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Some new inequalities of Hermite−Hadamard type via Katugampola fractional integral

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Abstract. In this study, we present the midpoint and trapezoid inequalities for an $F-$convex function in terms of Katugampola fractional integral operators. We obtained new results involving Katugampola-fractional integral operators for differentiable mapping ϕ whose second derivatives in the absolute values are $F-$convex. Also established connections between our results with several renowned results in literature. Results proved in this paper may stimulate further research in this area.

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1. Introduction

The theory of convex analysis has been one of the most interesting and useful areas for studying a wide range of problems in pure and applied sciences over the last few decades. Innovative methods and techniques have yielded different directions for the study of convex analysis. The convex function plays a significant role in optimization theory and in other fields of sciences and is usually defined in the following way:

Definition 1.1. *The function $\phi : [\theta_1, \theta_2] \to \mathbb{R}$, is said to be convex, if we have*

$$\phi(\lambda x + (1-\lambda) y) \leq \lambda \phi(x) + (1-\lambda)\phi(y)$$

for all $x, y \in [\theta_1, \theta_2]$ *and* $\lambda \in [0, 1]$.

If a real function ϕ of real variables defined on the interval $[\theta_1, \theta_2]$ is convex, then the double inequality

$$\phi\left(\frac{\theta_1+\theta_2}{2}\right) \leq \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\phi(x)dx \leq \frac{\phi(\theta_1)+\phi(\theta_2)}{2}. \tag{1. 1}$$

holds. Is known in the literature as Hermite-Hadamard's inequality. For concave, both inequalities are true in the opposite direction.

Definition 1.2. *([1])Let* $F : \mathbb{R} \to \mathbb{R}$ *be a given function. A function* $\phi : I \to \mathbb{R}$ *is called* $F-$*convex if*

$$\phi(\epsilon\theta_1 + (1-\epsilon)\theta_2) \leq \epsilon\phi(\theta_1) + (1-\epsilon)\phi(\theta_2) - \epsilon(1-\epsilon)F(\theta_1-\theta_2)$$

for all $\theta_1, \theta_2 \in I$ *and* $\epsilon \in [0, 1]$.

It is worth mentioning here that the class of $F-$convex functions was defined in the context of a strongly convex function. But they also unified several other important concepts of convexity ([14]) :
• Putting $F(x) = -cx^2$, we recaptures the $c-$convex functions introduced by Vial (see [21]).
• Putting $F(x) = -c|x|$ with $c > 0$, we recaptures approximate convex functions introduced by H.V. Ngai et al. (see [15]).
• For $F(x) = -c|x|^p$ with $c > 0$ and $p > 0$, we get approximately convex functions of order p introduced by Nikodem and Pales (see [16]).
• for $F(x) = -|x|\,\omega(|x|)$ with nondecreasing, upper-semicontinuous function $\omega : [0, \infty) \to (0, \infty]$ such that $\omega(0) = 0$ we obtain the definition of semi-convex functions introduced by G.Alberti et al. (see [2]).

Definition 1.3. *([9])Let* $[\theta_1, \theta_2] \subset \mathbb{R}$ *be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order* $\alpha > 0$ *of* $\phi \in X_c^p(\theta_1, \theta_2)$ *are defined by,*

$${}^{\rho}I^{\alpha}_{(\theta_1)+}\phi(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\theta_1}^{x}\frac{\phi(\epsilon)}{(x^{\rho}-\epsilon^{\rho})^{1-\alpha}}\epsilon^{\rho-1}d\epsilon$$

and

$${}^{\rho}I^{\alpha}_{(\theta_2)-}\phi(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{\theta_2}\frac{\phi(\epsilon)}{(\epsilon^{\rho}-x^{\rho})^{1-\alpha}}\epsilon^{\rho-1}d\epsilon$$

with $\theta_1 < x < \theta_2$ *and* $\rho > 0$*, if the integral exist.*

The following definition, the well-known definition of the Riemann–Liouville fractional integral:

Definition 1.4. *Let* $\phi \in L_1[\theta_1, \theta_2]$ *. The Riemann-Liouville integrals* $J^{\alpha}_{\theta_1^+}\phi$ *and* $J^{\alpha}_{\theta_2^-}\phi$ *of order* $\alpha > 0$ *with* $\theta_1 \geq 0$ *are defined by*

$$J^{\alpha}_{(\theta_1)+}\phi\ (x) = \frac{1}{\Gamma(\alpha)}\int_{\theta_1}^{x}(x-\epsilon)^{\alpha-1}\phi(\epsilon)\,d\epsilon\,, \qquad x > \theta_1$$

and

$$J^{\alpha}_{(\theta_2)-}\phi\ (x) = \frac{1}{\Gamma(\alpha)}\int_x^{\theta_2} (\epsilon - x)^{\alpha-1}\phi(\epsilon)\, d\epsilon\ , \qquad x < \theta_2$$

respectively, where $\Gamma(\alpha)$*-Euler gamma function. Here is* $J^0_{\theta_1^+}\phi(x) = J^0_{\theta_2^-}\phi(x) = \phi(x)$. *For* $\alpha = 1$ *these integrals reduce to the classical Riemann integral.*

Theorem 1.5. *Let* $\alpha > 0$ *and* $\rho > 0$*. Then for* $x > \theta_1$*,*
1. $\lim\limits_{\rho\to 1} {}^{\rho}I^{\alpha}_{\theta_1^+}\phi(x) = J^{\alpha}_{\theta_1^+}\phi(x)$*,*
2. $\lim\limits_{\rho\to 0} {}^{\rho}I^{\alpha}_{\theta_1^+}\phi(x) = H^{\alpha}_{\theta_1^+}\phi(x)$*,*
Similar results also hold for right-sided operators.

Undoubtedly, the Hadamard inequality occupies a special place in the field of applied and computational mathematics. Quite a few studies of the last few decades have been devoted to this inequality (some of them [3, 5, 6, 12, 13, 18, 19] and references therein).

Extensions of this inequality for the various convex functions using the Katugampola fractional integration operators can be found, for example, in [1, 4, 7, 8, 10, 11, 17, 20, 22].

Bayraktar and Nápoles in [4], for the functions (h, m, s)-convex modified of the second type, various extensions of the classic Hadamard inequality is obtained using Katugampola integrals. In [8], Kermausuor and Nwaeze established some new midpoint and trapezoidal type inequalities for prequasiinvex functions via the Katugampola fractional integrals. Yaldız and Akdemir in their study [22] obtained Hermite-Hadamard-type inequalities for functions whose first derivatives are $s-$convex using generalized Katugampola fractional integrals. In [1] Adamek obtained various Hermite-Hadamard-type inequalities for $F-$convex functions. Mumcu et al. in study [11], presented new Hadamard type inequalities for harmonically convex functions via Katugampola fractional integrals. Authors, in this paper [12], presented new Hermite-Hadamard inequalities for the $h-$convex functions within the framework of the generalized integral. Nápoles et al. in study [13], was obtained new Hermite-Hadamard-type integral inequalities for convex and quasi-convex functions in a generalized context. Recently, in [3, 18, 10], the authors obtained an estimate for the upper bound of this inequality for functions whose second derivative belongs to different convexity classes.

The main goal of the study is to obtain new Hadamard-type integral inequalities for functions whose second derivatives are $F-$convex through the Katugampola fractional integral operator.

2. Result for Midpoint Type Inequality

Lemma 2.1. *Let* $\theta_1, \theta_2 \in \mathbb{R}$, $0 \le \theta_1 < \theta_2, \rho > 0$, $\phi : [\theta_1^\rho, \theta_2^\rho] \to \mathbb{R}$ *and* $\phi \in C^2(\theta_1^\rho, \theta_2^\rho)$*. If* $\phi'' \in L[\theta_1^\rho, \theta_2^\rho]$ *and for all* $\alpha > 1$ *fractional integrals exist, then the equality*

$$\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho - \theta_1^\rho)^{\alpha-1}} \cdot \mathbf{K} - \phi\left(\frac{\theta_1^\rho + \theta_2^\rho}{2}\right) = \frac{\rho(\theta_2^\rho - \theta_1^\rho)^2}{\alpha 2^{2-\alpha}}\,(I_1 + I_2) \qquad (2.\ 2)$$

holds. Here

$$\mathbf{K}={}^{\rho}I^{\alpha-1}_{\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)^{\frac{1}{\rho}}+}\phi(\theta_2^\rho)+{}^{\rho}I^{\alpha-1}_{\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)^{\frac{1}{\rho}}-}\phi(\theta_1^\rho),$$

$$I_1=\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\rho\alpha+\rho-1}\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon,$$

$$I_2=\int_{\sqrt[\rho]{\frac{1}{2}}}^1(1-\epsilon^\rho)^\alpha\,\epsilon^{\rho-1}\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon.$$

