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The SSC of the Generalised Jahangir's Graph $J_{m,k}$ and its Algebraic Characterizations

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Abstract. In this article, we present important combinatorial and algebraic properties of spanning simplicial complex (SSC) of the generalised Jahangir's graph $J_{m,k}$. We describe the relation to find $f-$vectors associated to $\Delta_s(J_{m,k})$ and determine the Hilbert series for the SR-ring $K\big[\Delta_s(J_{m,k})\big]$. In the end, we present the associated primes of the facet ideal $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$ and the Cohen-Macaulay characterization of the SR-ring of $\Delta_s(J_{m,k})$.

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1. **Introduction**

The notion of the SSC on the set of edges of a connected, finite and simple graph was established by Anwar et al. in [2]. They characterized all spanning trees of unicyclic graph $U_{n,m}$ and computed $h-vector$ and Hilbert series of the SR-ring $K\big[\Delta_s(U_{n,m})\big]$. They showed that the SSC $\Delta_s(U_{n,m})$ is shifted and shellable. The algebraic properties of SSC of some other classes of graph were discussed in [8, 9, 15, 14]. These classes include $r-$cyclic graph $G_{n,r}$, cyclic graph having n edges and r cycles with exactly one common edge between every two consecutive cycles $G^1_{n,r}$, $n-$cyclic graphs $G_{t_1,t_2,\ldots,t_n}$ with a common edge and wheel graph W_n. In [12], Raza et al. explored few important combinatorial and algebraic characterizations of the SSC of Jahangir's graph $\mathcal{J}_{m,2}$. They presented the formula for $f-$vectors, computed Hilbert series of SR-ring $K\big[\Delta_s(\mathcal{J}_{m,2})\big]$ and showed that it is is Cohen Macaulay. Also, the authors proposed the following open problem in [12]. Since computing the spanning trees of a graph is a difficult problem to handle therefore it is obvious that computing SSC of general graph is also a NP-hard problem. This gives the motivation to study the algebraic and combinatorial properties associated to SSC of a general graph i.e, generalised Jahangir's graph.

Open Problem 1.1. *The combinatorial and algebraic properties of SSC of the generalised Jahangir's graph $J_{m,k}$ and results discussed in* [12] *can be studied for any integer $k \geq 1$. The important results of the open problem include:*

- *$f-$vectors of SSC of the generalised Jahangir's graph $J_{m,k}$.*
- *The Hilbert series for the SR-ring $K\big[\Delta_s(J_{m,k})\big]$.*
- *Cohen-Macaulayness of the SR-ring $K\big[\Delta_s(J_{m,k})\big]$.*

The current work carries motivation from the above open problem.

Definition 1.2. [10] *The generalised Jahangir's graph $J_{m,k}$ is comprised of a cycle $C_{m(k+1)}$ having $m(k+1)+1$ vertices where $m \geq 3$ and $k \geq 1$. The cycle $C_{m(k+1)}$ has an extra vertex adjoining to all m vertices of the cycle at span of $k+1$ to each other on the cycle. The edge set of the generalised Jahangir's graph $J_{m,k}$ is*

(1. 1)

$$\mathcal{E}(J_{m,k}) = \{s_{11}, s_{12}, \ldots, s_{1(k+2)}, s_{21}, s_{22}, \ldots, s_{2(k+2)}, \ldots, s_{m1}, s_{m2}, \ldots, s_{m(k+2)}\}$$

The edge set for the cycle C_t where $t \in \{1, 2, \ldots, m-1\}$ of the generalised Jahangir's graph $J_{m,k}$ is $\{s_{t1}, s_{t2}, s_{t3}, \ldots, s_{t(k+2)}, s_{(t+1)1}\}$ and $\{s_{m1}, s_{m2}, s_{m3}, \ldots, s_{m(k+2)}, s_{11}\}$ is the edge set of the cycle C_m. The shared edge between any two successive cycles C_{t-1} and C_t is s_{t1} where $t \in \{1, 2, 3, \ldots, m-1\}$ and the shared edge between the first cycle C_1 and the last cycle C_m is s_{11}.

2. Preliminaries

Some important definitions which are required to make this paper self expatiatory are described in this section. All the graphs considered here are connected, simple and finite graphs.

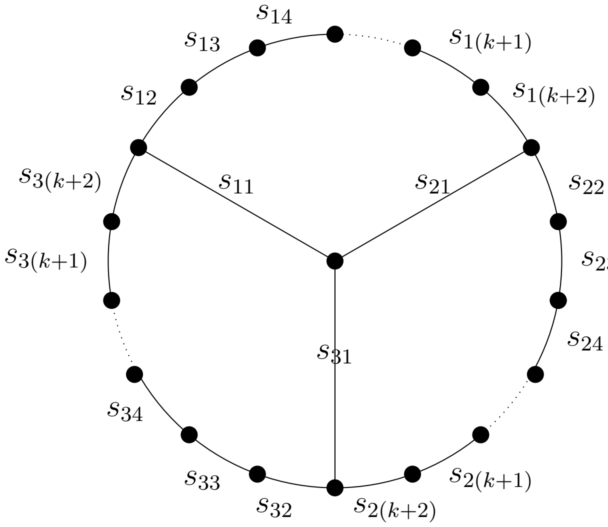
Definition 2.1. [6] A subtree of a graph G having all vertices of the graph is called *spanning tree.* The set having all edge sets of the spanning trees of G is represented by $s(G)$ and we can write;

$$s(G) := \{\mathcal{E}(T_j) \subset \mathcal{E}(G), \text{ where } (T_j) \text{ is a spanning tree of } G\}$$

Lemma 2.2. [12] *The number of edges in a spanning tree of the graph G having m cycles is $|\mathcal{E}(G)| - m$.*

By Lemma 2.2, the spanning trees of the graph $J_{m,k}$ are formed by deleting exactly m edges from the graph following the cutting down method as described below:

Exactly one edge is to be removed from the unshared edges of any cycle of the graph $J_{m,k}$. If a shared edge is deleted from two or more than two neighboring cycles then a new cycle is attained. Only one edge is to be deleted from the remaining unshared edges of newly formed cycle. All shared edges can not be deleted at a time to avoid disconnection in the graph. The spanning set of Jahangir's graph $J_{3,2}$ is given in Appendix A.



