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Inference for the Unit-Gompertz distribution based on record data

Azaz Ahmed*
Department of Statistics,
National College of Business Administration and Economics Lahore,
Govt. Islamia College Railway Road Lahore,
Email: azazpu17@gmail.com

Nuzhat Aftab
College of Statistical Sciences,
University of the Punjab, Q.A Campus, Lahore,
Govt. Commerce College Allama Iqbal Town Lahore,
Email: m.nuzhat.aftab@gmail.com

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Abstract.: Practically record values are applied in situations concerning meteorology, hydrology, and sports events. A keen interest of a sports statistician may be to predict the future record value for a specific event. There are practical situations in which the record values from the available data are lost due to specific reasons. This need for record values stimulates us to construct the probability model that predicts the future record value and provides an estimation procedure in case of censored data. In the present study, the mechanism of sample moments of lower record values using the Type-II censoring is developed by assuming that the record characteristics follow the Unit Gompertz distribution. Utilizing this mechanism, the moments of lower record values are evaluated and tabulated for the specific values of the parameters. These tabulated values are applied to estimate the location and scale parameters of the underlying distribution by the method of ordered least squares. Furthermore, the point prediction and prediction interval for future record values are produced. Finally, the methodology is applied to real-life data to explain the procedure as well as show the effectiveness of forecasting out-of-sample data through the sample moments of the lower record values, depending on the characteristic parameters of the Unit Gompertz distribution.

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1. INTRODUCTION

The data available in meteorology, hydrology, health, sporting, and athletic events are in form of record values. There is always a need to live a safe life to encounter these records concerning the flood, rainfall, earthquake, and several deaths due to COVID-19. The one possible source in the progress of science and technology is the Record of achievements that helps us to predict the activity in different areas. Moreover, deficiencies in handling data or a few months of record instead of a complete year may result in censored data. This censored data may be classified as Type-I and Type-II. If an experiment is stopped after observing a fixed number of observations then it is Type-II censoring while Type-I censoring is placed at a fixed point [14]. This significance of the records either available in censored or complete provokes the requirement to study the probability models of records. Chandler is the first who study the record values in 1952. Then many authors have contributed to this field. For example; the lower record value-based prediction is conferred by [2], the key development in record values and their associated statistics are discussed by [21], [12] developed the recurrence relation among the moments of record values when the underlying distribution of record data is extreme value distribution, similarly to study the more literature in detail based on the record values see, [3], [6], [25], [26], [27], [8], [24] and [23]. Currently, a study on the record values based on the flexible probability model in a Bayesian context is being analyzed by [15]. Using the Inverse Weibull distribution [5] studies the Bayesian analysis of record data following the different loss functions.

The estimation from the censored samples has received considerable attention from the researchers. For example, Balakrishnan et al.[13] applied the progressively Type-II censored scheme when the sample is taken from exponential distribution using ordered observations. The right progressively Type-II censoring is applied to evaluate the recurrence relations for moments when a sample is observed from the Burr distribution by [22]. The same censoring scheme is applied to other probability models to develop the recurrence relations as well as the estimation of the parameters by [9],[10] and [11]. Similar work is discussed by [4] for the doubly truncated generalized exponential distribution. There is extensive literature available on the estimation using the order statistics but nominal in the record statistics context.

In recent years, unit distributions are progressively developing. Practically, the unit distributions are used for quantities ranging from 0 to 1 such as the data of proportions, probabilities, and percentages. These data are widely concentrated for many purposes. For example, in quality control, a quality control manager wants to monitor the proportion of the minerals in Nestle water and the percentage composition of different vitamins in the particular manufacturing food item. A chemist may be interested in the proportion composition of sand, silt, and clay at different water depths. An economist focuses on percentages of expenditure on different commodity groups for consumer demand analysis. There is a long list on the development of unit distributions, for example, [17] constructed the quantile regression model based on Transmuted unit Rayleigh distribution. [16] suggested unit-Chen distribution with quantile regression: [18] introduced the new distribution with two tunning parameters specified on unit interval. The distribution considered in this study developed by [20] namely the unit-Gompertz (UG) distribution. Due to the distinct features of the hazard rate of the UG i.e. upside-down bathtubs and bathtub-shaped, it has preeminence on the existing probability models (exponential distribution, Lindley distribution, gamma distribution, lognormal distribution, Weibull distribution, and Chen distribution) in reliability analysis. Moreover, the UG distribution is increasingly used in many different fields to correctly describe the dynamics of cumulative population growths, like for instance the expansion of the current SARS-CoV-2 outbreak.

