty$ *or* $\lim_{\ell\to\infty} \frac{\vartheta_{\ell+1}}{\vartheta_\ell} = 1$;

A_4. $\sigma_\ell \in \left(0, \frac{1-2\zeta^2}{\|\Upsilon\|^2}\right)$ *such that* $\sum_{\ell=0}^{\infty} \sigma_\ell < \infty$.

If the solution set of SCFP Γ *in (5. 21) is nonempty, then the sequence* $\{a_\ell\} \to \nu^* \in \Gamma$.

Proof. We will break down the proof into five distinct steps.

Step I. The sequence $\{a_\ell\}$ is bounded.

Indeed, for any $\nu^* \in \Gamma$, it follows that $\Upsilon(\nu^*) \in \mathcal{Q} = \mathcal{F}_p(\mathcal{R})$. Since S satisfies condition (2. 1) and $\mathcal{F}_p(\mathrm{S})$ is closed. Let $\nu^* \in \mathcal{F}_p(\mathrm{S})$. Using (5. 22) & Lemma 2.12, we can deduce the following:

$$\begin{aligned}
\|d_\ell - \nu^*\|^2 =& \|a_\ell + \sigma_\ell \Upsilon^*(\mathcal{R} - I)\Upsilon(a_\ell) - \nu^*\|^2 \\
\le& \|\sigma_\ell \Upsilon^*(\mathcal{R} - I)A(a_\ell)\|^2 + 2\big\langle a_\ell - \nu^*, j(\sigma_\ell \Upsilon^*(\mathcal{R} - I)A(a_\ell)\big\rangle + \\
& 2\zeta^2\|a_\ell - \nu^*\|^2 \\
\le& \sigma_\ell^2\|\Upsilon\|^2\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 + 2\zeta^2\|a_\ell - \nu^*\|^2 + \\
& 2\sigma_\ell\big\langle \Upsilon a_\ell - \mathcal{R}(\Upsilon(a_\ell)) + \mathcal{R}(\Upsilon(a_\ell)) - \mathcal{R}(\Upsilon(\nu^*)), j((\mathcal{R} - I)A(a_\ell)\big\rangle \\
\le& \sigma_\ell^2\|\Upsilon\|^2\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 + 2\zeta^2\|a_\ell - \nu^*\|^2 - 2\sigma_\ell\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 \\
& + 2\sigma_\ell\big\langle \mathcal{R}(\Upsilon(a_\ell)) - \mathcal{R}(\Upsilon(\nu^*)), j((\mathcal{R} - I)A(a_\ell)\big\rangle \\
\le& \sigma_\ell^2\|\Upsilon\|^2\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 + 2\zeta^2\|a_\ell - \nu^*\|^2 - 2\sigma_\ell\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 \\
& + \sigma_\ell^2\|\mathcal{R}(\Upsilon(a_\ell)) - \mathcal{R}(\Upsilon(\nu^*))\|^2 \\
\le& \sigma_\ell^2\|\Upsilon\|^2\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 + 2\zeta^2\|a_\ell - \nu^*\|^2 - 2\sigma_\ell\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 \\
& \sigma_\ell\big(\|\mathcal{R}(\Upsilon(a_\ell)) - \mathcal{R}(\Upsilon(\nu^*))\|^2 + \|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2\big).
\end{aligned}$$

Since, by condition (C_α), we have

$$\begin{aligned}
\frac{1}{2}\|\Upsilon(\nu^*) - \mathcal{R}(\Upsilon(\nu^*))\| \le& \|\Upsilon(\nu^*) - \mathcal{R}(\Upsilon(a_\ell))\| \quad \text{implies} \\
\|\mathcal{R}(\Upsilon(a_\ell)) - \mathcal{R}(\Upsilon(\nu^*))\| \le& \alpha\|\Upsilon(a_\ell) - \mathcal{R}(\Upsilon(\nu^*))\| + \alpha\|\Upsilon(\nu^*) - \mathcal{R}(\Upsilon(a_\ell))\| + \\
& (1 - 2\alpha)\|\Upsilon(a_\ell) - \Upsilon(\nu^*)\| \\
\le& \|\Upsilon(a_\ell) - \Upsilon(\nu^*)\|.
\end{aligned}$$