The generalised Jahangir's graph $J_{3,k}$
Figure 1

Definition 2.3. [4] A *simplicial complex* (SC) Δ is defined as a collection of the subsets of a finite set $[n] = \{1, 2, 3, \dots, n\}$ such that $\{j\} \in \Delta$ for all $j \in [n]$ and if Δ has a set F then it has all subsets of F along with the null set. A member F of the SC is called its face and its dimension is 1 less than the number of vertices in F i.e. $\dim(F) = |F| - 1$. The maximal faces of a SC are known as its facets. The dimension of a SC is defined as follows:

$$\dim \Delta = \max\{\dim F | F \in \Delta\}$$

A SC Δ having facets $\{F_0, F_1, \dots, F_p\}$ is represented in terms of its facets as follows:

$$\Delta = \left\langle F_0, F_1, \dots, F_p \right\rangle$$

The f–vector of a SC is defined as a $\mathcal{D} + 1$-tuple where $\mathcal{D}$ is the dimension of the SC. In other words the f–vector can be represented as follows:

$$f(\Delta) = (f_0, f_1, \dots, f_{\mathcal{D}})$$

where f_i denotes the number of i–dimensional faces of Δ.

Definition 2.4. [2] Let the collection of the edge sets of all the spanning trees of the graph G be represented by $s(G) = \{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_t\}$. A simplicial complex $\Delta_s(G)$ defined on the edge set $\mathcal{E}(G)$ such that its facets are the members of $s(G)$, is called *spanning simplicial complex (SSC)* of G. Mathematically, it can be expressed as follows:

$$\Delta_s(G) = \left\langle \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_t \right\rangle$$

The SSC of the Jahangir's graph $J_{3,k}$ shown in Figure 1 is given by:

$$\Delta_s(G) = \left\langle s(J_{3,k}) \right\rangle$$

3. Spanning trees of $J_{m,k}$ and its Stanley-Reisner $K\left[\Delta_s(J_{m,k})\right]$

In the following section, a significant characterization of the generalised Jahangir's graph $J_{m,k}$ and its spanning tress $s(J_{m,k})$ is described with its f-vectors and dimension of $\Delta_s(J_{m,k})$. In the end, the Hilbert series of the SR-ring $K\left[\Delta_s(J_{m,k})\right]$ is computed in a Theorem 3.13.

The generalised Jahangir's graph $J_{m,k}$ consists of m consecutive cycles along with few more cycles which are formed by the deletion of the shared edges. If $C_{p_1}, C_{p_2}, \ldots, C_{p_t}$ are the consecutive cycles of the graph $J_{m,k}$, then the new cycle formed by deleting shared edges is represented by $C_{p_1,p_2,\ldots,p_t}$. The cycle $C_{p_1,p_2,\ldots,p_t}$ is obtained by deleting shared edges from the cycles C_{p_i} and $C_{p_{i-1}}$, where $1 \leq i \leq m-1$. Also the cycle $C_{p_1,p_2,\ldots,p_t}$ is formed by removing shared edges from the cycles C_{p_i} and C_{p_1}, where $1 \leq i \leq m$. The number of edges in the new cycles is explained in detail in Example 3.1. The number of distinct edges in the new cycle is denoted by $\mathcal{B}_{p_1,p_2,\ldots,p_t} = \left|C_{p_1,p_2,\ldots,p_t}\right|$. The total count of cycles in the graph $J_{m,k}$ and the cardinality of the edges in these cycles is determined in the following lemma.

Lemma 3.1. *Let $J_{m,k}$ be the generalised Jahangir's graph having successive cycles $C_1, C_2, \ldots, C_m$ of equal length and an outer cycle C_0. Then the total number of cycles in the graph is*

$$\Gamma = m^2 + 1$$

such that

$$\mathcal{B}_{p_1,p_2,\ldots,p_t} = \begin{cases} t(k+1)+2 & 1 \leq t \leq m-1 \\ t(k+1)+1 & t = m \end{cases}$$

Proof. Since the generalised Jahangir's graph $J_{m,k}$ has m successive cycles with new cycles $C_{p_1,p_2,\ldots,p_t}$ attained by removing the shared edges from the neighboring cycles $C_{p_1}, C_{p_2}, \ldots, C_{p_t}$. Therefore, the resulting cycles are $C_{m,1}, C_{m-1,m}, \ldots, C_{2,3}, C_{1,2}$ when one shared edge is removed, $C_{m,1,2}, C_{m-1,m,1}, C_{m-2,m-1,m}, \ldots, C_{1,2,3}$ when two shared edges are removed and in the similar manner we have

$C_{1,2,3,\dots,m}, C_{2,3,4,\dots,m,1}, \dots, C_{m,1,2,\dots,m-1}$. when $m-1$ shared edges are removed, along with an outer cycle C_0 having edge set $\mathcal{E}(J_{m,k}) \setminus \{s_{i1}\}_{i=1}^{m}$. Adding the above cycles in m cycles we get the collection of the cycles of the generalised Jahangir's graph $J_{m,k}$,

$$C_{p_1,p_2,\dots,p_t}, \text{where } p_q \in \{1,2,\dots,m\} \ \& \ 1 \le t \le m,$$

with $p_{q+1} = p_q + 1$ when $p_q \neq m$ and $p_{q+1} = 1$ when $p_q = m$.
By counting simply the cycles $C_{p_1,p_2,\dots,p_t}$ are m in number for $p_t \le m$ when t is fixed, giving the total cycles of the generalised Jahangir's graph $J_{m,k}$ including the outer cycle C_0 equals to $\Gamma = m^2 + 1$. To compute the number of edges in the cycle $C_{p_1,p_2,\dots,p_t}$ we have following cases:

Case(i): When $1 \le t \le m-1$
Since the cycle $C_{p_1,p_2,\dots,p_t}$ is formed by removing shared edges from the successive cycles $C_{p_1}, C_{p_2}, \dots, C_{p_t}$ which are $(t-1)$ in number. Therefore, we get the number of edges in $C_{p_1,p_2,\dots,p_t}$ by adding the orders of all $C_{p_1}, C_{p_2}, \dots, C_{p_t}$ and deducting $2(t-1)$, since the shared edges are considered twice in the count. This gives our desired result as follows:

$$\mathcal{B}_{p_1,p_2,\dots,p_t} = \left|C_{p_1,p_2,\dots,p_t}\right| = \sum_{n=1}^{t} \left|C_{p_n}\right| - 2(t-1) = (k+3)t - 2(t-1) = t(k+1) + 2$$

.

Case(ii): When $t = m$
Similarly, for $t = m$ when only one shared edge is left:

$$\mathcal{B}_{p_1,p_2,\dots,p_t} = \left|C_{p_1,p_2,\dots,p_t}\right| = \sum_{n=1}^{t} \left|C_{p_n}\right| - 2(t-1) - 1 = (k+3)t - 2(t-1) - 1 = t(k+1) + 1.$$

□

The following example is included to explain the the above lemma.