Therefore, this study aims at developing the procedure using the sample moments of the lower record values, depending on the characteristics and parameters of the aforementioned distribution. In the case of censored data, the effectiveness of the procedure of estimation is examined by utilizing the Type-II censored moments. One parameter, two parameters, and three parameters of the UG density function are considered. For the record prediction in the future, a prediction interval is evaluated. Finally, an illustrative example is given which explains the procedures developed in this paper.

Using the UG distribution, the basic terminology of record statistics is discussed in the next section.

1.1. **Basic terminology of record statistics.** Consider a sequence of independent and identically continuous random variables $\{Y_j, j \geq 1\}$. A is said to be a lower record value if $Y_j < Y_i$ for all $j < i$. If the inequality sign is reversed then it is known as an upper record value.

The lower record values from the information collected are lost for some reason. For instance, If the initial observations are lost (m_1), then the censoring is said to be left and if the final observations are lost (m_2), then it is known as right censoring. The number of censored observations is fixed. Therefore, let $Y_{L(m_1+1)}, Y_{L(m_1+2)}, Y_{L(m_1+3)}, ..., Y_{L(r-m_2)}$ be the available part of the lower records from the sample size r is Type-II censored sample. If $m_1 = m_2 = 0$, we have a complete sample. The probability density function (PDF) $f(y)$ of the i^{th} and joint PDF of i^{th} and j^{th} records based on Type-II censored sample are, respectively, given by

$$\begin{aligned} f_i(y) &= \frac{1}{\Gamma(i)}[H(y)]^{i-1}f(y) \quad ; \quad -\infty < y < +\infty \\ &\qquad i = m_1+1,\, m_1+2,\, m_1+3,\, \ldots, r-m_2 \\ &\qquad 0 \leq m_1, m_2 \leq r-1 \end{aligned} \tag{1. 1}$$

and

$$\begin{aligned} f_{i,j}(x,y) &= \frac{1}{\Gamma(i)\Gamma(j-i)}[H(y)]^{i-1}[H(x)-H(y)]^{j-i-1}h(y)f(x) \\ &\qquad -\infty << x < y < +\infty \\ &\qquad m_1+1 \leq i \leq j < r-m_2 \\ &\qquad 0 \leq m_1, m_2 \leq r-1 \end{aligned} \tag{1. 2}$$

where $h(y) = -\frac{d}{dy}H(y) = f(y)/F(y)$ and $H(y) = -\ln[F(y)]$ are the hazard rate and cumulative hazard function respectively and $F(y)$ is the cumulative distribution function (CDF).

The density function of the UG distribution and the CDF are respectively as

$$f(y) = \beta y^{-\beta-1}\, e^{-(y^{-\beta}-1)}; 0 < y < 1; \beta > 0, \tag{1. 3}$$

$$F(y) = e^{-(y^{-\beta}-1)}, 0 < y < 1; \beta > 0, \tag{1. 4}$$

The density function of 1. 3 in scale form is

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{-\beta-1} e^{-((\frac{x}{\alpha})^{-\beta}-1)};\ 0 < x < \alpha \tag{1. 5}$$

while the location-scale UG distribution has its density function given by

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x-\theta}{\alpha}\right)^{-\beta-1} e^{-((\frac{x-\theta}{\alpha})^{-\beta}-1)};\ \theta < x < \alpha \tag{1. 6}$$

where β, θ and α are the shape, location and scale parameters, respectively.

2. The moments of the UG distribution with lower record values

This section serves to derive the single and product moment of lower record values from the UG distribution using special functions. The hypergeometric series, which contains numerous other special functions as particular or limiting examples, represents the Gaussian or ordinary hypergeometric function $2F1(a, b, c, z)$ as a special function in mathematics. The details of special functions are given in [1]. These functions are as follows.