Proof. By integrating I_1 by parts twice, we get

$$\begin{aligned}
I_1&=\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+\rho-1}\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon \qquad (2.\ 3)\\
&=\frac{1}{\rho}\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho}\left(\rho\epsilon^{\rho-1}\right)\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon\\
&=\frac{1}{\rho}\left[\frac{\epsilon^{\alpha\rho}}{\theta_1^\rho-\theta_2^\rho}\phi'(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)|_0^{\sqrt[\rho]{\frac{1}{\epsilon}}}-\frac{\alpha\rho}{\theta_1^\rho-\theta_2^\rho}\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho-1}\phi'(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon\right]\\
&=\frac{\phi'\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)}{2^\alpha(\theta_1^\rho-\theta_2^\rho)\rho}-\frac{\alpha}{\theta_1^\rho-\theta_2^\rho}\left[\frac{\epsilon^{\alpha\rho-\rho}\phi(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)}{\rho(\theta_1^\rho-\theta_2^\rho)}|_0^{\sqrt[\rho]{\frac{1}{2}}}\right.\\
&\qquad\left.-\frac{\rho\alpha-\rho}{\rho\,(\theta_1^\rho-\theta_2^\rho)}\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho-\rho-1}\phi(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon\right]\\
&=\frac{\phi'\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)}{2^\alpha(\theta_1^\rho-\theta_2^\rho)\rho}-\frac{\alpha\phi\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)}{\rho(\theta_1^\rho-\theta_2^\rho)^2 2^{\alpha-1}}+\frac{\alpha\,(\alpha-1)}{(\theta_1^\rho-\theta_2^\rho)^2}\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho-\rho-1}\phi(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon
\end{aligned}$$

and similarly, for the second integral I_2, we can obtain

$$\begin{aligned}
I_2&=\int_{\sqrt[\rho]{\frac{1}{2}}}^1(1-\epsilon^\rho)^\alpha\,\epsilon^{\rho-1}\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon \qquad (2.\ 4)\\
&=\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+\rho-1}\phi''((1-\epsilon^\rho)\theta_1^\rho+\epsilon^\rho\theta_2^\rho)d\epsilon\\
&=\frac{\phi'\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)}{\rho(\theta_2^\rho-\theta_1^\rho)2^\alpha}-\frac{\alpha\phi\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)}{\rho(\theta_2^\rho-\theta_1^\rho)^2 2^{\alpha-1}}+\frac{\alpha\,(\alpha-1)}{(\theta_2^\rho-\theta_1^\rho)^2}\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho-\rho-1}\phi((1-\epsilon^\rho)\theta_1^\rho+\epsilon^\rho\theta_2^\rho)d\epsilon.
\end{aligned}$$

If in the integrals from (2. 3) and (2. 4) respectively change the variables $\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho=u^\rho$ and $(1-\epsilon^\rho)\theta_1^\rho+\epsilon^\rho\theta_2^\rho=v^\rho$, then, for these integrals, we get

a) for the first integral

$$I_1 = \frac{\alpha(\alpha-1)}{(\theta_1^\rho - \theta_2^\rho)^2} \int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho-\rho-1} \phi(\epsilon^\rho \theta_1^\rho + (1-\epsilon^\rho)\theta_2^\rho) d\epsilon$$

$$= \frac{\alpha(\alpha-1)}{\rho(\theta_1^\rho - \theta_2^\rho)^2} \int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho-2\rho} \phi(\epsilon^\rho \theta_1^\rho + (1-\epsilon^\rho)\theta_2^\rho) d\epsilon^\rho$$

$$= \frac{\alpha(\alpha-1)}{\rho(\theta_2^\rho - \theta_1^\rho)^2} \int_{\theta_2^\rho}^{\frac{\theta_1^\rho+\theta_2^\rho}{2}} \left(\frac{\theta_2^\rho - u^\rho}{\theta_2^\rho - \theta_1^\rho}\right)^{\alpha-2} \phi(u^\rho) d\left(\frac{u^\rho - \theta_2^\rho}{\theta_1^\rho - \theta_2^\rho}\right)$$

$$= \frac{\alpha(\alpha-1)}{\rho(\theta_2^\rho - \theta_1^\rho)^{\alpha+1}} \int_{\frac{\theta_1^\rho+\theta_2^\rho}{2}}^{\theta_2^\rho} (\theta_2^\rho - u^\rho)^{\alpha-2} \phi(u^\rho) du^\rho$$

$$= \frac{\alpha(\alpha-1)\Gamma(\alpha-1)}{\rho^{2-\alpha}(\theta_2^\rho - \theta_1^\rho)^{\alpha+1}} \cdot \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_{\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)^{\frac{1}{\rho}}+}^{\theta_2} (\theta_2^\rho - u^\rho)^{\alpha-2} \phi(u^\rho) u^{\rho-1} du$$

$$= \frac{\alpha\Gamma(\alpha)\rho^{\alpha-2}}{(\theta_2^\rho - \theta_1^\rho)^{\alpha+1}} \, {}^\rho I^{\alpha-1}_{\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)^{\frac{1}{\rho}}+} \phi(\theta_2^\rho).$$

b) Having done similar operations for the second integral, we obtain

$$I_2 = \frac{\alpha(\alpha-1)}{(\theta_1^\rho - \theta_2^\rho)^2} \int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho-\rho-1} \phi((1-\epsilon^\rho)\theta_1^\rho + \epsilon^\rho \theta_2^\rho) d\epsilon = \frac{\alpha\Gamma(\alpha)\rho^{\alpha-2}}{(\theta_2^\rho - \theta_1^\rho)^{\alpha+1}} \, {}^\rho I^{\alpha-1}_{\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)^{\frac{1}{\rho}}-} \phi(\theta_1^\rho).$$

By taking account last transformations, by summing equalities (2. 3) and (2. 4), we obtain

$$I_1 + I_2 = -\frac{\alpha 2^{2-\alpha}}{\rho(\theta_1^\rho - \theta_2^\rho)^2} \phi\left(\frac{\theta_1^\rho + \theta_2^\rho}{2}\right) + \frac{\alpha\Gamma(\alpha)\rho^{\alpha-2}}{(\theta_2^\rho - \theta_1^\rho)^{\alpha+1}} \left[{}^\rho I^{\alpha-1}_{\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)^{\frac{1}{\rho}}+} \phi(\theta_2^\rho) + {}^\rho I^{\alpha-1}_{\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)^{\frac{1}{\rho}}-} \phi(\theta_1^\rho) \right]. \tag{2. 5}$$

By multiplying the equality(2. 5) by the value $\frac{\rho(\theta_2^\rho - \theta_1^\rho)^2}{\alpha 2^{2-\alpha}}$, we complete the proof. □

Remark 2.2. *Under the condition of Lemma 2.1 if we take* $\rho = 1$ *in* (2. 2) *we obtained the equality proved by B. Bayraktar (see* [3] *Lemma 2.1 for* $m = 1$ *)*

$$\frac{\Gamma(\alpha)}{2^{2-\alpha}(\theta_2 - \theta_1)^{\alpha-1}} \left[I^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)^{\frac{1}{\rho}}+} \phi(\theta_2) + I^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)^{\frac{1}{\rho}}-} \phi(\theta_1) \right] - \phi\left(\frac{\theta_1+\theta_2}{2}\right) = \frac{(\theta_2-\theta_1)^2}{\alpha 2^{2-\alpha}} (I_1 + I_2).$$

Corollary 2.3. *Under the condition of Lemma 2.1 if we take* $\rho = 1$ *and* $\alpha = 2$ *in* (2. 2) *we obtained the equality:*

$$\frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \phi(x) dx - \phi\left(\frac{\theta_1+\theta_2}{2}\right) = \frac{(\theta_2-\theta_1)^2}{2} [I_1 + I_2],$$

where

$$I_1=\int_0^{\frac{1}{2}}\epsilon^2\phi''(\epsilon\theta_1+(1-\epsilon)\theta_2)d\epsilon \ \text{ and } \ I_2=\int_{\frac{1}{2}}^{1}(1-\epsilon)^2\phi''(\epsilon\theta_1+(1-\epsilon)\theta_2)d\epsilon.$$

Theorem 2.4. *Let $\phi:[\theta_1^\rho,\theta_2^\rho]\to\mathbb{R}$ and $\phi\in C^2(\theta_1^\rho,\theta_2^\rho)$. If $|\phi''|$ is $F-$convex on $[\theta_1^\rho,\theta_2^\rho]$, then the inequality*

$$\left|\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{K}-\phi\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)\right| \tag{2. 6}$$
$$\le\frac{(\theta_2^\rho-\theta_1^\rho)^2}{16\alpha}\left[\frac{2\left(|\phi(\theta_1^\rho)|+|\phi(\theta_2^\rho)|\right)}{\alpha+1}-\frac{\alpha+4}{(\alpha+2)(\alpha+3)}F(\theta_1^\rho-\theta_2^\rho)\right]$$

holds. Here $\rho>0$ and $\mathbf{K}$ is defined above in Lemma 2.1.

Proof. From (2. 2) by using the triangle inequalities, we can write

$$\left|\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{K}-\phi\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)\right|\le\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{\alpha 2^{2-\alpha}} \tag{2. 7}$$
$$\times\left[\left|\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+\rho-1}\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon\right|\right.$$
$$\left.+\left|\int_{\sqrt[\rho]{\frac{1}{2}}}^{1}(1-\epsilon^\rho)^\alpha\epsilon^{\rho-1}\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon\right|\right].$$

If we take $1-\epsilon^\rho=\epsilon^\rho$ in the second integral and assume the function $|\phi''|$ is $F-$convex, then for the right-hand side of (2. 7) we get:

$$\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{\alpha 2^{2-\alpha}}\left(\left|\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+\rho-1}\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon\right|\right.$$
$$\left.+\left|\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+\rho-1}\phi''((1-\epsilon^\rho)\theta_1^\rho+\epsilon^\rho\theta_2^\rho)d\epsilon\right|\right)$$
$$=\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{\alpha 2^{2-\alpha}}\left\{\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+\rho-1}\left[|\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)|+|\phi''((1-\epsilon^\rho)\theta_1^\rho+\epsilon^\rho\theta_2^\rho)|\right]d\epsilon\right\}$$
$$\le\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{\alpha 2^{2-\alpha}}\left\{\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+\rho-1}\left[|\phi(\theta_1^\rho)|+|\phi(\theta_2^\rho)|-2\cdot\epsilon^\rho(1-\epsilon^\rho)\cdot F(\theta_1^\rho-\theta_2^\rho)\right]d\epsilon\right\}.$$
$$=\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{\alpha 2^{2-\alpha}}\left[\frac{|\phi(\theta_1^\rho)|+|\phi(\theta_2^\rho)|}{\rho(\alpha+1)2^{\alpha+1}}-\frac{2\,(\alpha+4)}{\rho(\alpha+2)(\alpha+3)2^{\alpha+3}}F(\theta_1^\rho-\theta_2^\rho)\right]$$
$$=\frac{(\theta_2^\rho-\theta_1^\rho)^2}{16\alpha}\left[\frac{2\left(|\phi(\theta_1^\rho)|+|\phi(\theta_2^\rho)|\right)}{\alpha+1}-\frac{\alpha+4}{(\alpha+2)(\alpha+3)}F(\theta_1^\rho-\theta_2^\rho)\right]$$