Example 3.1. *Let $J_{3,1}$ be the Jahangir's graph with edge set $\{s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23}, s_{31}, s_{32}, s_{33}\}$. Then the cycles in Jahangir's graph $J_{3,1}$ are $C_0, C_1, C_2, C_3, C_{12}, C_{23}, C_{31}, C_{123}, C_{231}, C_{312}$ satisfying the Lemma 3.1 as follows:*

$$\Gamma = m^2 + 1 = 3^2 + 1 = 10$$

The order of each cycle of the Jahangir's graph $J_{m,k}$, when $1 \le t \le m-1$ is computed using $t(k+1)+2$. The order of the cycles C_1, C_2, C_3 is:

$$\mathcal{B}_1 = t(k+1) + 2 = 1(1+1) + 2 = 4$$

Similarly, the order of the cycles C_{12}, C_{23}, C_{31} is:

$$\mathcal{B}_{12} = t(k+1) + 2 = 2(1+1) + 2 = 6$$

The order of the cycles $C_{123}, C_{231}, C_{312}$, when one shared edge is left is calculated using $t(k+1)+1$ as follows:

$$\mathcal{B}_{123} = t(k+1) + 1 = 3(1+1) + 1 = 7$$

In the following propositions, we take any two cycles $C_{\mu_1,\mu_2,\ldots,\mu_p}$ and $C_{\nu_1,\nu_2,\ldots,\nu_q}$ where $p,q \in \{1,2,\ldots,m\}$ of the generalised Jahangir's graph $J_{m,k}$. We use a notation "$y \to z$" which shows that y immediate proceeds z.

Proposition 3.2. Let $\{\mu_1,\mu_2,\ldots,\mu_p\} \subseteq \{\nu_1,\nu_2,\ldots,\nu_q\}$. Then

$$\left|C_{\mu_1,\mu_2,\ldots,\mu_p} \cap C_{\nu_1,\nu_2,\ldots,\nu_q}\right| = \begin{cases} \mathcal{B}_{\mu_1,\mu_2,\ldots,\mu_p} - 2, & \mu_1,\mu_q \notin \{\nu_1,\nu_p\} \\ \mathcal{B}_{\mu_1,\mu_2,\ldots,\mu_p} - 1, & \mu_p \notin \{\nu_1,\nu_q\} \ \&\ \mu_1 \in \{\nu_1,\nu_q\} \\ \mathcal{B}_{\mu_1,\mu_2,\ldots,\mu_p} - 1, & \mu_1 \notin \{\nu_1,\nu_q\} \ \&\ \mu_p \in \{\nu_1,\nu_q\} \\ \mathcal{B}_{\mu_1,\mu_2,\ldots,\mu_p} - 2, & \mu_1 = \nu_1, \mu_p = \nu_q, \ or \\ & \mu_1 = \nu_q, \mu_p = \nu_1, p \neq q \\ \mathcal{B}_{\mu_1,\mu_2,\ldots,\mu_p}, & \mu_1 = \nu_1, \mu_p = \nu_q, \ or \\ & \mu_1 = \nu_q, \mu_p = \nu_1, p = q \neq m \end{cases}$$

Proof. Since the cycles $C_{\mu_1,\mu_2,\ldots,\mu_p}$ and $C_{\nu_1,\nu_2,\ldots,\nu_q}$ are formed by removing the shared edges from $C_{\mu_1}, C_{\mu_2},\ldots,C_{\mu_p}$ and $C_{\nu_1}, C_{\nu_2},\ldots,C_{\nu_q}$ respectively, therefore $\mu_1,\mu_q \notin \{\nu_1,\nu_p\}$ shows that the cycle $C_{\mu_1,\mu_2,\ldots,\mu_p}$ does not overlap the extreme edges of the cycle $C_{\nu_1,\nu_2,\ldots,\nu_q}$. Hence, the intersection of the cycles $C_{\mu_1,\mu_2,\ldots,\mu_p}$ and $C_{\nu_1,\nu_2,\ldots,\nu_q}$ contains only the unshared edges of the cycle $C_{\mu_1,\mu_2,\ldots,\mu_p}$ eliminating the two edges on its extreme ends. This implies that the order of the intersection is $\mathcal{B}_{\mu_1,\mu_2,\ldots,\mu_p} - 2$. In the second case, $\mu_p \notin \{\nu_1,\nu_q\}$ shows that the cycle C_{μ_p} does not overlap the extreme ends of the cycle $C_{\nu_1,\nu_2,\ldots,\nu_p}$ and $\mu_1 \in \{\nu_1,\nu_q\}$ implies that the cycle C_{μ_1} overlaps one of the extreme ends of the cycle $C_{\nu_1,\nu_2,\ldots,\nu_p}$. Therefore the intersection contains unshared edges of the cycle $C_{\mu_1,\mu_2,\ldots,\mu_p}$ along with one shared edge the cycle. Hence order of the intersection is $\mathcal{B}_{\mu_1,\mu_2,\ldots,\mu_p} - 1$. In the same way the remaining two cases can be concluded. □

Proposition 3.3. Let $\{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho\} \subseteq \{\mu_1,\mu_2,\ldots,\mu_p\}$ and $\underline{\mu}_t \in \{\mu_1,\mu_2,\ldots,\mu_p\}$ and $\underline{\mu}_{t-1} \to \underline{\mu}_t$ with $t \leq \varrho < p$. Then

$$\left|C_{\mu_1,\mu_2,\ldots,\mu_p} \cap C_{\nu_1,\nu_2,\ldots,\nu_q}\right| = \begin{cases} \mathcal{B}_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho} - 1, & \nu_q \to \mu_1 \ \&\ \underline{\mu}_1 = \nu_1 \\ \mathcal{B}_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho} - 2, & \nu_q \not\to \mu_1 \ \&\ \underline{\mu}_1 = \nu_1 \\ \mathcal{B}_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho} - 1, & \mu_p \to \nu_1 \ \&\ \underline{\mu}_\varrho = \nu_q \\ \mathcal{B}_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho} - 2, & \mu_p \not\to \nu_1 \ \&\ \underline{\mu}_\varrho = \nu_q \end{cases}$$

Proof. The proof of the proposition is divided into the following four cases.

Case(i): $\nu_q \to \mu_1 \ \&\ \underline{\mu}_1 = \nu_1$

When the neighboring cycle $C_{\underline{\mu}_1}$ from the cycle $C_{\mu_1,\mu_2,\ldots,\mu_p}$ is overlapping with the initial neighboring cycle C_{ν_1} from the cycle $C_{\nu_1,\nu_2,\ldots,\nu_q}$ and cycle C_{ν_q} immediately proceeds the cycle C_{μ_1}, then by Proposition 3.2 case (ii) intersection has unshared edges of the cycle $C_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho}$ including one of its shared edge. Hence the order of the intersection is $\mathcal{B}_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho} - 1$.