The integral $\frac{1}{\Gamma(a)}\int\limits_0^\infty t^{a-1}(1+t)^{b-a-1}e^{-zt} = U(a, b, z)$ is called the hypergeometric function and

The integral $\frac{1}{y^{b-1}\Gamma(a)\Gamma(b-a)}\int\limits_0^y x^{a-1}(1+x)^{-c}(y-x)^{b-a-1}dx = {}_2F_1(a, c, b, -y)$ is called regularized hypergeometric function.

The term single moment is meant the expected value of the random variable over the univariate probability distribution and the product moment is the expected value of the product of random variables over the bivariate probability distribution. To derive the single moment, the probability density function based on lower record values is obtained by using (1.2) into (1.3) and then taking the expected value over the lower record values, we have

$E(Y_{L(i)}) = \frac{\beta}{\Gamma(i)}\int\limits_0^1 y^{-\beta-1}(y^{-\beta-1}-1)^{i-1}e^{-(y^{-\beta}-1)}dy$; $m_1 + 1 \le i \le r - m_2$; $0 \le m_1, m_2 \le r - 1$

Substituting $t = y^{-\beta} - 1$, we have

$E(Y_{L(i)}) = \frac{1}{\Gamma(i)}\int\limits_0^\infty t^{i-1}(1+t)^{\frac{-1}{\beta}}e^{-t}dy$

It can be simplified as,

$$E(Y_{L(i)}) = U(i, 1 - \frac{1}{\beta} + i, 1);\ m_1 + 1 \le i \le r - m_2;\ 0 \le m_1, m_2 \le r - 1 \tag{2. 7}$$

It follows from (1.2) that the product moment of any two record statistics $Y_{L(i)}$ and $Y_{L(i)}$ $(m_1 + 1 \leq i \leq j < r - m_2)$ can be expressed as

$$E(Y_{L(i)}Y_{L(j)}) = \frac{1}{\Gamma(i)\Gamma(j-i)} \int_0^1 \int_y^1 xy\,(x^{-\beta} - 1)^{(i-1)}(y^{-\beta} - x^{-\beta})^{(j-i-1)} \times \beta x^{-\beta-1}\beta y^{-\beta-1}e^{-(y^{-\beta}-1)}dxdy$$

Making the transformation $u = x^{-\beta} - 1$, $v = y^{-\beta} - 1$ and simplifying the resulting terms gives

$$E(Y_{L(i)}Y_{L(j)}) = \frac{1}{\Gamma(i)\Gamma(j-i)} \int_0^\infty (v+1)^{\frac{-1}{\beta}} dv \times \int_0^v u^{(i-1)}(1+u)^{\frac{-1}{\beta}}(v-u)^{(j-i-1)}du \qquad (2.\ 8)$$

On substituting the expression of the regularized hypergeometric function in (2.8) and simplifying the resulting expression as

$$E(Y_{L(i)}Y_{L(j)}) = \int_0^\infty v^{j-1}e^{-v}(1+v)^{\frac{-1}{\beta}}\,{}_2F_1(i, \tfrac{1}{\beta}, j, -v)dv = g(i,j,\beta),\ m_1 + 1 \leq i \leq j < r - m_2 \qquad (2.\ 9)$$

The expression of means, variances, and covariances are evaluated using the (2.7) and (2.9) following the (1.3). These measures are presented in Table 1 for $\beta =$ 1.25, 1.75, and 2.75, $m_1 + 1 \leq i \leq 10 - m_2$ and $m_1 + 1 \leq i \leq j < 10 - m_2$ with $m_1 = m_2 = 0$. The values of the shape parameter are chosen with the concern of asymmetric to negative skewness.

3. Estimation of parameters of UG distribution with Type-II censoring

In this section, the parameters estimation of (1.5) and (1.6) are considered. The parameters are to be estimated by applying generalized least-squares theory when the available data in record values for the scale case and location and scale case, using Type-II singly and doubly censoring. The beauty of the generalized least-squares method is providing the best linear unbiased estimates (BLUEs).