By simple integration

$$\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+\rho-1}d\epsilon=\frac{1}{\rho(\alpha+1)2^{\alpha+1}} \ \text{ and } \ \int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^{\alpha\rho+2\rho-1}(1-\epsilon^\rho)d\epsilon=\frac{\alpha+4}{\rho(\alpha+2)(\alpha+3)2^{\alpha+3}}.$$

Taking into account the last inequality and substituting the values of the above integrals, we complete the proof. □

Remark 2.5. *In (2. 6), if we take* $\rho = 1$ *in* (2. 6)*, then we get the inequality.*

$$\left|\frac{\Gamma(\alpha)}{2^{2-\alpha}(\theta_2-\theta_1)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)^+}\phi(\theta_2)+J^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)^-}\phi(\theta_1)\right]-\phi\left(\frac{\theta_1+\theta_2}{2}\right)\right|$$
$$\leq \frac{(\theta_2^\rho-\theta_1^\rho)^2}{16\alpha}\left[\frac{2\left[|\phi''(\theta_1^\rho)|+|\phi''(\theta_2^\rho)|\right]}{\alpha+1}-\frac{\alpha+4}{(\alpha+2)(\alpha+3)}\cdot F(\theta_1^\rho-\theta_2^\rho)\right].$$

Remark 2.6. *In (2. 6), if we take* $F(x) = 0$ *and* $\rho = 1$*, then we obtained the inequality proved by B. Bayraktar [see* [3] *Theorem 2.1 for* $m = 1$ *and* $s = 1$ *]*

$$\left|\frac{\Gamma(\alpha)}{2^{2-\alpha}(\theta_2-\theta_1)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)+}\phi(\theta_2)+J^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)-}\phi(\theta_1)\right]-\phi\left(\frac{\theta_1+\theta_2}{2}\right)\right|$$
$$\leq \frac{(\theta_2-\theta_1)^2}{8\alpha\,(\alpha+1)}\,\left[|\phi''(\theta_1)|+|\phi''(\theta_2)|\right].$$

Corollary 2.7. *Under the condition of Remark 2.6 if we take* $\alpha = 2$ *we obtained*

$$\left|\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\phi(x)dx-\phi\left(\frac{\theta_1+\theta_2}{2}\right)\right|\leq\frac{(\theta_2-\theta_1)^2}{48}[|\phi''(\theta_1)|+|\phi''(\theta_2)|].$$

This inequality for convex functions was obtained in [18] *(Proposition* 1*) and* [3] *(Corollary* 2.1*).*

Theorem 2.8. *Let* $\phi : [\theta_1^\rho, \theta_2^\rho\,] \to \mathbb{R}$ *and* $\phi \in C^2(\theta_1^\rho, \theta_2^\rho)$ *with* $q \geq 1$*. If* $|\phi''|^q$ *is* $F-$*convex on* $[\theta_1^\rho, \theta_2^\rho\,]$*, then the inequality*

$$\left|\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{K}-\phi\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)\right|\leq\frac{(\theta_2^\rho-\theta_1^\rho)^2}{8\alpha\,(\alpha+1)}\cdot\mathbf{v}\cdot\mathbf{H}, \tag{2. 8}$$

holds. Here $\rho > 0$ *and* $\mathbf{K}$ *is defined above in Lemma 2.1*,

$$\mathbf{v}=\left(\frac{\alpha+1}{4\,(\alpha+2)}\right)^{\frac{1}{q}},$$
$$\mathbf{H}=\left[2|\phi''(\theta_1^\rho)|^q+\frac{2\,(\alpha+3)}{\alpha+1}|\phi''(\theta_2^\rho)|^q-\frac{\alpha+4}{\alpha+3}F(\theta_1^\rho-\theta_2^\rho)\right]^{\frac{1}{q}}$$
$$+\left[\frac{2\,(\alpha+3)}{\alpha+1}|\phi''(\theta_1^\rho)|^q+2|\phi''(\theta_2^\rho)|^q-\frac{\alpha+4}{\alpha+3}F(\theta_1^\rho-\theta_2^\rho)\right]^{\frac{1}{q}}.$$

Proof. From (2. 2) the validity of the following inequality is obvious:

$$\left|\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{K}-\phi\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)\right|\leq\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{\alpha 2^{2-\alpha}}(|I_1|+|I_2|). \tag{2. 9}$$

By using the fact that$|\phi''|^q$ is a $F-$convex by using power-mean integral inequality, for the I_1, we get

$$
\begin{aligned}
|I_1| &\le \int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho+\rho-1}|\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)|d\epsilon \qquad (2.\ 10)\\
&\le \left(\int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho+\rho-1}d\epsilon\right)^{1-\frac{1}{q}} \left(\int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho+\rho-1}|\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)|^q d\epsilon\right)^{\frac{1}{q}}\\
&\le \left(\int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho+\rho-1}d\epsilon\right)^{1-\frac{1}{q}}\\
&\quad \times \Bigg[|\phi''(\theta_1^\rho)|^q \int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho+2\rho-1}d\epsilon + |\phi''(\theta_2^\rho)|^q \int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho+\rho-1}(1-\epsilon^\rho)d\epsilon\\
&\quad -F(\theta_1^\rho-\theta_2^\rho)\int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\alpha\rho+2\rho-1}(1-\epsilon^\rho)d\epsilon\Bigg]^{\frac{1}{q}}.
\end{aligned}
$$

Calculating the integrals, we get

$$
\begin{aligned}
|I_1| &\le \left(\frac{1}{\rho(\alpha+1)2^{\alpha+1}}\right)^{1-\frac{1}{q}} \Bigg[\frac{|\phi''(\theta_1^\rho)|^q}{\rho(\alpha+2)2^{\alpha+2}} + \frac{(\alpha+3)|\phi''(\theta_2^\rho)|^q}{\rho(\alpha+1)\ (\alpha+2)2^{\alpha+2}}\\
&\quad -\frac{\alpha+4}{\rho(\alpha+2)(\alpha+3)2^{\alpha+3}}F(\theta_1^\rho-\theta_2^\rho)\Bigg]^{\frac{1}{q}}\\
&= \frac{1}{\rho(\alpha+1)2^{\alpha+1}}\left(\frac{\alpha+1}{4(\alpha+2)}\right)^{\frac{1}{q}}\\
&\quad \times\left[2|\phi''(\theta_1^\rho)|^q + \frac{2(\alpha+3)}{\alpha+1}|\phi''(\theta_2^\rho)|^q - \frac{\alpha+4}{\alpha+3}F(\theta_1^\rho-\theta_2^\rho)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Similarly, for the $|I_2|$, we get

$$
\begin{aligned}
|I_2| &\le \frac{1}{\rho(\alpha+1)2^{\alpha+1}}\left(\frac{\alpha+1}{4(\alpha+2)}\right)^{\frac{1}{q}} \qquad (2.\ 11)\\
&\quad \times\left[\frac{2(\alpha+3)}{\alpha+1}|\phi''(\theta_1^\rho)|^q + 2|\phi''(\theta_2^\rho)|^q - \frac{\alpha+4}{\alpha+3}F(\theta_1^\rho-\theta_2^\rho)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Taking into account inequality (2. 10), (2. 11) and (2. 9), we get the final result. □

Remark 2.9. *In (2. 8) if we take $\rho=1$, then we obtained the inequality*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha)}{2^{2-\alpha}(\theta_2-\theta_1)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)+}\phi(\theta_2)+J^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)-}\phi(\theta_1)\right]-\phi\left(\frac{\theta_1+\theta_2}{2}\right)\right|\\
&\le \frac{(\theta_2-\theta_1)^2}{8\alpha(\alpha+1)}\cdot \mathbf{v}\cdot\mathbf{H},
\end{aligned}
$$

where

$$\mathbf{v} = \left(\frac{\alpha+1}{4(\alpha+2)}\right)^{\frac{1}{q}},$$

$$\mathbf{H} = \left[2|\phi''(\theta_1)|^q + \frac{2(\alpha+3)}{\alpha+1}|\phi''(\theta_2)|^q - \frac{\alpha+4}{\alpha+3}\cdot F(\theta_1-\theta_2)\right]^{\frac{1}{q}}$$
$$+\left[\frac{2(\alpha+3)}{\alpha+1}|\phi''(\theta_1)|^q + 2|\phi''(\theta_2)|^q - \frac{\alpha+4}{\alpha+3}\cdot F(\theta_1-\theta_2)\right]^{\frac{1}{q}}.$$

Corollary 2.10. *In (2. 8) if we take $F(x) = 0$ and $\rho = 1$, then we obtained the inequality proved by B. Bayraktar (see* [3] *Theorem 2.2 for $s = 1$ and $m = 1$).*

Corollary 2.11. *In (2. 8), if we chose $\rho = 1, F(x) = 0$ and $\alpha = 2$, then we obtained inequality for the convex functions*

$$\left|\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\phi(x)dx - \phi\left(\frac{\theta_1+\theta_2}{2}\right)\right| \le \frac{(\theta_2-\theta_1)^2}{48}\cdot\mathbf{H}_1,$$

where

$$\mathbf{H}_1 = \left[\frac{5|\phi''(\theta_1)|^q + 3|\phi''(\theta_2)|^q}{8}\right]^{\frac{1}{q}} + \left[\frac{3|\phi''(\theta_1)|^q + 5|\phi''(\theta_2)|^q}{8}\right]^{\frac{1}{q}}.$$

This inequality for convex functions was obtained in [18] *(Proposition* 5*),* [3] *(Corollary* 2.2*).*