Case(ii): $\nu_q \nrightarrow \mu_1$ & $\underline{\mu}_1 = \nu_1$
Here C_{ν_q} and C_{μ_1} are not consecutive cycles then there is no shared edge in intersection. Therefore, by the Proposition 3.2 case (i) we have $\left|C_{\mu_1,\mu_2,\ldots,\mu_p} \cap C_{\nu_1,\nu_2,\ldots,\nu_q}\right| = \mathcal{B}_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho} - 2$.
Case(iii): $\mu_p \to \nu_1$ & $\underline{\mu}_\varrho = \nu_q$
In this case the neighboring cycle $C_{\underline{\mu}_p}$ from the cycle $C_{\mu_1,\mu_2,\ldots,\mu_p}$ and the neighboring cycle C_{ν_1} from the cycle $C_{\nu_1,\nu_2,\ldots,\nu_q}$ immediately proceed each other and the cycles $C_{\underline{\mu}_\varrho}$ and C_{ν_q} overlap each other, then by Proposition 3.2 case (iii) intersection contains the unshared edges of the cycle $C_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho}$ with one of its shared edge. Hence the order of the intersection is $\mathcal{B}_{\underline{\mu}_1,\underline{\mu}_2,\ldots,\underline{\mu}_\varrho} - 1$.
Case(iv): $\mu_p \nrightarrow \nu_1$ & $\underline{\mu}_\varrho = \nu_q$
$\mu_p \nrightarrow \nu_1$ implies that the cycles C_{μ_p} and C_{ν_1} are not consecutive cycles. Therefore intersection has no shared edge. Using the Proposition 3.2 case (i) we get required order of intersection. □

Remark 3.4. *In the Proposition 3.3, when the cycles $C_{\underline{\mu}_1}, C_{\underline{u}_2}, \ldots, C_{\underline{\mu}_{t_0-1}}, C_{\underline{\mu}_{t_0}}, \ldots, C_{\underline{\mu}_\varrho}$ are not ϱ successive neighboring cycles of the cycle $C_{\mu_1,\mu_2,\ldots,\mu_p}$ such that there is a $t_0 < \varrho < p$ and $C_{\underline{\mu}_{t_0-1}}$ and $C_{\underline{\mu}_{t_0}}$ are not successive cycles. Then the Proposition 3.3 is applied on the parts which are overlapped to compute the the order of the intersection of $C_{\mu_1,\mu_2,\ldots,\mu_p} \cap C_{\nu_1,\nu_2,\ldots,\nu_q}$. For example in Jahangir's graph $J_{6,1}$, the intersection of the cycles C_{1234} and C_{345612} is obtained by applying Proposition 3.3 on the overlapping portions of the cycles.*

Proposition 3.5. Let $p \leq q$ and $\{\mu_1, \mu_2, \ldots, \mu_p\} \cap \{\nu_1, \nu_2, \ldots, \nu_q\} = \phi$. Then

$$\left|C_{\mu_1,\mu_2,\ldots,\mu_p} \cap C_{\nu_1,\nu_2,\ldots,\nu_q}\right| = \begin{cases} 1, & \nu_q \nrightarrow \mu_1 \ \& \ \mu_p \to \nu_1 \\ 1, & \nu_q \to \mu_1 \ \& \ \mu_p \nrightarrow \nu_1 \\ 2, & \nu_q \to \mu_1 \ \& \ \mu_p \to \nu_1 \\ 0, & otherwise \end{cases}$$

Proof. Since the intersection of $\{\mu_1, \mu_2, \ldots, \mu_p\}$ and $\{\nu_1, \nu_2, \ldots, \nu_q\}$ is empty then the cycles $C_{\mu_1,\mu_2,\ldots,\mu_p}$ and $C_{\nu_1,\nu_2,\ldots,\nu_q}$ are non-overlapping cycles. If $C_{\nu_q} \nrightarrow C_{\mu_1}$ and $C_{\mu_p} \to C_{\nu_1}$, then the last cycle C_{μ_p} from the neighboring cycle $C_{\mu_1,\mu_2,\ldots,\mu_p}$ shares one edge with the initial cycle C_{ν_1} from the neighboring cycle $C_{\nu_1,\nu_2,\ldots,\nu_q}$. Hence, the intersection of $C_{\mu_1,\mu_2,\ldots,\mu_p}$ and $C_{\nu_1,\nu_2,\ldots,\nu_q}$ has only one edge. Remaining can be proved in the same way. □

Proposition 3.6. Let C_0 be the outer cycle of the generalised Jahangir's graph $J_{m,k}$. Then

$$\left|C_0 \cap C_{p_1,p_2,\ldots,p_t}\right| = \begin{cases} \mathcal{B}_{p_1,p_2,\ldots,p_t} - 2, & p_t \leq m \\ \mathcal{B}_{p_1,p_2,\ldots,p_t} - 1, & p_t = m \end{cases}$$

Proof. Since the cycle $C_{p_1,p_2,\dots,p_t}$ is obtained by removing the shared edges from the cycles $C_{p_1}, C_{p_2}, C_{p_3}, \dots, C_{p_t}$. If $p_t \leq m$, then the intersection will contain only unshared edges of the cycle $C_{p_1,p_2,\dots,p_t}$ excluding the two shared edges from its extreme ends giving order of the of intersection equals to $\mathcal{B}_{p_1,p_2,\dots,p_t} - 2$. Similarly, when $p_t = m$ the intersection will contain only unshared edge of $C_{p_1,p_2,\dots,p_t}$ excluding one shared edge on one extreme end giving the required order of intersection. This completes the proof. □

Here we set $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)})$ as a subset of $\mathcal{E}(J_{m,k})$ defined in Eq. 1. 1 obtained by removing $s_{\omega_1\lambda_1}, s_{\omega_2\lambda_2}, \dots, s_{\omega_m\lambda_m}$ edges from the edge set $\mathcal{E}(J_{m,k})$ having no cycle in it, where $\omega_\eta \in \{1, 2, \dots, m\}$ and $\lambda_\eta \in \{1, 2, 3, \dots, (k+2)\}$. The following three propositions describe the conditions required for $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)})$ to be the spanning tree of the graph $J_{m,k}$.

Proposition 3.7. A subset $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)})$ of $\mathcal{E}(J_{m,k})$ with $\omega_\eta\lambda_\eta \neq \omega_\eta 1$ will belong to $s(J_{m,k})$ if and only if

$$\mathcal{E}(J_{m,k})(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)}) = \mathcal{E}(J_{m,k}) \setminus \{s_{1\lambda_1}, s_{2\lambda_2}, \dots, s_{m\lambda_m}\}$$

Proof. For a spanning tree of the generalised Jahangir's graph $J_{m,k}$ with successive m cycles $C_1, C_2, \dots, C_m$ and shared edges $s_{11}, s_{21}, \dots, s_{m1}$, we have to delete exactly m edges without any disconnection and cycles in the graph by cutting down method. Therefore, when no shared edge is deleted we have to delete only one edge from unshared edges from every cycle. This completes the proof. □

Proposition 3.8. A subset $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)})$ of $\mathcal{E}(J_{m,k})$ with $\lambda_i = 1, \forall i$ will belong to $s(J_{m,k})$ if and only if

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)}) = \mathcal{E}(J_{m,k}) \setminus \{s_{\omega_1\lambda_1}, s_{\omega_2\lambda_2}, \dots, s_{\omega_m\lambda_m}\}$$

where $\{s_{\omega_1\lambda_1}, s_{\omega_2\lambda_2}, \dots, s_{\omega_m\lambda_m}\}$ will carry exactly one edge from $C_{(\omega_\eta-1)(\omega_\eta)} \setminus \{s_{(\omega_\eta+1)1}, s_{(\omega_\eta-1)1}\}$ except $s_{\omega_\eta 1}$.