3.1. **The Scale Case.** Let $X_{L(m_1+1)}, X_{L(m_2+2)}, ..., X_{L(r-m_2)}$ be the first $r - m_1 - m_2$ lower record values from the UG distribution with PDF $f(x/\alpha)$ and let $Y_{L(i)} = \frac{X_{L(i)}}{\alpha}$ $(m_1 + 1 \leq i \leq r - m_2)$ be the corresponding record values from distribution (reduced form) which is free from the scale parameters. Assume, $\mathbf{y} = (y_{L(m_1+1)}, y_{L(m_1+2)}, ..., y_{L(r-m_2)})^t$, $\mathbf{a} = (a_{m_1+1}, a_{m_2+2}, ..., a_{r-m_2})^t$ $\mathbf{1} = (1, 1, ..., 1)^t$ and$\mathbf{B} = ((\Omega_{i,j}), m_1 + 1 \leq i, j \leq r - m_2$.

Then the BLUE of the scale parameter α is given by

$$\hat{\alpha} = \sum_{i=m_1+1}^{r-m_2} c_i x_{L(i)} = [\frac{\mathbf{a^t B^{-1}}}{\mathbf{a^t B^{-1} a}}]\mathbf{x} \qquad (3.\ 10)$$

Where $c = [\frac{\mathbf{a^t B^{-1}}}{\mathbf{a^t B^{-1} a}}]$ are the coefficients and variance is given by

$$Var(\hat{\alpha}) = [\frac{1}{\mathbf{a^t B^{-1} a}}]\alpha^2. \qquad (3.\ 11)$$

TABLE 1. The means vector and variances covariances matrix for β = 1.25, 1.75 and 2.75

					β=1.25					
$E(Y_{L(i)}) = \mathbf{a}$	0.6537	0.4770	0.3745	0.3088	0.2635	0.2304	0.2052	0.1853	0.1693	0.1560
			$Cov(Y_{L(i)}, Y_{L(j)})B = \Omega_{i,j},\ m_1 + 1 \le i \le j < 10 - m_2$							
j/i	1	2	3	4	5	6	7	8	9	10
1	0.0388									
2	0.0217	0.0267								
3	0.0135	0.0165	0.0164							
4	0.0091	0.0011	0.0109	0.0103						
5	0.0065	0.0079	0.0078	0.0073	0.0067					
6	0.0049	0.0059	0.0058	0.0054	0.0050	0.0046				
7	0.0038	0.0046	0.0045	0.0042	0.0039	0.0036	0.0033			
8	0.0031	0.0037	0.0036	0.0033	0.0031	0.0028	0.0026	0.0024		
9	0.0025	0.0030	0.0029	0.0027	0.0025	0.0023	0.0021	0.0020	0.0018	
10	0.0021	0.0025	0.0024	0.0023	0.0021	0.0019	0.0018	0.0016	0.0015	0.0014
					β=1.75					
$E(Y_{L(i)}) = \mathbf{a}$	0.7308	0.5824	0.4902	0.4276	0.3821	0.3476	0.3202	0.2980	0.2795	0.2638
j/i	1	2	3	4	5	6	7	8	9	10
1	0.0262									
2	0.0158	0.0205								
3	0.0105	0.0135	0.0141							
4	0.0075	0.0096	0.0100	0.0098						
5	0.0056	0.0072	0.0074	0.0073	0.0070					
6	0.0044	0.0056	0.0058	0.0056	0.0054	0.0052				
7	0.0035	0.0045	0.0046	0.0045	0.0043	0.0041	0.0039			
8	0.0029	0.0037	0.0038	0.0037	0.0036	0.0034	0.0032	0.0031		
9	0.0025	0.0031	0.0032	0.0031	0.0030	0.0029	0.0027	0.0026	0.0025	
10	0.0021	0.0027	0.0027	0.0027	0.0026	0.0024	0.0023	0.0022	0.0021	0.0020
					β=2.75					
$E(Y_{L(i)}) = \mathbf{a}$	0.8141	0.7039	0.6310	0.5789	0.5393	0.5080	0.4824	0.4610	0.4427	0.4269
j/i	1	2	3	4	5	6	7	8	9	10
1	0.0139									
2	0.0090	0.0123								
3	0.0063	0.0086	0.0094							
4	0.0047	0.0064	0.0070	0.0071						
5	0.0037	0.0050	0.0054	0.0055	0.0055					
6	0.0030	0.0040	0.0044	0.0046	0.0044	0.0044				
7	0.0025	0.0033	0.0036	0.0037	0.0037	0.0036	0.0035			
8	0.0021	0.0028	0.0031	0.0031	0.0031	0.0030	0.0030	0.0029		
9	0.0018	0.0024	0.0026	0.0027	0.0027	0.0026	0.0026	0.0025	0.0025	
10	0.0016	0.0021	0.0023	0.0024	0.0023	0.0023	0.0022	0.0022	0.0021	0.0021

Evaluating the (2.7) and (2.9) and then using (3.11) we can obtain the estimate of the variance of scale parameter with all possible choices of censoring for sample size 5. Results are presented in Table 2. Each value is multiplied by α^2.