Theorem 2.12. *Let $\phi : [\theta_1^\rho, \theta_2^\rho] \to \mathbb{R}$ and $\phi \in C^2(\theta_1^\rho, \theta_2^\rho)$ with $0 \le \theta_1 < \theta_2$. If $|\phi''|^q$ is $F-$convex on $[\theta_1^\rho, \theta_2^\rho]$, then $\forall\alpha > 1$ and $q > 1$, $p > 1$, such that $\frac{1}{q} + \frac{1}{p} = 1$ the inequality*

$$\left|\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{K} - \phi\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)\right| \le \frac{(\theta_2^\rho-\theta_1^\rho)^2}{\alpha 2^{2-\alpha}}\cdot\mathbf{L}\cdot\mathbf{M}, \tag{2. 12}$$

holds. Here $\rho > 0$ and $\mathbf{K}$ is defined above in Lemma 2.1,

$$\mathbf{L} = \left(\frac{1}{(\alpha p+1)2^{\alpha p+1}}\right)^{\frac{1}{p}},$$

$$\mathbf{M} = \left[\frac{|\phi''(\theta_1^\rho)|^q + 3|\phi''(\theta_2^\rho)|^q}{8} - \frac{F(\theta_1^\rho-\theta_2^\rho)}{12}\right]^{\frac{1}{q}} + \left[\frac{3|\phi''(\theta_1^\rho)|^q + |\phi''(\theta_2^\rho)|^q}{8} - \frac{F(\theta_1^\rho-\theta_2^\rho)}{12}\right]^{\frac{1}{q}}.$$

Proof. From (2. 2) the validity of the following inequality is obvious:

$$\left|\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{K} - \phi\left(\frac{\theta_1^\rho+\theta_2^\rho}{2}\right)\right| \le \frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{\alpha 2^{2-\alpha}}(|I_1|+|I_2|). \tag{2. 13}$$

By using the fact that $|\phi''|^q$ is a $F-$convex by using Hölder integral inequality, for the $|I_1|+|I_2|$, we get

$$
\begin{aligned}
&|I_1|+|I_2| \qquad (2.\ 14)\\
&\leq \frac{1}{\rho}\left(\int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\rho\alpha}|\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)|d\epsilon^\rho+\int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\rho\alpha}|\phi''((1-\epsilon^\rho)\theta_1^\rho+\epsilon^\rho\theta_2^\rho)|d\epsilon^\rho\right)\\
&\leq \frac{1}{\rho}\left(\int_0^{\sqrt[\rho]{\frac{1}{2}}} (\epsilon^{\rho\alpha})^p d\epsilon^\rho\right)^{\frac{1}{p}}\\
&\times\left\{\left[|\phi''(\theta_1^\rho)|^q\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^\rho d\epsilon^\rho+|\phi''(\theta_2^\rho)|^q\int_0^{\sqrt[\rho]{\frac{1}{2}}}(1-\epsilon^\rho)d\epsilon^\rho-F(\theta_1^\rho-\theta_2^\rho)\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^\rho(1-\epsilon^\rho)d\epsilon^\rho\right]^{\frac{1}{q}}\right.\\
&\left.+\left[|\phi''(\theta_1^\rho)|^q\int_0^{\sqrt[\rho]{\frac{1}{2}}}(1-\epsilon^\rho)d\epsilon^\rho+|\phi''(\theta_2^\rho)|^q\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^\rho d\epsilon^\rho-F(\theta_1^\rho-\theta_2^\rho)\int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^\rho(1-\epsilon^\rho)d\epsilon^\rho\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

With simple integration

$$
\left(\int_0^{\sqrt[\rho]{\frac{1}{2}}}(\epsilon^\rho)^{\alpha p}d\epsilon^\rho\right)^{\frac{1}{p}}=\left(\frac{1}{(\alpha p+1)\,2^{\alpha p+1}}\right)^{\frac{1}{p}};\ \int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^\rho d\epsilon^\rho=\frac{1}{8};
$$

$$
\int_0^{\sqrt[\rho]{\frac{1}{2}}}(1-\epsilon^\rho)d\epsilon^\rho=\frac{3}{8};\ \int_0^{\sqrt[\rho]{\frac{1}{2}}}\epsilon^\rho(1-\epsilon^\rho)d\epsilon^\rho=\frac{1}{12};
$$

and after simplification, on the right hand side of the last inequality, we get

$$
\begin{aligned}
|I_1|+|I_2| &\leq \frac{1}{\rho}\cdot\left(\frac{1}{(\alpha p+1)\,2^{\alpha p+1}}\right)^{\frac{1}{p}}\\
&\times\left\{\left[\frac{|\phi''(\theta_1^\rho)|^q+3|\phi''(\theta_2^\rho)|^q}{8}-\frac{F(\theta_1^\rho-\theta_2^\rho)}{12}\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\frac{3|\phi''(\theta_1^\rho)|^q+|\phi''(\theta_2^\rho)|^q}{8}-\frac{F(\theta_1^\rho-\theta_2^\rho)}{12}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Taking into account the last inequality, the final result follows from (2. 13). □

Corollary 2.13. *In (2. 12), if we take* $\rho=1$ *in* (2. 12)*, then we obtained the inequality*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha)}{2^{2-\alpha}(\theta_2-\theta_1)^{\alpha-1}}\left[J^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)+}\phi(\theta_2)+J^{\alpha-1}_{\left(\frac{\theta_1+\theta_2}{2}\right)-}\phi(\theta_1)\right]-\phi\left(\frac{\theta_1+\theta_2}{2}\right)\right|\\
&\leq\frac{(\theta_2-\theta_1)^2}{\alpha 2^{2-\alpha}}\cdot\mathbf{L}\cdot\mathbf{M},
\end{aligned}
$$

where $\mathbf{L}$ *idefined in Theorem 2.12,*

$$
\mathbf{M}=\left[\frac{|\phi''(\theta_1)|^q+3|\phi''(\theta_2)|^q}{8}-\frac{F(\theta_1-\theta_2)}{12}\right]^{\frac{1}{q}}+\left[\frac{3|\phi''(\theta_1)|^q+|\phi''(\theta_2)|^q}{8}-\frac{F(\theta_1-\theta_2)}{12}\right]^{\frac{1}{q}}.
$$

Theorem 2.14. *Let* $\phi : [\theta_1^\rho, \theta_2^\rho\ ] \to [0, +\infty)$ *and* $\phi \in C^2\ (\theta_1^\rho, \theta_2^\rho)$ *with* $0 \leq \theta_1 < \theta_2$*. If* $|\phi''|^q$ *is* $F-$*convex on* $[\theta_1^\rho, \theta_2^\rho\ ]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, then the inequality*

$$\left|\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho - \theta_1^\rho)^{\alpha-1}} \cdot \mathbf{K} - \phi\left(\frac{\theta_1^\rho + \theta_2^\rho}{2}\right)\right| \leq \frac{(\theta_2^\rho - \theta_1^\rho)^2}{\alpha 2^{2-\alpha}} \cdot (\nu + \mathbf{P}), \tag{2. 15}$$

holds. Here $\rho > 0$ *and* $\mathbf{K}$ *is defined above in Lemma 2.1*,

$$\mathbf{P} = \frac{1}{q}\left[\frac{|\phi''(\theta_1^\rho)|^q + |\phi''(\theta_2^\rho)|^q}{2} - \frac{F(\theta_1^\rho - \theta_2^\rho)}{6}\right],$$

$$\nu = \frac{1}{p\,(\alpha p + 1)\,2^{\alpha p+1}}.$$

Proof. From (2. 2) the validity of the following inequality is obvious:

$$\left|\frac{\Gamma(\alpha)\rho^{\alpha-1}}{2^{2-\alpha}(\theta_2^\rho - \theta_1^\rho)^{\alpha-1}} \cdot \mathbf{K} - \phi\left(\frac{\theta_1^\rho + \theta_2^\rho}{2}\right)\right| \leq \frac{\rho(\theta_2^\rho - \theta_1^\rho)^2}{\alpha 2^{2-\alpha}}(|I_1| + |I_2|). \tag{2. 16}$$

By using the fact that $|\phi''|^q$ is a $F-$convex by using the well know Young inequality $\left(xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q,\ \text{where}\ \ x, y \in [0, +\infty)\right)$ for the first integral I_1, we get

$$|I_1| \leq \int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\rho\alpha+\rho-1}|\phi''(\epsilon^\rho\theta_1^\rho + (1-\epsilon^\rho)\theta_2^\rho)|d\epsilon = \frac{1}{\rho}\int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^{\rho\alpha}|\phi''(\epsilon^\rho\theta_1^\rho + (1-\epsilon^\rho)\theta_2^\rho)|d\epsilon^\rho \tag{2. 17}$$

$$\leq \frac{1}{\rho}\left[\frac{1}{p}\int_0^{\sqrt[\rho]{\frac{1}{2}}} (\epsilon^\rho)^{\alpha p}\, d\epsilon^\rho + \frac{1}{q}\int_0^{\sqrt[\rho]{\frac{1}{2}}} |\phi''(\epsilon^\rho\theta_1^\rho + (1-\epsilon^\rho)\theta_2^\rho)|^q d\epsilon^\rho\right]$$

$$\leq \frac{1}{\rho}\left[\frac{1}{p\,(\alpha p+1)\,2^{\alpha p+1}} + \frac{1}{q}\left(\begin{array}{c} |\phi''(\theta_1^\rho)|^q \int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^\rho d\epsilon^\rho + |\phi''(\theta_1^\rho)|^q \int_0^{\sqrt[\rho]{\frac{1}{2}}} (1-\epsilon^\rho)\, d\epsilon^\rho \\ -F(\theta_1^\rho - \theta_2^\rho)\int_0^{\sqrt[\rho]{\frac{1}{2}}} \epsilon^\rho(1-\epsilon^\rho)d\epsilon^\rho \end{array}\right)\right]$$

$$= \frac{1}{\rho}\left[\frac{1}{p\,(\alpha p+1)\,2^{\alpha p+1}} + \frac{1}{q}\left(\frac{|\phi''(\theta_1^\rho)|^q + 3\,|\phi''(\theta_1^\rho)|^q}{8} - \frac{F(\theta_1^\rho - \theta_2^\rho)}{12}\right)\right].$$