So,

$$\begin{aligned}
\|d_\ell - \nu^*\|^2 \le& \sigma_\ell^2\|\Upsilon\|^2\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 + 2\zeta^2\|a_\ell - \nu^*\|^2 - 2\sigma_\ell\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 \\
& \sigma_\ell\big(\|\Upsilon(a_\ell) - \Upsilon(\nu^*)\|^2 + \|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2\big). \\
\le& \big(2\zeta^2 + \sigma_\ell\|\Upsilon\|^2\big)\|a_\ell - \nu^*\|^2 - \sigma_\ell\big(1 - \sigma_\ell\|\Upsilon\|^2\big)\|(\mathcal{R} - I)\Upsilon(a_\ell)\|^2 \\
\le& \|a_\ell - \nu^*\|.
\end{aligned}$$

As, from Lemma 3.3, we obtain

$$\|c_\ell - \nu^*\| \le \|a_\ell - \nu^*\|$$

$$\begin{aligned}\|b_\ell - \nu^*\| =& \|(1-\upsilon_\ell c_\ell + \upsilon_\ell \breve{\rho} c_\ell - \nu^*\| \\ \leq& (1-\upsilon_\ell)\|c_\ell - \nu^*\| + \upsilon_\ell \wp \|c_\ell - \nu^*\| \\ \leq& (1-\upsilon_\ell(1-\wp))\|c_\ell - \nu^*\| \\ \leq& (1-\upsilon_\ell(1-\wp))\|a_\ell - \nu^*\|\end{aligned}$$

Now,

$$\begin{aligned}\|a_{\ell+1} - \nu^*\| =& \|(1-\vartheta_\ell)\breve{\rho} c_\ell + \vartheta_\ell \breve{\rho} b_\ell - \nu^*\| \\ \leq& (1-\vartheta_\ell)\wp\|c_\ell - \nu^*\| + \vartheta_\ell \wp \|b_\ell - \nu^*\| \\ \leq& (1-\vartheta_\ell)\wp\|a_\ell - \nu^*\| + \wp\vartheta_\ell(1-\upsilon_\ell(1-\wp))\|a_\ell - \nu^*\| \\ \leq& \wp(1-\vartheta_\ell\upsilon_\ell(1-\wp))\|a_\ell - \nu^*\| \ \leq \ \|a_\ell - \nu^*\|\end{aligned}$$

Hence, the sequence $\{a_\ell\}$ is bounded. Therefore, $\{b_\ell\}, \{c_\ell\}, \{d_\ell\}$ and $\{\breve{\rho}(a_\ell)\}$ is also bounded.

Step II. Now, we have to show that

$$\lim_{\ell\to\infty} \|a_{\ell+1} - a_\ell\| = 0.$$

By (5. 22), we have

$$\begin{aligned}\|a_{\ell+2} - a_{\ell+1}\| =& \|(1-\vartheta_{\ell+1})\breve{\rho} c_{\ell+1} + \vartheta_{\ell+1}\breve{\rho} b_{\ell+1} - ((1-\vartheta_\ell)\breve{\rho} c_\ell + \vartheta_\ell \breve{\rho} b_\ell)\| \\ \leq& (1-\vartheta_{\ell+1})\|\breve{\rho}(c_{\ell+1}) - \breve{\rho}(c_\ell)\| + |\vartheta_{\ell+1} - \vartheta_\ell|\|\breve{\rho}(b_\ell) - \breve{\rho}(c_\ell)\| + \\ & \vartheta_{\ell+1}\|\breve{\rho}(b_{\ell+1}) - \breve{\rho}(b_\ell)\| \\ \leq& \wp(1-\vartheta_{\ell+1})\|c_{\ell+1} - c_\ell\| + \wp|\vartheta_{\ell+1} - \vartheta_\ell|\|b_\ell - c_\ell\| + \wp\vartheta_{\ell+1}\|b_{\ell+1} - b_\ell\|.\end{aligned}$$