It is examined that left censoring provides a more precise estimator than right censoring for the scale parameter, see Table 2. By increasing the m_1 when m_2 is fixed (or vice-versa) then both estimators lose their efficiency.

TABLE 2. Estimation under Scale-Case

r	m_1	m_2	$Var(\hat{\alpha})$
5	0	0	0.1754
5	0	1	0.1876
5	0	2	0.1997
5	0	3	0.2116
5	1	0	0.2417
5	2	0	0.2727
5	3	0	0.2903
5	1	1	0.2670
5	1	2	0.2938
5	2	1	0.3069

3.2. **The Location and Scale Case.** Let $X_{\mathrm{Lm_1+1)}}$,$X_{L(m_1+2)}$,...,$X_{\mathrm{L(r\text{-}}m_2)}$ be the first $r-m_1-m_2$ lower record values (1.6). Then the BLUE of location and scale parameters are given by

$$\hat{\theta} = \sum_{i=m_1+1}^{r-m_2} c_i X_{L(i)} \quad (3.\ 12)$$

$$\hat{\alpha} = \sum_{i=m_1+1}^{r-m_2} b_i X_{L(i)} \quad (3.\ 13)$$

where $\mathbf{c} = \frac{\mathbf{a^tB^{-1}a1^tB^{-1}-a^tB^{-1}1a^tB^{-1}}}{(\mathbf{a^tB^{-1}a})(\mathbf{1^tB^{-1}1})-(\mathbf{a^tB^{-1}1})^2}$, $\mathbf{b} = \frac{\mathbf{1^tB^{-1}1a^tB^{-1}-1^tB^{-1}a1^tB^{-1}}}{(\mathbf{a^tB^{-1}a})(\mathbf{1^tB^{-1}1})-(\mathbf{a^tB^{-1}1})^2}$, and are the vector of the elements c_i and b_i respectively. The variances and covariances of the above estimators are given by

$$Var\left(\hat{\theta}\right)/\alpha^2 = \frac{\mathbf{a^tB^{-1}a}}{(\mathbf{a^tB^{-1}a})\,(\mathbf{1^tB^{-1}1}) - (\mathbf{a^tB^{-1}1})^2} \quad (3.\ 14)$$

$$Var\left(\hat{\alpha}\right)/\alpha^2 = \frac{\mathbf{1^tB^{-1}1}}{(\mathbf{a^tB^{-1}a})\,(\mathbf{1^tB^{-1}1}) - (\mathbf{a^tB^{-1}1})^2} \quad (3.\ 15)$$

$$Cov\left(\hat{\theta}, \hat{\alpha}\right)/\alpha^2 = -\frac{\mathbf{a^tB^{-1}1}}{(\mathbf{a^tB^{-1}a})\,(\mathbf{1^tB^{-1}1}) - (\mathbf{a^tB^{-1}1})^2} \quad (3.\ 16)$$

By evaluating (2.7) and (2.9) and then using (3.14)-(3.16), we can obtain the estimate of the variance of location and scale parameters and their covariances with all possible choices of censoring for sample size 5. Results are presented in Table 3 using the singly and doubly censored samples. Each value may be multiplied by α^2 From Table 3, it is observed that left censoring provides a more precise estimator than right censoring for the location parameter while for the scale parameter, the situation is reversed. By increasing the m_1 when m_2 is fixed (or vice-versa) then both estimators lose their efficiency.