So, we have

$$|I_1| \leq \frac{1}{p\,(\alpha p+1)\,\rho 2^{\alpha p+1}} + \frac{1}{\rho q}\left(\frac{|\phi''(\theta_1^\rho)|^q + 3\,|\phi''(\theta_1^\rho)|^q}{8} - \frac{F(\theta_1^\rho - \theta_2^\rho)}{12}\right).$$

Similarly, for the second integral, we can get

$$|I_2| \leq \frac{1}{p\,(\alpha p+1)\,\rho 2^{\alpha p+1}} + \frac{1}{\rho q}\left[\frac{3\,|\phi''(\theta_1^\rho)|^q + |\phi''(\theta_1^\rho)|^q}{8} - \frac{F(\theta_1^\rho - \theta_2^\rho)}{12}\right].$$

By summing last inequalities, we have

$$|I_1| + |I_2| \leq \frac{1}{2^{\alpha p+1}p\,(\alpha p+1)\,\rho} + \frac{1}{\rho q}\left[\frac{|\phi''(\theta_1^\rho)|^q + |\phi''(\theta_2^\rho)|^q}{2} - \frac{F(\theta_1^\rho - \theta_2^\rho)}{6}\right].$$

Taking into account the last inequality, the final result follows from (2. 17). □

Corollary 2.15. *In (2. 15) if we take $\rho = 1$ in (2. 15), then we obtained the inequality*

$$\left|\frac{\Gamma(\alpha)}{2^{2-\alpha}(\theta_2-\theta_1)^{\alpha-1}}\left[J^{\alpha-1}_{(\frac{\theta_1+\theta_2}{2})+}\phi(\theta_2)+J^{\alpha-1}_{(\frac{\theta_1+\theta_2}{2})-}\phi(\theta_1)\right]-\phi\left(\frac{\theta_1+\theta_2}{2}\right)\right|$$
$$\leq \frac{(\theta_2-\theta_1)^2}{\alpha 2^{2-\alpha}}\cdot(\nu+\mathbf{P}),$$

where

$$\mathbf{P}=\frac{1}{q}\left[\frac{|\phi''(\theta_1)|^q+|\phi''(\theta_2)|^q}{2}-\frac{F(\theta_1-\theta_2)}{6}\right],$$
$$\nu=\frac{1}{p}\left(\frac{1}{(\alpha p+1)2^{\alpha p+1}}\right).$$

3. Result for Trapezoid Type Inequality

Lemma 3.1. *Let $\theta_1,\theta_2\in\mathbb{R}, 0\leq\theta_1<\theta_2, \rho>0$, $\phi:[\theta_1^\rho,\theta_2^\rho]\to\mathbb{R}$ and $\phi\in C^2\ (\theta_1^\rho,\theta_2^\rho)$. For all $\alpha>1$ if the fractional integrals exist, then the following equality is holds*

$$\frac{\phi(\theta_1^\rho)+\phi(\theta_2^\rho)}{2}-\frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{2(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{U}=\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{2}(I_1+I_2), \tag{3. 18}$$

where

$$\mathbf{U}=\frac{\rho\,(\alpha+1)}{\theta_2^\rho-\theta_1^\rho}\left[{}^\rho I^\alpha_{(\theta_1^\rho)+}\phi(\theta_2^\rho\ -\ {}^\rho I^\alpha_{(\theta_2^\rho)-}\phi(\theta_1^\rho)\right]-\left[{}^\rho I^{\alpha-1}_{(\theta_1^\rho)+}\phi(\theta_2^\rho)+\ {}^\rho I^{\alpha-1}_{(\theta_2^\rho)-}\phi(\theta_1^\rho)\right],$$

$$I_1 = \int_0^1 \epsilon^{\alpha\rho+\rho-1}(1-\epsilon^\rho)\phi''(\theta_1^\rho\epsilon^\rho+(1-\epsilon^\rho)\theta_2^\rho)\,d\epsilon,$$
$$I_2 = \int_0^1 \epsilon^{2\rho-1}(1-\epsilon^\rho)^\alpha\phi''(\theta_1^\rho\epsilon^\rho+(1-\epsilon^\rho)\theta_2^\rho)\,d\epsilon\,.$$

Proof. In the integral I_1, at the beginning, we make the change of variables: $\epsilon^\rho = z$, And then, integrating twice by parts, we get:

$$\begin{aligned}
I_1 &= \frac{1}{\rho}\int_0^1 z^{\alpha}\,(1-z)\phi''(z\theta_1^{\rho}+(1-z)\theta_2^{\rho})\,dz\\
&= -\frac{1}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})}\int_0^1 (\alpha z^{\alpha-1}-(\alpha+1)z^{\alpha})\phi'(z\theta_1^{\rho}+(1-z)\,\theta_2^{\rho})\,dz\\
&= -\frac{1}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})}\left[\frac{\alpha z^{\alpha-1}-(\alpha+1)z^{\alpha}}{\theta_2^{\rho}-\theta_1^{\rho}}\phi(z\theta_1^{\rho}+(1-z)\,\theta_2^{\rho})|_0^1\right.\\
&-\frac{\alpha\,(\alpha-1)}{\theta_2^{\rho}-\theta_1^{\rho}}\int_0^1 z^{\alpha-2}\phi(z\theta_1^{\rho}+(1-z)\,\theta_2^{\rho})\,dz\\
&\left.+\frac{(\alpha+1)\,\alpha}{\theta_2^{\rho}-\theta_1^{\rho}}\int_0^1 z^{\alpha-1}\phi(z\theta_1^{\rho}+(1-z)\,\theta_2^{\rho})\,dz\right]\\
&= \frac{1}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})^2}\phi(\theta_1^{\rho})+\frac{\alpha\,(\alpha-1)}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})^2}\int_0^1 z^{\alpha-2}\phi(z\theta_1^{\rho}+(1-z)\,\theta_2^{\rho})\,dz\\
&-\frac{(\alpha+1)\,\alpha}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})^2}\int_0^1 z^{\alpha-1}\phi(z\theta_1^{\rho}+(1-z)\,\theta_2^{\rho})\,dz
\end{aligned}$$

switching back to the variable ϵ, we get

$$\begin{aligned}
I_1 &= \frac{1}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})^2}\phi(\theta_1^{\rho})+\frac{\alpha\,(\alpha-1)}{(\theta_2^{\rho}-\theta_1^{\rho})^2}\int_0^1 \epsilon^{\rho(\alpha-2)}\phi(\epsilon^{\rho}\theta_1^{\rho}+(1-\epsilon^{\rho})\,\theta_2^{\rho})\epsilon^{\rho-1}\,d\epsilon\\
&-\frac{(\alpha+1)\,\alpha}{(\theta_2^{\rho}-\theta_1^{\rho})^2}\int_0^1 \epsilon^{\rho(\alpha-1)}\phi(\epsilon^{\rho}\theta_1^{\rho}+(1-\epsilon^{\rho})\,\theta_2^{\rho})\,\epsilon^{\rho-1}d\epsilon.
\end{aligned}$$

In the resulting integrals, we make the change of variables: $\epsilon^{\rho}\theta_1^{\rho}+(1-\epsilon^{\rho})\,\theta_2^{\rho}=v^{\rho}\implies \epsilon^{\rho}=\frac{\theta_2^{\rho}-v^{\rho}}{\theta_2^{\rho}-\theta_1^{\rho}}$ and $\rho\epsilon^{\rho-1}d\epsilon=-\frac{\rho v^{\rho-1}dv}{\theta_2^{\rho}-\theta_1^{\rho}}$.

When $\epsilon^{\rho}=0\implies v^{\rho}=\theta_2^{\rho}$ and $v=\theta_2$, similarly $\epsilon^{\rho}=1\implies v^{\rho}=\theta_1^{\rho}\implies v=\theta_1$.

$$\begin{aligned}
I_1 &= \frac{1}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})^2}\phi(\theta_1^{\rho})+\frac{\alpha\,(\alpha-1)}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})^2}\int_{\theta_1^{\rho}}^{\theta_2^{\rho}}\left(\frac{\theta_2^{\rho}-v^{\rho}}{\theta_2^{\rho}-\theta_1^{\rho}}\right)^{\alpha-2}\phi(v^{\rho})\frac{\rho v^{\rho-1}dv}{\theta_2^{\rho}-\theta_1^{\rho}}\\
&-\frac{(\alpha+1)\,\alpha}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})^2}\int_{\theta_1^{\rho}}^{\theta_2^{\rho}}\left(\frac{\theta_2^{\rho}-v^{\rho}}{\theta_2^{\rho}-\theta_1^{\rho}}\right)^{\alpha-1}\phi(v^{\rho})\,\frac{\rho v^{\rho-1}dv}{\theta_2^{\rho}-\theta_1^{\rho}}\\
&= \frac{1}{\rho\,(\theta_2^{\rho}-\theta_1^{\rho})^2}\phi(\theta_1^{\rho})+\frac{\alpha\,(\alpha-1)\,\Gamma(\alpha-1)\rho^{\alpha-2}}{(\theta_2^{\rho}-\theta_1^{\rho})^{\alpha+1}}\,{}^{\rho}I^{\alpha-1}_{(\theta_1^{\rho})+}\phi(\theta_2^{\rho})\\
&-\frac{(\alpha+1)\,\alpha\Gamma(\alpha)\rho^{\alpha-1}}{(\theta_2^{\rho}-\theta_1^{\rho})^{\alpha+2}}\,{}^{\rho}I^{\alpha}_{(\theta_1^{\rho})+}\phi(\theta_2^{\rho}).
\end{aligned}$$

Finally, for the integral I_1 we got

$$I_1 = \frac{1}{\rho\left(\theta_2^\rho - \theta_1^\rho\right)^2}\phi(\theta_1) + \frac{\Gamma(\alpha+1)\rho^{\alpha-2}}{\left(\theta_2^\rho - \theta_1^\rho\right)^{\alpha+1}}\, {}^\rho I^{\alpha-1}_{(\theta_1^\rho)+}\phi(\theta_2^\rho) - \frac{\Gamma(\alpha+2)\rho^{\alpha-1}}{\left(\theta_2^\rho - \theta_1^\rho\right)^{\alpha+2}}\, {}^\rho I^{\alpha}_{(\theta_1^\rho)+}\phi(\theta_2^\rho). \tag{3. 19}$$