Proof. To get a spanning tree of the generalised Jahangir's graph $J_{m,k}$ using cutting down method when exactly one shared edge $s_{\omega_\eta 1}$ is deleted, we have to delete exactly $m-1$ edges from the remaining edges. However, we have to delete only one edge from unshared edges of the cycle $C_{(\omega_\eta-1)(\omega_\eta)}$ except $s_{\omega_\eta 1}$ to keep the graph connected. This completes the proof.

□

Proposition 3.9. A subset $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)})$ of $\mathcal{E}(J_{m,k})$ will belong to $s(J_{m,k})$, where $\omega_\eta\lambda_\eta = \omega_\eta 1$ and $\eta \in \{\rho_1, \rho_2, \dots, \rho_r\} \subset \{1, 2, \dots, m\}$, if and only if the following hold:

(1) If the shared edges $s_{\omega_{\rho_1}1}, s_{\omega_{\rho_2}1}, \dots, s_{\omega_{\rho_r}1}$ are from successive cycles, then

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)}) = \mathcal{E}(J_{m,k}) \setminus \{s_{\omega_1\lambda_1}, s_{\omega_2\lambda_2}, \dots, s_{\omega_m\lambda_m}\}$$

such that $\{s_{\omega_1\lambda_1}, s_{\omega_2\lambda_2}, \ldots, s_{\omega_m\lambda_m}\}$ will have only one edge from the cycle $C_{\omega_{\rho_0}\omega_{\rho_1}\ldots\omega_{\rho_r}}$ except $s_{\omega_{\rho_1}1}, s_{\omega_{\rho_2}1}, \ldots, s_{\omega_{\rho_r}1}$, where ω_{ρ_0} immediately proceeds ω_{ρ_1}.

(2) If none of the shared edges $s_{\omega_{\rho_1}1}, s_{\omega_{\rho_2}1}, \ldots, s_{\omega_{\rho_r}1}$ are from successive cycles, then

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}) = \mathcal{E}(J_{m,k}) \setminus \{s_{\omega_1\lambda_1}, s_{\omega_2\lambda_2}, \ldots, s_{\omega_m\lambda_m}\}$$

such that for each edge $s_{\omega_{\rho_t}1}$, the Proposition 3.8 is satisfied.

(3) If some of the shared edges $s_{\omega_{\rho_1}1}, s_{\omega_{\rho_2}1}, \ldots, s_{\omega_{\rho_r}1}$ are from successive cycles, then

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}) = \mathcal{E}(J_{m,k}) \setminus \{s_{\omega_1\lambda_1}, s_{\omega_2\lambda_2}, \ldots, s_{\omega_m\lambda_m}\}$$

such that for the shared edges of successive cycles and for the remaining of the shared edges the Proposition 3.9.1 and 3.9.2 hold respectively.

Proof. For the first case, when $(\rho_r - \rho_1)$ edges are deleted from the r successive cycles $C_{\omega_{\rho_1}}, C_{\omega_{\rho_2}}, \ldots, C_{\omega_{\rho_r}}$, then the rest of the edges are $m - (\rho_r - \rho_1)$. Therefore, to get the spanning tree of the graph $J_{m,k}$ exactly one edge must be deleted from the unshared edges of the cycle $C_{\omega_{\rho_0},\omega_{\rho_1},\ldots,\omega_{\rho_r}}$ and the rest of $m - (\rho_r - \rho_1)$ cycles in the graph $J_{m,k}$. This proves the first case of the proposition. Using Propositions 3.7 and 3.8 the remaining the cases of the proposition can be proved. This concludes the proof of the proposition. □

Remark 3.10. *Let the different categories of the subsets of the edge set $\mathcal{E}(J_{m,k})$ of the graph $J_{m,k}$ mentioned in the Propositions 3.7, 3.8 and 3.9 be denoted by $\Omega_{J1}, \Omega_{J2}, \Omega_{J3a}$, $\Omega_{J3b}, \Omega_{J3c}$ respectively. Then, the spanning set $s(J_{m,k})$ of the generalised Jahangir's graph can be represented as:*

$$s(J_{m,k}) = \Omega_{J1} \cup \Omega_{J2} \cup \Omega_{J3a} \cup \Omega_{J3b} \cup \Omega_{J3c}$$

We describe a main characterization of the f–vectors of the graph $J_{m,k}$ in the next result.

Proposition 3.11. Let $\Delta_s(J_{m,k})$ be the SSC of the generalised Jahangir's graph $J_{m,k}$. Then $\mathcal{D} = \dim(\Delta_s(J_{m,k})) = m(k+1) - 1$ with f–vector $f(\Delta_s(J_{m,k})) = (f_0, f_1, \ldots, f_{\mathcal{D}})$ and

$$f_j = \begin{pmatrix} m(k+2) \\ j+1 \end{pmatrix} + \sum_{t=1}^{\Gamma} (-1)^t \left[\sum_{\{j_1,j_2,\ldots,j_t\} \in C_J^t} \begin{pmatrix} m(k+2) - \sum_{w=1}^{t} \mathcal{B}_{j_w} + \sum_{\{j_\mu,j_\nu\} \subseteq \{j_p\}_{p=1}^{t}} |C_{j_\mu} \cap C_{j_\nu}| \\ j+1 - \sum_{w=1}^{t} \mathcal{B}_{j_w} + \sum_{\{j_\mu,j_\nu\} \subseteq \{j_p\}_{p=1}^{t}} |C_{j_\mu} \cap C_{j_\nu}| \end{pmatrix} \right]$$

where $0 \le j \le \mathcal{D}$.

$J = \{j_1, j_2, \ldots, j_n\}$ where $1 \le n \le m$ & $j_i \in \{1, 2, \ldots, m\}$ such that $j_{i+1} = 1$ when $j_i = m$ and $j_{i+1} = j_i + 1$ when $j_i \neq m$ and C_J^t are the subsets of J having order t.