By using triangular inequality, it is easy to show that

$$\begin{aligned}\|c_{\ell+1} - c_\ell\| \leq \|a_{\ell+1} - a_\ell\| + \|a_\ell - c_\ell\| \\ \|b_{\ell+1} - b_\ell\| \leq \|a_{\ell+1} - a_\ell\| + \|a_\ell - b_\ell\|,\end{aligned}$$

which imply that

$$\begin{aligned}\|a_{\ell+2} - a_{\ell+1}\| \leq& \wp(1-\alpha)\|a_{\ell+1} - a_\ell\| + \wp(1-\vartheta_{\ell+1})\|a_\ell - b_\ell\| + \\ & \wp|\vartheta_{\ell+1} - \vartheta_\ell|\|b_\ell - c_\ell\| + \wp\vartheta_{\ell+1}\|a_{\ell+1} - a_\ell\| + \wp\vartheta_{\ell+1}\|a_\ell - b_\ell\| \\ \leq& \wp\|a_{\ell+1} - a_\ell\| + \wp|\vartheta_{\ell+1} - \vartheta_\ell|\|b_\ell - c_\ell\| + \wp\|a_\ell - b_\ell\|.\end{aligned}$$

For appropriate constants C_1 and C_2, we have

$$\|a_{\ell+2} - a_{\ell+1}\| \leq \wp\|a_{\ell+1} - a_\ell\| + \wp|\vartheta_{\ell+1} - \vartheta_\ell|C_1 + \wp C_2$$

Take $x_\ell = \|a_{\ell+1} - a_\ell\|$, $\wp_\ell = \wp$, $y_\ell = \wp|\vartheta_{\ell+1} - \vartheta_\ell|C_1$, $z_\ell = \wp C_2$. From Lemma 2.7, we have

$$\lim_{\ell\to\infty} \|a_{\ell+1} - a_\ell\| = 0.$$

Step III. Next, we prove that

$$\lim_{\ell\to\infty} d(a_\ell, \mathrm{S}(a_\ell)) = 0.$$

As from Lemma 3.4, we have $\lim_{\ell\to\infty} \|a_\ell - u_\ell\| = 0$. Which implies

$$\lim_{\ell\to\infty} d(a_\ell, \mathrm{S}(a_\ell)) \leq \lim_{\ell\to\infty} \|a_\ell - u_\ell\| = 0,$$

this implies

$$\lim_{n\to\infty} d(a_\ell, \mathrm{S}(a_\ell)) = 0.$$

Step IV. In this step, we demonstrate that for any $b^* \in \mathcal{F}_p(\mathrm{S})$, we have the following:

$$\limsup_{\ell\to\infty} \big\langle g(b^*) - b^*, j(a_{\ell+1} - b^*)\big\rangle \leq 0.$$

Given that $\{a_\ell\}$ is a bounded sequence and Ω is reflexive, we can conclude that subsequence $\{a_{\ell_j}\}$ of $\{a_\ell\}$ that converges to $\nu^* \in \Omega$. Suppose that

$$h(a) = \varpi_\ell \|a_\ell - a\|^2 \;\; \forall a \in \mathrm{B}_s.$$

Then, the function $h(a)$ is continuous and convex on B_s. Define

$$\mathrm{B}_{s_1} = \big\{a \in \mathrm{B}_s; h(a) = \inf_{b^*\in K} h(b^*)\big\}.$$

By considering the properties of $h(a)$, we can deduce that B_{s_1} is a nonempty, bounded, and closed convex subset of B_s. For every $a \in \mathrm{B}_{s_1}$, the compactness of $\mathrm{S}(a)$ guarantees the existence of $u_n \in \mathrm{S}(a)$ such that:

$$\|a_\ell - u_\ell\| = d(a_\ell, \mathrm{S}(a)) \text{ and } u_\ell \to u \in \mathrm{S}(a).$$

As, $\lim_{\ell\to\infty} d(a_\ell, \mathrm{S}(a_\ell)) = 0$, $\frac{1}{2} d(a_\ell, \mathrm{S}(a_\ell)) \leq \|a_\ell - a\|$ for sufficiently large ζ, such that $\ell \leq \zeta$. By (2. 1), we get $\mathrm{H}(\mathrm{S}(a_\ell), \mathrm{S}(a)) \leq \|a_\ell - a\|$
Now,

$$\begin{aligned}
h(u) =& \varpi_\ell \|a_\ell - u\|^2 \\
\leq& \varpi_\ell (\|a_\ell - u_\ell\| + \|u_\ell - u\|)^2 \\
=& \varpi_\ell \big(d(a_\ell, \mathrm{S}(a))\big)^2 \\
\leq& \varpi_\ell \big(d(a_\ell, \mathrm{S}(a)) + \mathrm{H}(\mathrm{S}(a_\ell), \mathrm{S}(a))\big)^2 \\
\leq& \varpi_\ell \|a_\ell - a\|^2 = h(a).
\end{aligned}$$