Similarly, for the second integral, we get:

$$I_2 = \frac{1}{\rho\left(\theta_2^\rho - \theta_1^\rho\right)^2}\phi(\theta_2^\rho) + \frac{\Gamma(\alpha+1)\rho^{\alpha-2}}{\left(\theta_2^\rho - \theta_1^\rho\right)^{\alpha+1}}\, {}^\rho I^{\alpha-1}_{(\theta_2^\rho)-}\phi(\theta_1^\rho) - \frac{\Gamma(\alpha+2)\rho^{\alpha-1}}{\left(\theta_2^\rho - \theta_1^\rho\right)^{\alpha+2}}\, {}^\rho I^{\alpha}_{(\theta_2^\rho)-}\phi(\theta_1^\rho). \tag{3. 20}$$

We add (3. 19) and (3. 20) and multiply both sides of the resulting equality by the expression $\frac{\rho(\theta_2^\rho - \theta_1^\rho)^2}{2}$, we get (3. 18). Proof is completed. □

Remark 3.2. *If we take $\rho = 1$ in* (3. 18)*, we obtained the equality proved by B. Bayraktar [see* [3] *Lemma 3.1]*

Remark 3.3. *If we take $\rho = 1$ and $\alpha = 2$ in* (3. 18)*, we obtained the equality*

$$\frac{\phi(\theta_1) + \phi(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2}\phi(x)dx = \frac{(\theta_2 - \theta_1)^2}{2}(I_1 + I_2),$$

where

$$I_1 = \int_0^1 \epsilon^2(1-\epsilon)\phi''(\epsilon\theta_1 + (1-\epsilon)\theta_2)d\epsilon \text{ and } I_2 = \int_0^1 \epsilon(1-\epsilon)^2\phi''(\epsilon\theta_1 + (1-\epsilon)\theta_2)d\epsilon.$$

Theorem 3.4. *Let $\phi : [\theta_1^\rho, \theta_2^\rho\ ] \to \mathbb{R}$ and $\phi \in C^2\ (\theta_1^\rho, \theta_2^\rho)$ with $0 \le \theta_1 < \theta_2$. If $|\phi''|$ is $F-$convex on $[\theta_1^\rho, \theta_2^\rho\ ]$, then the inequality*

$$\left|\frac{\phi(\theta_1^\rho) + \phi(\theta_2^\rho)}{2} - \frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{2(\theta_2^\rho - \theta_1^\rho)^{\alpha-1}}\cdot \mathbf{U}\right| \le \frac{(\theta_2^\rho - \theta_1^\rho)^2}{2\,(\alpha+2)\,(\alpha+3)}\left[\frac{\alpha+3}{\alpha+1}\left(|\phi''(\theta_1^\rho)| + |\phi''(\theta_2^\rho)|\right) - \frac{4F(\theta_1^\rho - \theta_2^\rho)}{\alpha+4}\right] \tag{3. 21}$$

holds. Here $\rho > 0$ and $\mathbf{U}$ is defined above in Lemma 3.1.

Proof. From (3. 18) and the properties of the module, taking into account that the function $|\phi''|$ is $F-$convex for I_1, we get

$$\begin{aligned} |I_1| &\leq \int_0^1 \epsilon^{\rho\alpha+\rho-1}(1-\epsilon^\rho)\,|\phi''(\theta_1^\rho\epsilon^\rho+(1-\epsilon^\rho)\theta_2^\rho)|\,d\epsilon \\ &= \frac{1}{\rho}\int_0^1 z^\alpha(1-z)\,|\phi''(\theta_1^\rho z+(1-z)\theta_2^\rho)|\ dz \\ &\leq \frac{1}{\rho}\left[|\phi''(\theta_1^\rho)|\int_0^1 z^{\alpha+1}(1-z)dz+|\phi''(\theta_2^\rho)|\int_0^1 z^\alpha(1-z)^2dz \right. \\ &\qquad\left. -F(\theta_1^\rho-\theta_2^\rho)\int_0^1 z^{\alpha+1}(1-z)^2dz\right] \\ &= \frac{1}{\rho}\left[\frac{|\phi''(\theta_1^\rho)|}{(\alpha+2)(\alpha+3)}+\frac{2\,|\phi''(\theta_2^\rho)|}{(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{2F(\theta_1^\rho-\theta_2^\rho)}{(\alpha+2)(\alpha+3)(\alpha+4)}\right] \\ &= \frac{1}{\rho(\alpha+2)(\alpha+3)}\left[|\phi''(\theta_1^\rho)|+\frac{2\,|\phi''(\theta_2^\rho)|}{\alpha+1}-\frac{2F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right]. \end{aligned}$$

Similarly, for the second integral, we can write

$$|I_2| \leq \frac{1}{\rho(\alpha+2)(\alpha+3)}\left[\frac{2\,|\phi''(\theta_1^\rho)|}{\alpha+1}+|\phi''(\theta_2^\rho)|-\frac{2F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right].$$

By summing last inequalities, we get

$$|I_1|+|I_2| \leq \frac{1}{\rho(\alpha+2)(\alpha+3)}\left[\frac{\alpha+3}{\alpha+1}\left(|\phi''(\theta_1^\rho)|+|\phi''(\theta_2^\rho)|\right)-\frac{4F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right]. \quad (3.\ 22)$$

We multiply both sides of inequality (3. 22) by the expression $\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{2}$, we get:

$$\frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{2}\left(|I_1|+|I_2|\right) \leq \frac{(\theta_2^\rho-\theta_1^\rho)^2}{2(\alpha+2)(\alpha+3)}\left[\frac{\alpha+3}{\alpha+1}\left(|\phi''(\theta_1^\rho)|+|\phi''(\theta_2^\rho)|\right)-\frac{4F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right].$$

Taking into account the last inequality, the validity of (3. 21) follows. The proof is complete. □

Remark 3.5. *In (3. 21) if we take $F(x)=0$ and $\rho=1$ we obtained the inequality proved by B. Bayraktar [see* [3] *Theorem 3.1 for $s=1$ and $m=1$]*

Corollary 3.6. *In (3. 21) if we choose $\alpha=2$, $\rho=1$ and $F(x)=0$, then we obtain inequality for the convex functions:*

$$\left|\frac{\phi(\theta_1)+\phi(\theta_2)}{2}-\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\phi(x)dx\right| \leq \frac{(\theta_2-\theta_1)^2}{24}\left[|\phi''(\theta_1)|+|\phi''(\theta_2)|\right].$$

This inequality for convex functions obtained in [18]*(Proposition* 2 *) and* [3] *(Corollari* 3.1*).*

Theorem 3.7. *Let $\phi:[\theta_1^\rho,\theta_2^\rho\,]\to\mathbb{R}$ and $\phi\in C^2\left(\theta_1^\rho,\theta_2^\rho\right)$ with $0\leq\theta_1<\theta_2$. $|\phi''|^q$ is $F-$convex on $[\theta_1^\rho,\theta_2^\rho\,]$, then for all $\alpha,q>1$ the inequality:*

$$\left|\frac{\phi(\theta_1^\rho)+\phi(\theta_2^\rho)}{2}-\frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{2(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{U}\right| \leq \frac{(\theta_2^\rho-\theta_1^\rho)^2}{2(\rho+1)^{\frac{1}{p}}}\cdot\mathbf{G} \quad (3.\ 23)$$

*is holds. Here $\rho > 0$ and $\mathbf{U}$ **is** defined above in Lemma 3.1,*

$$\mathbf{G} = \left[\frac{1}{(2+q\alpha)(3+q\alpha)}\right]^{\frac{1}{q}} \left\{ \left[\frac{2|\phi''(\theta_1^\rho)|^q}{1+q\alpha} + |\phi''(\theta_2^\rho)|^q - \frac{2F(\theta_1^\rho - \theta_2^\rho)}{4+q\alpha}\right]^{\frac{1}{q}} + \left[|\phi''(\theta_1^\rho)|^q + \frac{2|\phi''(\theta_2^\rho)|^q}{1+q\alpha} - \frac{2F(\theta_1^\rho - \theta_2^\rho)}{4+q\alpha}\right]^{\frac{1}{q}} \right\}.$$

Proof. From (3. 18) and by using triangular inequality, we obtained:

$$\left|\frac{\phi(\theta_1^\rho)+\phi(\theta_2^\rho)}{2} - \frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{2(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}} \cdot \mathbf{U}\right| \le \frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{2}(|I_1|+|I_2|)\,. \tag{3. 24}$$

By using the Hölder integral inequality and since $|\phi''|^q$ is $F-$convex function for the $|I_1|$, we obtain

$$\begin{aligned}
|I_1| &= \left|\int_0^1 \epsilon^\rho(1-\epsilon^\rho)^\alpha \epsilon^{\rho-1}\phi''(\epsilon^\rho\theta_1^\rho + (1-\epsilon^\rho)\theta_2^\rho)d\epsilon\right| \\
&= \frac{1}{\rho}\left|\int_0^1 (\epsilon^\rho)^{\frac{1}{p}}(\epsilon^\rho)^{\frac{1}{q}}(1-\epsilon^\rho)^\alpha\ \phi''(\epsilon^\rho\theta_1^\rho + (1-\epsilon^\rho)\theta_2^\rho)d\epsilon^\rho\right| \\
&\le \frac{1}{\rho}\int_0^1 (\epsilon^\rho)^{\frac{1}{p}}(\epsilon^\rho)^{\frac{1}{q}}(1-\epsilon^\rho)^\alpha\ |\phi''(\epsilon^\rho\theta_1^\rho + (1-\epsilon^\rho)\theta_2^\rho)|\,d\epsilon^\rho \\
&\le \frac{1}{\rho}\left(\int_0^1 \epsilon^\rho d\epsilon^\rho\right)^{\frac{1}{p}}\left[|\phi''(\theta_1^\rho)|^q\int_0^1 (\epsilon^\rho)^2(1-\epsilon^\rho)^{\alpha q}d\epsilon^\rho \right. \\
&\quad \left. +|\phi''(\theta_2^\rho)|^q\int_0^1 \epsilon^\rho(1-\epsilon^\rho)^{\alpha q+1}d\epsilon^\rho - F(\theta_1^\rho-\theta_2^\rho)\int_0^1 (\epsilon^\rho)^2(1-\epsilon^\rho)^{\alpha q+1}d\epsilon^\rho\right]^{\frac{1}{q}}
\end{aligned}$$

or

$$|I_1| \le \frac{1}{\rho(\rho+1)^{\frac{1}{p}}}\left[B(3,1+q\alpha)|\phi''(\theta_1^\rho)|^q + B(2,2+q\alpha)|\phi''(\theta_2^\rho)|^q - F(\theta_1^\rho-\theta_2^\rho)B(3,\alpha q+2)\right]^{\frac{1}{q}}.$$