TABLE 3. Estimation under Location and Scale-Case

r	m_1	m_2	$Var(\hat{\theta})$	$Var(\hat{\alpha})$	$Cov(\hat{\theta},\hat{\theta})$
5	0	0	0.0061	0.3012	0.0277
5	0	1	0.0127	0.3453	0.0447
5	0	2	0.0323	0.4459	0.0891
5	0	3	0.1167	0.8005	0.2621
5	1	0	0.0083	0.5627	0.0515
5	2	0	0.0116	0.8839	0.0841
5	3	0	0.0206	1.6326	0.1661
5	1	1	0.0185	0.7174	0.0913
5	1	2	0.0604	1.2254	0.2371
5	2	1	0.0323	1.4269	0.1902

TABLE 4. Quantiles of the statistic T_r^p with $\beta = 0.75$

r/p	1%	2.5%	5%	10%	90%	95%	97.5%	99%
2	0.0045	0.0112	0.0233	0.0515	5.2667	11.4181	22.3619	56.0393
3	0.0039	0.0107	0.0224	0.0467	2.1166	3.3796	5.4254	9.3270
4	0.0048	0.0100	0.0205	0.0419	1.5148	2.2275	3.1194	4.8191
5	0.0050	0.0106	0.0223	0.0437	1.344	1.9932	2.8237	4.1038
6	0.0047	0.0118	0.0233	0.0468	1.2168	1.7036	2.4405	3.3184

4. PREDICTION OF FUTURE RECORD

This section deal with that how to do the prediction of future record value. Suppose the lower record values $X_{L(1)}$,$X_{L(2)}$,..., $X_{L(r-1)}$ are available. If one has a keen interest to predict the next lower record then the predicted value of this record can be obtained using the best linear unbiased predictor as

$$X^*_{L(r)} = \hat{\theta} + \hat{\alpha} a_r \tag{4. 17}$$

the standard error of (4.17) can be obtained as,

$$S.E(X^*_{L(r)}) = \sqrt{\mathrm{var}(\hat{\theta}) + a_r^2\,\mathrm{var}(\hat{\alpha}) + 2a_r cov(\hat{\theta},\hat{\alpha})},$$

where $\hat{\theta}$ and $\hat{\alpha}$ are the BLUE of θ and α using the first record values. As the interval prediction is more reliable than the point prediction. Therefore, a pivotal quantity is developed to construct the prediction interval for the future record value as,

$$T_r^p = \frac{X_{L(r-1)} - X_{L(r)}}{\hat{\alpha}} \tag{4. 18}$$

The exact sampling distribution of the statistic is difficult to construct. Therefore, a Monte Carlo simulation study is conducted with R-language software in which some percentage points of the statistic T_r^p have been evaluated by generating the $X_(L(r-1))$ values from (1.4) with $\beta = 0.75$ and resulted values are presented in Table 4 for $n = 2(1)6$. These tabulated values are used to calculate the $100(1-p)\%$ prediction intervals for the future record value $X^*_{L(r)}$.

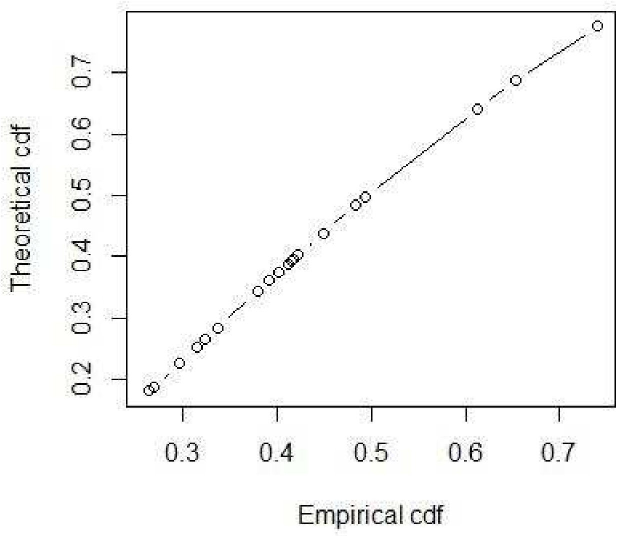

FIGURE 1. the empirical and the theoretical CDF for $\beta = 0.75$

4.1. **Application.** In this section, the methodology is explained by applying real data related to fields of weather phenomena like flood, drought, rainfall, etc. The maximum flood levels over the twenty four-years periods (1890-1969) of the Susquehanna River situated in Pennsylvania were recorded and given in [19]. Due to the small sample size, Figure 1 represents a rough indication of the goodness of fit which is obtained by plotting the empirical CDF vs the CDF of the UG distribution with $\beta = 0.75$. Thus the UG distribution for $\beta = 0.75$ provides an approximately good fit to this data that confirms the applicability of the UG distribution in said phenomena. The first five lower records of maximum flood level are observed as 0.654, 0.613, 0.315, 0.297, 0.269.