Proof. Let the edge set of the generalised Jahangir's graph $J_{m,k}$ be $\mathcal{E}(J_{m,k})$ as defined in Eq. 1. 1 . The different classes of the spanning trees $s(J_{m,k})$ according to the Propositions 3.7,3.8,3.9 and the Remark 3.10 are $\Omega_{J1}, \Omega_{J2}, \Omega_{J3a}, \Omega_{J3b}$ and Ω_{J3c}. Therefore, by the Definition 2.4 the SSC of the generalised Jahangir's graph $J_{m,k}$ can be written as

$$\Delta_s(J_{m,k}) = \Big\langle \Omega_{J1} \cup \Omega_{J2} \cup \Omega_{J3a} \cup \Omega_{J3b} \cup \Omega_{J3c} \Big\rangle$$

Since the Propositions 3.7,3.8 and 3.9 explain that the facets $\hat{\mathcal{E}}_{(\omega_1\lambda_1, \omega_2\lambda_2, \ldots, \omega_m\lambda_m)} = \mathcal{E}(T_{(\omega_1\lambda_1, \omega_2\lambda_2, \ldots, \omega_m\lambda_m)})$ are formed by the deletion of the m edges from the edge set $\mathcal{E}(J_{m,k})$ of the generalised Jahangir's graph $J_{m,k}$. Therefore, the cardinality of all the facets is same and equals to $m(k+1)$ which shows that all facets have same dimension equal to $m(k+1)-1$. Hence the dimension of $\dim(\Delta_s(J_{m,k}))$ is $m(k+1)-1$. The definition of $\Delta_s(J_{m,k})$ shows that it has only those subsets of the edge set $\mathcal{E}(J_{m,k})$ which do not carry any cycles in them. The Lemma 3.1 gives the total number of cycles in $J_{m,k}$ which is equal to $\Gamma = m^2+1$. Here we take a subset $\mathcal{F}$ of the edge set $\mathcal{E}(J_{m,k})$ such that it has no cycle in it and its cardinality $j+1$. In fact the total count of these subsets is f_j, where $0 \le j \le m(k+1)-1$. This number can be found by the inclusion exclusion principle. Hence,
$f_j =$ Total count of subsets of $\mathcal{E}(J_{m,k})$ having cardinality $j+1$ not carrying any of the cycles C_0 and $C_{p_1,p_2,\ldots,p_t}$, where $1 \le t \le m$ & $p_q \in \{1, 2, \ldots, m\}$ such that $p_{q+1} = p_q + 1$ when $p_q \neq m$ & $p_{q+1} = 1$ when $p_q = m$. By Inclusion Exclusion Principle and above notations we get $f_j = \Big($ Total count of the subsets of $\mathcal{E}(J_{m,k})$ having cardinaltiy $j+1\Big) - \sum\limits_{\{j_1\}\in C_J^1} \Big($ Total count of the subset of $\mathcal{E}(J_{m,k})$ carrying C_{j_w} for $w=1$ and cardinality $j+1\Big) + \sum\limits_{\{j_1,j_2\}\in C_J^2} \Big($ Total count of the subset of $\mathcal{E}(J_{m,k})$ carrying both $C_{j_w}, \forall\ 1 \le w \le 2$ and cardinality $j+1\Big) - \cdots + (-1)^{\Gamma} \sum\limits_{\{j_1,j_2,\ldots,j_\Gamma\}\in C_J^\Gamma} \Big($Total count of the subset of $\mathcal{E}(J_{m,k})$ carrying each C_{j_w} together for all $1 \le w \le \Gamma$ and cardinality $j+1\Big)$.

This implies that
$$f_j = \binom{m(k+2)}{j+1} - \Big[\sum_{\{j_1\}\in C_J^1} \binom{m(k+2) - \mathcal{B}_{j_1}}{j+1-\mathcal{B}_{j_1}} \Big] +$$

$$\left[\sum_{\{j_1,j_2\}\in C_J^2}\begin{pmatrix} m(k+2)-\sum_{w=1}^{2}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{2}}\left|C_{j_\mu}\cap C_{j_\nu}\right| \\ j+1-\sum_{w=1}^{2}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{2}}\left|C_{j_\mu}\cap C_{j_\nu}\right|\end{pmatrix}\right]$$
$$-\cdots+(-1)^{\Gamma}$$
$$\left[\sum_{\{j_1,j_2,\dots,j_\Gamma\}\in C_J^\Gamma}\begin{pmatrix} m(k+2)-\sum_{w=1}^{\Gamma}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{\Gamma}}\left|C_{j_\mu}\cap C_{j_\nu}\right| \\ j+1-\sum_{w=1}^{\Gamma}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{\Gamma}}\left|C_{j_\mu}\cap C_{j_\nu}\right|\end{pmatrix}\right]$$

This implies that

$$f_j=\begin{pmatrix} m(k+2) \\ j+1\end{pmatrix}+\sum_{t=1}^{\Gamma}(-1)^t$$
$$\left[\sum_{\{j_1,j_2,\dots,j_t\}\in C_J^t}\begin{pmatrix} m(k+2)-\sum_{w=1}^{t}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{t}}\left|C_{j_\mu}\cap C_{j_\nu}\right| \\ i+1-\sum_{w=1}^{t}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{t}}\left|C_{j_\mu}\cap C_{j_\nu}\right|\end{pmatrix}\right]$$ □

EXAMPLE 3.2. *Let $\Delta_s(J_{3,2})$ be a SSC of the generalised Jahangir's graph $J_{m,k}$. Then the $\dim(\Delta_s(J_{3,2}))=8$ and $\Gamma=3^2+1=10$. Therefore, f-vector $f(\Delta_s(J_{3,2}))=(f_0,f_1,\dots,f_8)$ and*

$$f_j=\begin{pmatrix} 12 \\ j+1\end{pmatrix}-\left[\sum_{\{j_1\}\in C_J^1}\begin{pmatrix} 12-\mathcal{B}_{j_1} \\ j+1-\mathcal{B}_{j_1}\end{pmatrix}\right]+$$
$$\left[\sum_{\{j_1,j_2\}\in C_J^2}\begin{pmatrix} 12-\sum_{w=1}^{2}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{2}}\left|C_{j_\mu}\cap C_{j_\nu}\right| \\ j+1-\sum_{w=1}^{2}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{2}}\left|C_{j_\mu}\cap C_{j_\nu}\right|\end{pmatrix}\right]$$
$$-\cdots+(-1)^{10}$$
$$\left[\sum_{\{j_1,j_2,\dots,j_9\}\in C_J^{10}}\begin{pmatrix} 12-\sum_{w=1}^{12}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{10}}\left|C_{j_\mu}\cap C_{j_\nu}\right| \\ j+1-\sum_{w=1}^{12}\mathcal{B}_{j_w}+\sum_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^{10}}\left|C_{j_\mu}\cap C_{j_\nu}\right|\end{pmatrix}\right]$$

where $0\leq j\leq 8$.

Definition 3.12. Let Δ be a SSC defined on a finite set of vertices $[y_1,y_2,\dots,y_n]$. A monomial ideal $I_{\mathcal{N}}(\Delta)$ formed by the square free monomials in $S=k[y_1,y_2,\dots,y_n]$ associated to the non faces of the SSC is called SR-ideal by assigning variable to each vertex of the SSC. *The face ring or SR-ring $k[\Delta]=S/I_{\mathcal{N}}(\Delta)$ is well known to be a standard graded algebra. The further details of the Hilbert series $\hbar_\chi(A)$ and* *the Hilbert function $\hbar(A,\chi)$ of the graded algebra can be seen in* [11] *and* [13].