By using properties of Euler's Beta and Gamma functions, we get

$$|I_1| \le \frac{1}{\rho(\rho+1)^{\frac{1}{p}}}\left[\frac{1}{(2+q\alpha)(3+q\alpha)}\right]^{\frac{1}{q}}\left[\frac{2|\phi''(\theta_1^\rho)|^q}{1+q\alpha} + |\phi''(\theta_2^\rho)|^q - \frac{2F(\theta_1^\rho-\theta_2^\rho)}{4+q\alpha}\right]^{\frac{1}{q}}. \tag{3. 25}$$

Similarly from $|I_2|$, we have

$$|I_2| \le \frac{1}{\rho(\rho+1)^{\frac{-1}{p}}}\left[\frac{1}{(2+q\alpha)(3+q\alpha)}\right]^{\frac{1}{q}}\left[|\phi''(\theta_1^\rho)|^q + \frac{2|\phi''(\theta_2^\rho)|^q}{1+q\alpha} - \frac{2F(\theta_1^\rho-\theta_2^\rho)}{4+q\alpha}\right]^{\frac{1}{q}}. \tag{3. 26}$$

Adding inequalities (3. 25) and(3. 26), we get

$$|I_1| + |I_2| \le \frac{1}{\rho(\rho+1)^{\frac{1}{p}}}\cdot\mathbf{H}.$$

Put this value in (3. 24) inequality, we get final result. □

Corollary 3.8. *In (3. 23) if we take $\rho = 1$, we obtained the inequality*

$$\left|\frac{\phi(\theta_1)+\phi(\theta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha-1}}\cdot \mathbf{U}\right| \leq \frac{(\theta_2-\theta_1)^2}{2^{1+\frac{-1}{p}}}\cdot \mathbf{G}, \tag{3. 27}$$

where

$$\mathbf{U} = \frac{\alpha+1}{\theta_2-\theta_1}\left[I^{\alpha}_{\theta_1^+}\phi(\theta_2) \;-\; I^{\alpha}_{\theta_2^-}\phi(\theta_1)\right] - \left[I^{\alpha-1}_{\theta_1^+}\phi(\theta_2) + \; I^{\alpha-1}_{\theta_2^-}\phi(\theta_1)\right],$$

$$\mathbf{G} = \left[\frac{1}{(2+q\alpha)\,(3+q\alpha)}\right]^{\frac{1}{q}}\left\{\left[\frac{2|\phi''(\theta_1)|^q}{1+q\alpha} + |\phi''(\theta_2)|^q - \frac{2F(\theta_1-\theta_2)}{4+q\alpha}\right]^{\frac{1}{q}} + \left[|\phi''(\theta_1)|^q + \frac{2|\phi''(\theta_2)|^q}{1+q\alpha} - \frac{2F(\theta_1-\theta_2)}{4+q\alpha}\right]^{\frac{1}{q}}\right\}.$$

Corollary 3.9. *In (3. 23) if we take $\rho = 1$ and $F(x) = c|x|^2$ in (3. 23) we obtained the inequality for strongly convex function:*

$$\left|\frac{\phi(\theta_1)+\phi(\theta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha-1}}\cdot \mathbf{U}\right| \leq \frac{(\theta_2-\theta_1)^2}{2^{1+\frac{1}{p}}}\cdot \mathbf{G}, \tag{3. 28}$$

where U defined above in Corollary 3.8 and

$$\mathbf{G} = \left[\frac{1}{(2+q\alpha)\,(3+q\alpha)}\right]^{\frac{1}{q}}\left\{\left[\frac{2|\phi''(\theta_1)|^q}{1+q\alpha} + |\phi''(\theta_2)|^q - \frac{2c(\theta_1-\theta_2)^2}{4+q\alpha}\right]^{\frac{1}{q}} + \left[|\phi''(\theta_1)|^q + \frac{2|\phi''(\theta_2)|^q}{1+q\alpha} - \frac{2c(\theta_1-\theta_2)^2}{4+q\alpha}\right]^{\frac{1}{q}}\right\}.$$

Remark 3.10. *In (3. 23) if we take $\rho = 1$ and $F(x) = 0$, we obtained the inequality proved by B. Bayraktar [see* [3] *Theorem* 3.3 *for $s = 1$ and $m = 1$]*

Theorem 3.11. *Let $\phi : [\theta_1^\rho, \theta_2^\rho\,] \to \; [0, +\infty)$ and $\phi \in C^2\;(\theta_1^\rho\;, \theta_2^\rho)$ with $0 \leq \theta_1 < \theta_2$. $|\phi''|^q$ is a $F-$convex on $[\theta_1^\rho, \theta_2^\rho\,]$, with $\epsilon^\rho \in (0, 1)$, then for all $\alpha, q > 1$ the inequality*

$$\left|\frac{\phi(\theta_1^\rho)+\phi(\theta_2^\rho)}{2} - \frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{2(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot \mathbf{U}\right| \leq \frac{(\theta_2^\rho-\theta_1^\rho)^2}{2p}\left(1 + \;\omega\cdot \mathbf{G_1}\right) \tag{3. 29}$$

holds. Here $\rho > 0$ and $\mathbf{U}$ is defined above in Lemma 3.1,

$$\mathbf{G_1} = \frac{\alpha q+3}{1+q\alpha}\left(|\phi''(\theta_1^\rho)|^q + |\phi''(\theta_2^\rho)|^q\right) - \frac{4F(\theta_1^\rho-\theta_2^\rho)}{4+q\alpha},$$
$$\omega = \frac{1}{(\alpha q+2)(\alpha q+3)}.$$

Proof. From (3. 18) and by using triangular inequality, we obtained:

$$\left|\frac{\phi(\theta_1^\rho)+\phi(\theta_2^\rho)}{2} - \frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{2(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot \mathbf{U}\right| \leq \frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{2}\left(|I_1| + |I_2|\right). \tag{3. 30}$$

And by using the well know Young integral inequality and since $|\phi''|^q$ is a $F-$convex function, we get

$$\begin{aligned}|I_1| &= \frac{1}{\rho}\left|\int_0^1 \epsilon^\rho(1-\epsilon^\rho)^\alpha \phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon^\rho\right| \\ &\le \frac{1}{\rho}\int_0^1 \epsilon^{\frac{\rho}{p}}\epsilon^{\frac{\rho}{q}}(1-\epsilon^\rho)^\alpha\left|\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)\right|d\epsilon^\rho \\ &\le \frac{1}{\rho p}\left(\int_0^1 (\epsilon^{\frac{\rho}{p}})^p d\epsilon^\rho\right)+\frac{1}{\rho q}\left[|\phi''(\theta_1^\rho)|^q\int_0^1 \epsilon^{2\rho}(1-\epsilon^\rho)^{\alpha q}d\epsilon^\rho\right. \\ &\quad \left.+|\phi''(\theta_2^\rho)|^q\int_0^1 \epsilon^\rho(1-\epsilon^\rho)^{\alpha q+1}d\epsilon^\rho - F(\theta_1^\rho-\theta_2^\rho)\int_0^1 \epsilon^\rho(1-\epsilon^\rho)^{\alpha q}\epsilon^\rho(1-\epsilon^\rho)d\epsilon\right],\end{aligned}$$

or by using Euler's Beta and Gamma functions and its properties, we have

$$\begin{aligned}|I_1| &\le \frac{1}{2\rho p} \qquad (3.\ 31) \\ &\quad + \frac{1}{\rho q}\left[B(3,1+q\alpha)|\phi''(\theta_1^\rho)|^q + |\phi''(\theta_2^\rho)|^q B(2,2+q\alpha) - F(\theta_1^\rho-\theta_2^\rho)B\left(3,\alpha q+2\right)\right] \\ &= \frac{1}{2\rho p}+\frac{1}{\rho q}\left[\frac{1}{(2+q\alpha)\,(3+q\alpha)}\right]\left[\frac{2|\phi''(\theta_1^\rho)|^q}{1+q\alpha}+|\phi''(\theta_2^\rho)|^q-\frac{2F(\theta_1^\rho-\theta_2^\rho)}{4+q\alpha}\right].\end{aligned}$$

Similarly, for the $|I_2|$, we get the inequalities

$$|I_2| \le \frac{1}{2\rho p}+\frac{1}{\rho q}\left[\frac{1}{(2+q\alpha)\,(3+q\alpha)}\right]\left[|\phi''(\theta_1^\rho)|^q+\frac{2|\phi''(\theta_2^\rho)|^q}{1+q\alpha}-\frac{2F(\theta_1^\rho-\theta_2^\rho)}{4+q\alpha}\right]. \qquad (3.\ 32)$$

Adding inequalities (3. 31) and(3. 32) we get

$$\begin{aligned}|I_1|+|I_2| &\le \frac{1}{p\rho}\left(1+\frac{1}{(2+q\alpha)\,(3+q\alpha)}\cdot\left[\frac{\alpha q+3}{1+q\alpha}\left(|\phi''(\theta_1^\rho)|^q+|\phi''(\theta_2^\rho)|^q\right)-\frac{4F(\theta_1^\rho-\theta_2^\rho)}{4+q\alpha}\right]\right) \\ &\le \frac{1}{p\rho}\left(1+\ \omega\cdot\mathbf{G_1}\right).\end{aligned}$$