Hence, $u \in \mathrm{S}(a) \cap \mathrm{B}_{s_1}$ or $\mathrm{S}(a) \cap \mathrm{B}_{s_1} \neq \emptyset$ for all $a \in \mathrm{B}_{s_1}$. Let $\nu^* \in \mathcal{F}_p(\mathrm{S})$, then b^* is unique element in B_s, such that

$$\|\nu^* - b^*\| = d(\nu^*, \mathrm{B}_{s_1}) = \inf_{a\in \mathrm{B}_{s_1}} \|\nu^* - a\|.$$

As, $\mathrm{S}(b^*) \cap \mathrm{B}_{s_1} \neq \emptyset$, let $u \in \mathrm{S}(b^*) \cap \mathrm{B}_{s_1}$, implies

$$\|\nu^* - u\| = d(\mathrm{S}(\nu^*), u) \leq \mathrm{H}(\mathrm{S}(\nu^*), \mathrm{S}(b^*)) \leq \|\nu^* - b^*\|.$$

Therefore, $b^* = u \in \mathrm{S}(b^*)$. Utilizing the uniqueness of $y \in \mathrm{B}_{s_1}$, along with Lemma 2.10 and the definition of B_{s_1}, we can conclude that for any $v \in \mathrm{B}_s$, we have

$$\varpi_\ell \big\langle v - b^*, j(a_\ell - b^*)\big\rangle \leq 0.$$

Particularly,

$$\varpi_\ell \big\langle g(b^*) - b^*, j(a_\ell - b^*)\big\rangle \leq 0.$$

By leveraging the fact that $\lim_{\ell\to\infty} \|a_{\ell+1} - a_\ell\| = 0$ and the norm-weak* uniform continuity of the duality mapping j in a Banach space equipped with a uniformly Gâteaux differential norm, we obtain:

$$\lim_{\ell\to\infty} \big(\big\langle g(b^*) - b^*, j(a_{\ell+1} - b^*)\big\rangle - \big\langle g(b^*) - b^*, j(a_\ell - b^*)\big\rangle\big) = 0.$$

Therefore, the sequence $\big\{\big\langle g(b^*) - b^*, j(a_\ell - b^*)\big\rangle\big\}$ satisfies the conditions of Proposition 2.9 so we must have

$$\limsup_{\ell\to\infty} \big\langle g(b^*) - b^*, j(a_{\ell+1} - b^*)\big\rangle \leq 0. \tag{5. 23}$$

Step V. Finally, we need to prove that the sequence $a_\ell \to \nu^* \in \mathcal{F}_p(\mathrm{S})$.
Now, $\forall\ \ell \in \mathbb{N}$, define a set

$$\lambda_\ell = \max\big\{\big\langle g(b^*) - b^*, j(a_{\ell+1} - b^*)\big\rangle, 0\big\} \geq 0.$$

It is straightforward that $\lim_{\ell\to\infty} \lambda_\ell = 0$. Now, utilizing (5. 23) & Lemma 2.8, we obtain:

$$\begin{aligned}
\|a_{\ell+1} - \nu^*\|^2 =& \|(1-\vartheta_\ell)\breve{\rho}c_\ell + \vartheta_\ell\breve{\rho}b_\ell - \nu^*\|^2 \\
\leq& (1-\vartheta_\ell)^2\|\breve{\rho}c_\ell - \nu^*\|^2 + 2\vartheta_\ell\big\langle\breve{\rho}b_\ell - \nu^*, j(a_{\ell+1} - \nu^*)\big\rangle \\
\leq& (1-\vartheta_\ell)^2\wp\|c_\ell - \nu^*\|^2 + 2\vartheta_\ell\big\langle\breve{\rho}b_\ell - \breve{\rho}(\nu^*), j(a_{\ell+1} - \nu^*)\big\rangle + \\
& 2\vartheta_\ell\big\langle\breve{\rho}(\nu^*) - \nu^*, j(a_{\ell+1} - \nu^*)\big\rangle \\
\leq& (1-\vartheta_\ell)^2\wp\|a_\ell - \nu^*\|^2 + 2\vartheta_\ell\wp(\|b_\ell - \nu^*\|\|a_{\ell+1} - \nu^*\|) + \\
& 2\vartheta_\ell\big\langle\breve{\rho}(\nu^*) - \nu^*, j(a_{\ell+1} - \nu^*)\big\rangle \\
\leq& (1-\vartheta_\ell)^2\wp\|a_\ell - \nu^*\|^2 + \vartheta_\ell\wp(\|a_\ell - \nu^*\|^2 + \|a_{\ell+1} - \nu^*\|^2) + \\
& 2\vartheta_\ell\big\langle\breve{\rho}(\nu^*) - \nu^*, j(a_{\ell+1} - \nu^*)\big\rangle \\
\leq& \frac{(1-\vartheta_\ell)^2 + \wp\vartheta_\ell}{1 - \wp\vartheta_\ell}\|a_\ell - \nu^*\|^2 + \\
& \frac{2\vartheta_\ell}{1 - \wp\vartheta_\ell}\big\langle\breve{\rho}(\nu^*) - \nu^*, j(a_{\ell+1} - \nu^*)\big\rangle \\
\leq& \frac{(1+\vartheta_\ell^2)\wp - \wp\vartheta_\ell}{1 - \wp\vartheta_\ell}\|a_\ell - \nu^*\| + \frac{2\vartheta_\ell}{1 - \wp\vartheta_\ell}\lambda_{\ell+1}.
\end{aligned}$$

For some suitable constant $M > 0$, we have

$$\|a_{\ell+1} - \nu^*\|^2 \leq \wp\|a_\ell - \nu^*\| + \vartheta_\ell\big(\frac{2\vartheta_\ell}{1 - \wp\vartheta_\ell}\lambda_{\ell+1} + \wp M\vartheta_\ell\big).$$

Taking $x_\ell = \|a_\ell - \nu^*\|$, $\wp_\ell = \wp$, and $y_\ell = \vartheta_\ell\big(\frac{2\vartheta_\ell}{1-\wp\vartheta_\ell}\lambda_{\ell+1} + \wp M\vartheta_\ell\big)$ and $z_\ell = 0$, and

$$\limsup_{\ell\to\infty} \frac{y_\ell}{\wp_\ell} = \limsup_{\ell\to\infty} \frac{\vartheta_\ell\big(\frac{2\vartheta_\ell}{1-\wp\vartheta_\ell}\lambda_{\ell+1} + \wp M\vartheta_\ell\big)}{\wp}.$$

As by condition A_1 that $\vartheta_\ell \to 0$ as $\ell \to \infty$, we have

$$\limsup_{\ell\to\infty} \frac{y_\ell}{\wp_\ell} = \limsup_{\ell\to\infty} \frac{\vartheta_\ell\left(\frac{2\vartheta_\ell}{1-\wp\vartheta_\ell}\lambda_{\ell+1} + \wp M\vartheta_\ell\right)}{\wp} = 0.$$

Based on Lemma 2.7, we can conclude that $\lim_{\ell\to\infty} \|a_\ell - \nu^*\| = 0$. Hence, a_ℓ converges strongly to ν^*. □

6. CONCLUSION

In our study, we examined the iterative scheme proposed by Thakur et al. for a pair of mappings, one single-valued and the other multivalued, satisfying the condition (C_α) in Banach spaces. We conducted a comparative analysis of the convergence rate of this scheme with those of Ishikawa, Agarwal (S), and Thakur et al.'s iterative schemes. Our analysis resulted in the establishment of convergence results for our proposed scheme.

To further support our findings, we presented an example illustrating that our scheme achieves faster convergence compared to the aforementioned iterative schemes. By validating our results with this example, we highlight the superior performance of our proposed scheme.

Additionally, we explored the application of our iterative scheme in solving the SCFP Problem. Through this application, we showcased the practical utility of our scheme in addressing this specific problem.

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