In the following theorem, we present our main result.

Theorem 3.13. Let $\Delta_s(J_{m,k})$ be the SSC of $J_{m,k}$. Then the Hilbert series $\hbar_\chi(A)$ of the SR-ring $k\big[\Delta_s(J_{m,k})\big]$ is given as follows:

$$\hbar(k[\Delta_s(J_{m,k})],\chi) = 1+\sum_{j=0}^{\mathcal{D}}\frac{\binom{n}{j+1}\chi^{+1}}{(1-\chi)^{j+1}}+\sum_{j=0}^{\mathcal{D}}\sum_{k=1}^{\Gamma}(-1)^k$$

$$\left[\sum_{\{j_1,i_2,\ldots,j_k\}\in C_J^k}\begin{pmatrix} m(k+2)-\sum\limits_{w=1}^{k}\mathcal{B}_{j_w}+\sum\limits_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^k}|C_{j_\mu}\bigcap C_{j_\nu}| \\ j+1-\sum\limits_{w=1}^{k}\mathcal{B}_{j_w}+\sum\limits_{\{j_\mu,j_\nu\}\subseteq\{j_p\}_{p=1}^k}|C_{j_\mu}\bigcap C_{j_\nu}|\end{pmatrix}\right]\frac{\chi^{j+1}}{(1-\chi)^{j+1}}$$

Proof. Let Δ be a SC with f−factors $f(\Delta)=(f_0,f_1,\ldots,f_{\mathcal{D}})$ and dimension $\mathcal{D}$. Then by [13], the Hilbert series of the SR-ring $k[\Delta]$ is given as follows:

$$\hbar(k[\Delta],\chi)=1+\sum_{j=0}^{\mathcal{D}}\frac{f_j\chi^{j+1}}{(1-\chi)^{j+1}}$$

We get the required result by using the values of f−factors from Proposition 3.11 in above expression.

□

The following section describes the associated primes of the facet ideal $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$ of the SSC $\Delta_s(J_{m,k})$.

4. The facet ideal $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$ and its associated primes

Lemma 4.1. *If $\Delta_s(J_{m,k})$ be the SSC of the generalized Jahangir's graph $J_{m,k}$, then*

$$I_{\mathcal{F}}(\Delta_s(J_{m,k}))=\left(\bigcap_{1\le\gamma\le m}(x_{\gamma1})\right)\bigcap\left(\bigcap_{\gamma\in\{0,1,2,\ldots,m-1\}}(x_{\gamma1}x_{\gamma2}x_{(\gamma-1)(k+2)})\right)$$

$$\bigcap\left(\bigcap_{1\le\gamma\le m,2\le\alpha\le(k+2)-\beta}(x_{\gamma\alpha}x_{\gamma(\alpha+\beta)})\right)\bigcap\left(\{x_{\gamma1}\}_{\gamma=1}^{m}\setminus\{x_{\alpha1}\bigcup x_{\alpha2}x_{(\alpha-1)(k+2)}\}\right).$$

Proof. Let $\Delta_s(J_{m,k})$ be the SSC of the generalized Jahangir's graph $J_{m,k}$ having m successive cycles of same length and $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$ be the facet ideal. It is well known from [5], that the minimal vertex cover of a SC Δ and minimal prime ideal of the facet ideal $I_{\mathcal{F}}(\Delta)$ have $1-1$ correspondence. Hence, the minimal vertex cover of the $\Delta_s(J_{m,k})$ will provide the primary decomposition of the facet ideal $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$. Using the definition of $\Delta_s(J_{m,k})$ and Propositions 3.7,3.8 and 3.9 we get $\{s_{\gamma1}\}$ for all $\gamma\in\{1,2,3,\ldots,m\}$ as a minimal vertex cover as $\{s_{\gamma1}\}\in\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}$ for any $\omega_\eta\in\{1,2,\ldots,m\}$ and $\lambda_\eta\in\{1,2,3,\ldots,(k+2)\}$. Also, $\{s_{\gamma1},s_{\gamma2},s_{(\gamma-1),(k+2)}\}$ with $\gamma\in\{0,1,2,\ldots,m-1\}$ is a minimal vertex cover of $\Delta_s(J_{m,k})$ as at least one of the member of the set

$\{s_{\gamma 1}, s_{\gamma 2}, s_{(\gamma-1)(k+2)}\}$ belongs to $\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)}$. Moreover, $\{s_{\gamma\alpha}, s_{\gamma(\alpha+\beta)}\}$ where $1 \leq \gamma \leq m$ and $2 \leq \alpha \leq (k+2) - \beta$ and $\{s_{\gamma 1}\}_{\gamma=1}^{m} \setminus \{s_{\alpha 1} \cup s_{\alpha 2}, s_{(\alpha-1)(k+2)}\}$ are also minimal vertex covers of $\Delta_s(J_{m,k})$ as they have non empty intersection with $\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)}$. □

5. The Cohen-Macaulay Characterization of the SR-ring $K[\Delta_s(J_{m,k})]$

The following section describes the Cohen-Macaulay characterization of the SR-ring $K[\Delta_s(J_{m,k})]$ using the criteria given in [1].

Definition 5.1. [1] Let I be a monomial ideal such that $G(I) = \{g_1, g_2, \dots, g_t\}$ is an ordered system of generators. Then I is said to have *linear residuals* if $\mathrm{Res}(I_\tau) = \{w_1, w_2, \dots, w_{\tau-1}\}$ such that it is minimally generated by linear monomials, $\forall\, 1 \leq \tau \leq t$ when

$$w_k = \frac{m_\tau}{\gcd(m_k, m_\tau)}$$

Theorem 5.2. [1] The necessary and sufficient condition for shellablity of the SC Δ is that its facet ideal $I_{\mathcal{F}}(\Delta)$ has linear residuals.

Corollary 5.3. [1] Let Δ be a pure SC of dimension $\mathcal{D}$ over a finite set $[n]$. Then its facet ideal $I_{\mathcal{F}}(\Delta)$ has linear residuals if the Stanley-Reisner ring $k[\Delta]$ is Cohen Macaulay.

In this theorem we describe our main result.

Theorem 5.4. The SR-ring $K[\Delta_s(J_{m,k})]$ of the SSC $\Delta_s(J_{m,k})$ is Cohen-Macaulay.