Put this value in (3. 30) inequality, we get final result. □

Corollary 3.12. *In (3. 29) if we take $\rho=1$, then we obtained the following inequality:*

$$\left|\frac{\phi(\theta_1)+\phi(\theta_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha-1}}\cdot\mathbf{U}\right| \le \frac{(\theta_2-\theta_1)^2}{2p}\left(1+\ \omega\cdot\mathbf{G_1}\right), \qquad (3.\ 33)$$

where

$$\begin{aligned}\mathbf{U} &= \frac{\alpha+1}{\theta_2-\theta_1}\left[I^\alpha_{\theta_1^+}\phi(\theta_2)\ -\ I^\alpha_{\theta_2^-}\phi(\theta_1)\right]-\left[I^{\alpha-1}_{\theta_1^+}\phi(\theta_2)+\ I^{\alpha-1}_{\theta_2^-}\phi(\theta_1)\right], \\ \mathbf{G_1} &= \frac{\alpha q+3}{1+q\alpha}\left(|\phi''(\theta_1)|^q+|\phi''(\theta_2)|^q\right)-\frac{4F(\theta_1-\theta_2)}{4+q\alpha}, \\ \omega &= \frac{1}{(\alpha q+2)(\alpha q+3)}.\end{aligned}$$

Corollary 3.13. *In (3. 29) if we take $\rho = 1$ and $F(x) = c|x|^2$, then we obtained the following inequality for strongly $F-$convex function:*

$$\left|\frac{\phi(\theta_1)+\phi(\theta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha-1}}\cdot\mathbf{U}\right| \leq \frac{(\theta_2-\theta_1)^2}{2p}\left(1+\ \omega\cdot\mathbf{G_2}\right). \tag{3. 34}$$

where $\mathbf{U}$ and ω defined in (3. 33),

$$\mathbf{G_2} = \frac{\alpha q+3}{1+q\alpha}\left(|\phi''(\theta_1)|^q + |\phi''(\theta_2)|^q\right) - \frac{4c(\theta_1-\theta_2)^2}{4+q\alpha}.$$

Theorem 3.14. *Let $\phi : [\theta_1^\rho, \theta_2^\rho\ ] \to \mathbb{R}$ and $\phi \in C^2\ (\theta_1^\rho\ , \theta_2^\rho)$ with $0 \leq \theta_1 < \theta_2$.If $|\phi''|^q$ is a $F-$convex on $[\theta_1^\rho, \theta_2^\rho\ ]$, with $\epsilon^\rho \in (0,1)$, then for all $\alpha, q > 1$ the inequality*

$$\left|\frac{\phi(\theta_1^\rho)+\phi(\theta_2^\rho)}{2} - \frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{2(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{U}\right| \leq \frac{(\theta_2^\rho-\theta_1^\rho)^2}{2}\cdot\xi\cdot\mathbf{V}, \tag{3. 35}$$

is holds. Here $\rho > 0$ and $\mathbf{U}$ is defined in Lemma 3.1,

$$\mathbf{V} = \left[\frac{2|\phi''(\theta_1^\rho)|^q}{\alpha+1} + |\phi''(\theta_2^\rho)|^q - \frac{2F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right]^{\frac{1}{q}} + \left[|\phi''(\theta_1^\rho)|^q + \frac{2|\phi''(\theta_2^\rho)|^q}{\alpha+1} - \frac{2F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right]^{\frac{1}{q}},$$

$$\xi = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha+1}{\alpha+3}\right)^{\frac{1}{q}}.$$

Proof. From Lemma 3.1 and from triangular inequality, we obtained

$$\left|\frac{\phi(\theta_1^\rho)+\phi(\theta_2^\rho)}{2} - \frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{2(\theta_2^\rho-\theta_1^\rho)^{\alpha-1}}\cdot\mathbf{U}\right| \leq \frac{\rho(\theta_2^\rho-\theta_1^\rho)^2}{2}\left(|I_1|+|I_2|\right). \tag{3. 36}$$

By using the fact that$|\phi''|^q$ is a $F-$convex by using power-mean integral inequality, for the I_1, we get

$$\begin{aligned}
|I_1| &= \left|\int_0^1 \epsilon^\rho(1-\epsilon^\rho)^\alpha\epsilon^{\rho-1}\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon\right| \\
&\leq \frac{1}{\rho}\int_0^1 \epsilon^\rho(1-\epsilon^\rho)^\alpha\left|\phi''(\epsilon^\rho\theta_1^\rho+(1-\epsilon^\rho)\theta_2^\rho)d\epsilon^\rho\right| \\
&\leq \frac{1}{\rho}\left(\int_0^1(\epsilon^\rho(1-\epsilon^\rho)^\alpha d\epsilon^\rho\right)^{1-\frac{1}{q}}\left[|\phi''(\theta_1^\rho)|^q\int_0^1\epsilon^{2\rho}(1-\epsilon^\rho)^\alpha d\epsilon^\rho\right. \\
&\quad\left.+|\phi''(\theta_2^\rho)|^q\int_0^1\epsilon^\rho(1-\epsilon^\rho)^{\alpha+1}d\epsilon^\rho - F(\theta_1^\rho-\theta_2^\rho)\int_0^1\epsilon^\rho(1-\epsilon^\rho)^\alpha\epsilon^\rho(1-\epsilon^\rho)d\epsilon^\rho\right]^{\frac{1}{q}},
\end{aligned}$$

or by using Euler's Beta function and it properties, we have

$$|I_1| \leq \frac{1}{\rho}\left(\frac{1}{(\alpha+1)(\alpha+2)}\right)^{1-\frac{1}{q}} \tag{3. 37}$$
$$\times\left[B(3,\alpha+1)|\phi''(\theta_1^\rho)|^q + B(2,\alpha+2)|\phi''(\theta_2^\rho)|^q - F(\theta_1^\rho-\theta_2^\rho)B\left(3,\alpha+2\right)\right]^{\frac{1}{q}}$$
$$= \frac{1}{\rho}\left(\frac{1}{(\alpha+1)(\alpha+2)}\right)^{1-\frac{1}{q}}\left(\frac{1}{(\alpha+2)\left(\alpha+3\right)}\right)^{\frac{1}{q}}\left[\frac{2|\phi''(\theta_1^\rho)|^q}{\alpha+1} + |\phi''(\theta_2^\rho)|^q - \frac{2F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right]^{\frac{1}{q}}$$
$$= \frac{1}{\rho(\alpha+1)(\alpha+2)}\left(\frac{\alpha+1}{\alpha+3}\right)^{\frac{1}{q}}\left[\frac{2|\phi''(\theta_1^\rho)|^q}{\alpha+1} + |\phi''(\theta_2^\rho)|^q - \frac{2F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right]^{\frac{1}{q}}.$$

Similarly from $|I_2|$, we get

$$|I_2| \leq \frac{1}{\rho(\alpha+1)(\alpha+2)}\left(\frac{\alpha+1}{\alpha+3}\right)^{\frac{1}{q}}\left[|\phi''(\theta_1^\rho)|^q + \frac{2|\phi''(\theta_2^\rho)|^q}{\alpha+1} - \frac{2F(\theta_1^\rho-\theta_2^\rho)}{\alpha+4}\right]^{\frac{1}{q}}. \tag{3. 38}$$

Adding inequalities (3. 37) and(3. 38), we have

$$|I_1| + |I_2| \leq \frac{1}{\rho(\alpha+1)(\alpha+2)}\left(\frac{\alpha+1}{\alpha+3}\right)^{\frac{1}{q}} \cdot \mathbf{V}.$$

Put this value in (3. 36) inequality, we get final result. □

Corollary 3.15. *In (3. 35) if we take $\rho = 1$, then we obtained the inequality*

$$\left|\frac{\phi(\theta_1)+\phi(\theta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha-1}} \cdot \mathbf{U}\right| \leq \frac{(\theta_2-\theta_1)^2}{2} \cdot \xi \cdot \mathbf{V}, \tag{3. 39}$$

where $\mathbf{U}$ *defined in (3.12),*

$$\mathbf{V} = \left[\frac{2|\phi''(\theta_1)|^q}{\alpha+1} + |\phi''(\theta_2)|^q - \frac{2F(\theta_1-\theta_2)}{\alpha+4}\right]^{\frac{1}{q}} + \left[|\phi''(\theta_1)|^q + \frac{2|\phi''(\theta_2)|^q}{\alpha+1} - \frac{2F(\theta_1-\theta_2)}{\alpha+4}\right]^{\frac{1}{q}},$$

$$\xi = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha+1}{\alpha+3}\right)^{\frac{1}{q}}.$$

Corollary 3.16. *In* (3. 39) *if we take $\rho = 1$ and $F(x) = c|x|^2$, then we obtained the inequality for strongly convex function:*

$$\left|\frac{\phi(\theta_1)+\phi(\theta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha-1}} \cdot \mathbf{U}\right| \leq \frac{(\theta_2-\theta_1)^2}{2} \cdot \xi \cdot \mathbf{V}, \tag{3. 40}$$

where $\mathbf{U}$ *defined in (3.12),*

$$\mathbf{V} = \left[\frac{2|\phi''(\theta_1)|^q}{\alpha+1} + |\phi''(\theta_2)|^q - \frac{2c(\theta_1-\theta_2)^2}{\alpha+4}\right]^{\frac{1}{q}}$$
$$+ \left[|\phi''(\theta_1)|^q + \frac{2|\phi''(\theta_2)|^q}{\alpha+1} - \frac{2c(\theta_1-\theta_2)^2}{\alpha+4}\right]^{\frac{1}{q}},$$
$$\xi = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha+1}{\alpha+3}\right)^{\frac{1}{q}}.$$

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