For illustration, we consider the first 4 lower record values and predict the 5th lower record as, Using (3.12) and (3.13), we determine the BLUE of θ and α as
$\hat{\theta} = (-0.2874 \times 0.654) - (0.1393 \times 0.613) - (0.3594 \times 0.315) + (1.7852 \times 0.297) = 0.1439$
$\hat{\alpha} = (2.2944 \times 0.654) + (0.5235 \times 0.613) + (1.2304 \times 0.315) - (4.0483 \times 0.297) = 1.0067$
The prediction for the future lower record values as
$X^{*}_{L(r)} = 0.1439 + 1.0067 \times 0.114 = 0.259$
which is close to the observed 5th record value. Furthermore, a one-sided 10% prediction interval is calculated using Table 6 $(0.297, 0.297 - 0.044 \times 1.0067) = (0.297, 0.238)$.

Moreover, historic flood data may contain censored samples because of the deficiencies in recording the flood data, for example, the lower record values of a few years may be lost due to the deficiencies in handling data. Or flood data may contain a few months of record

TABLE 5. The summary result of the application

	Scale-Case			Location and Scale- Case		
Cases	$\hat{\alpha}$	$Var(\hat{\alpha})$	$\hat{\theta}$	$\hat{\alpha}$	$Var(\hat{\theta})$	$Var(\hat{\alpha})$
(i)	1.6855	0.4983	0.1567	0.9735	0.0058	0.2855
(ii)	1.3438	0.3607	0.0218	1.2836	0.0531	0.7348
(iii)	2.0379	1.1325	0.2067	0.5355	0.0033	0.2535
(iv)	1.9848	1.0519	0.0559	1.7094	0.0541	2.0964

instead of a complete year. This incomplete year of the maximum flood is a censored event. Suppose in the first five lower record values of maximum flood level the two observations on either end of the data are censored due to the aforementioned reasons. The estimation of the parameter of (1.5) and (1.6) with known the shape parameter is considered for the following situations (i) complete sample i.e., $m_1 = 0,\ m_2 = 0$ (ii) right censoring i.e., (iii) left censoring i.e., $m_1 = 2,\ m_2 = 0$ (iv) doubly censoring i.e., $m_1 = 1,\ m_2 = 1$ and the result is presented in a summarized way in Table 5.

It is examined that the direction of censoring has a different effect on the estimation of parameters based on lower record values for the data following the UG distribution. In Scale-Case loss of information from the left side has a more adverse effect on the efficiency. In contrast to left and doubling sampling, right censoring increases the estimator's effectiveness. On the other hand, location and scale-case the left censoring provides more efficient estimates as compared to left and double censoring, see Table 5.

5. CONCLUSION

The Unit distributions are widely applied for the data of proportions, probabilities and percentages. These data are concentrated in different fields such as the proportion of the minerals and vitamins in particular food, the proportion composition of sand, silt and clay at different water depths. Among the Unit distributions, the unit-Gompertz (UG) distribution is recently developed by [20] which has superiority over exponential, Lindley, gamma, lognormal, Weibull and Chen in reliability analysis due to its hazard rates. By knowing the importance of record values, we feel a necessity to discuss UG distribution in the record statistic context. Therefore, in this paper, we established the moments by considering the parent distribution as Unit Gompertz. These moments are then utilized to derive the estimator of the parameters of (1.5) and (1.6) in a situation when data on the record values are subject to be censored. Type-II singly and doubly censored record values are considered. It is concluded that for scale-case, right censoring provides a more precise estimator than left censoring. For the location and scale case, for the location parameter left censoring provides a more precise estimator than right censoring while for the scale parameter, the situation is reversed. By increasing the m_1 when m_2 is fixed (or vice-versa) then both estimators lose their efficiency. For the validity of future record value, a prediction interval is constructed by utilizing BLUE. Finally, we present an illustrative example to explain the inference procedures developed in this paper.

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