Proof. To prove that SR-ring $K[\Delta_s(J_{m,k})]$ is Cohen-Macaulay, we will prove that the facet ideal $I_{\mathcal{F}}\big(\Delta_s(J_{m,k}\big)$ has linear residuals in $S = k[y_{11}, y_{12}, y_{13}, \dots, y_{1(k+2)}, y_{21}, y_{22}, y_{23}, \dots, y_{2(k+2)}, \dots, y_{m1}, y_{m2}, y_{m3}, \dots, y_{m(k+2)}]$ using the Corollary 5.3. Since the spanning tress of the generalised Jahangir's graph $J_{m,k}$ are given by:

$$s(J_{m,k}) = \Omega_{J1} \cup \Omega_{J2} \cup \Omega_{J3a} \cup \Omega_{J3b} \cup \Omega_{J3c}$$

Therefore, the SSC of $J_{m,k}$ is

$$\Delta_s(J_{m,k}) = \left\langle \hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)} \right\rangle$$

where,
$\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)} = \mathcal{E}(J_{m,k}) \backslash \{s_{\omega_1\lambda_1}, s_{\omega_2\lambda_2}, \dots, s_{\omega_m\lambda_m}\} \big| \hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)} \in s(J_{m,k})$ Hence, the facet ideal of $\Delta_s(J_{m,k})$ is

$$I_{\mathcal{F}}(\Delta_s(J_{m,k})) = \Big(y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)}} \mid \hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_m\lambda_m)} \in s(J_{m,k}) \Big)$$

The facet ideal $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$ is a monomial ideal with degree of each monomial equal to $m(k+1)-1$. The product of all variables in S other than $y_{\omega_1\lambda_1}, y_{\omega_2\lambda_2}, \dots, y_{\omega_m\lambda_m}$ gives the monomials in $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$.

Now we will prove that the facet ideal $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$ has linear residuals according to the orders of its monomials given as follows:

(5. 2)
$$\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}} \mid \lambda_{\sigma_1} \neq 1;\ 1 \leq \sigma_1 \leq m\ \&\ \lambda_t = 1;\ t \neq \sigma_1\},$$
$$\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}} \mid \lambda_{\sigma_1}, \lambda_{\sigma_2} \neq 1;\ 1 \leq \sigma_1, \sigma_2 \leq m\ \&\ \lambda_t = 1;\ t \neq \sigma_1, \sigma_2\},$$
$$\ldots$$
$$\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}} \mid \lambda_{\sigma_1}, \lambda_{\sigma_2}, \ldots, \lambda_{\sigma_m} \neq 1; 1 \leq \sigma_1, \sigma_2 \ldots \sigma_m \leq m\}$$

In Eq. 5. 2 , the monomials $\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}} \mid \lambda_{\sigma_1} \neq 1;\ 1 \leq \sigma_1 \leq m\ \&\ \lambda_t = 1;\ t \neq \sigma_1\}$ have the pattern $y_{\hat{\mathcal{E}}_{(11,21,\ldots,(m-1)1,\omega_m\lambda_m)}}$, $y_{\hat{\mathcal{E}}_{(11,21,\ldots,\omega_{(m-1)}\lambda_{(m-1)},m1)}}, \ldots, y_{\hat{\mathcal{E}}_{(11,\omega_2\lambda_2,31,\ldots,(m-1)1,m1)}}, y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,21,\ldots,(m-1)1,m1)}}$, where $\lambda_t \in \{2, 3, \ldots, (k+2)\}$ and $1 \leq \omega_t \leq m$. Similarly, the other monomials in Eq. 5. 2 . Now we substitute

$$\text{Res}(y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}}) = \{\frac{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}}}{\gcd(g_t, y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}})}\}$$

where g_t proceeds $y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}}$ w.r.t to the order in Eq. 5. 2 . In $\text{Res}(y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}})$ substituting $\sigma_1 = m$ gives,

$$\text{Res}(y_{\hat{\mathcal{E}}_{(11,21,\ldots,(m-1)1,\omega_m\lambda_m)}}) = \{\frac{y_{\hat{\mathcal{E}}_{(11,21,\ldots,(m-1)1,\omega_m\lambda_m)}}}{\gcd(g_t, y_{\hat{\mathcal{E}}_{(11,21,\ldots,(m-1)1,\omega_m\lambda_m)}})}\}$$

Here g_t are all the monomials having the form $y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}}$ in S, where $\lambda_m \neq m\ \&\ \omega_m = 2, 3, \ldots, (k+2)$. Since $y_{\hat{\mathcal{E}}_{(11,21,\ldots,(m-1)1,\omega_m\lambda_m)}}$ and g_t has difference at only one point. Therefore, there are only linear terms in $\text{Res}(y_{\hat{\mathcal{E}}_{(11,21,\ldots,(m-1)1,\omega_m\lambda_m)}})$. This shows that only the linear monomials generate the $\text{Res}(y_{\hat{\mathcal{E}}_{(11,21,\ldots,(m-1)1,\omega_m\lambda_m)}})$ minimally.
Following the similar procedure, the order of all the monomials in Eq. 5. 2 of the facet ideal $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$ ensures that $\text{Res}(y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}})$ has only linear monomials for all $y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_m\lambda_m)}} \in I_{\mathcal{F}}(\Delta_s(J_{m,k}))$. Hence, the facet $I_{\mathcal{F}}(\Delta_s(J_{m,k}))$ has linear residuals and by Corollary 5.3 $\Delta_s(J_{m,k})$ is Cohen-Macaulay. □

6. Conclusions

The current paper generalizes the results discussed in [12]. Since computing the spanning trees of an arbitrary graph is an NP-hard problem, therefore, attributing the combinatorial and algebraic properties of SSC for an arbitrary graph caries the same level of hardness. However, there are many classes of simple finite connected graph for which the problem is still open for e.g., prism graph, peterson graph, circular graphs etc. These give open scopes for the results discussed in this article.

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Appendix A

$s(J_{3,2}) = \{ \{s_{11}, s_{21}, s_{31}, s_{12}, s_{13}, s_{22}, s_{23}, s_{32}, s_{33}\}, \{s_{11}, s_{21}, s_{31}, s_{12}, s_{13}, s_{22}, s_{23}, s_{32}, s_{34}\},$
$\{s_{11}, s_{21}, s_{31}, s_{12}, s_{13}, s_{22}, s_{23}, s_{33}, s_{34}\}, \{s_{11}, s_{21}, s_{31}, s_{12}, s_{13}, s_{22}, s_{24}, s_{32}, s_{33}\},$
$\{s_{11}, s_{21}, s_{31}, s_{12}, s_{13}, s_{22}, s_{24}, s_{32}, s_{34}\}, \{s_{11}, s_{21}, s_{31}, s_{12}, s_{13}, s_{22}, s_{24}, s_{33}, s_{34}\},$
$\{s_{11}, s_{21}, s_{31}, s_{12}, s_{13}, s_{23}, s_{24}, s_{32}, s_{33}\}, \{s_{11}, s_{21}, s_{31}, s_{12}, s_{13}, s_{23}, s_{24}, s_{32}, s_{34}\},$
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