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Coupled fixed point results and application to integral equations

Satyendra Kumar Jain
Department of Mathematics, St. Aloysius College Jabalpur , (M.P.), India.
Email: satyendra.jain.26@gmail.com

Gopal Meena*
Department of Applied Mathematics, Jabalpur Engineering College, Jabalpur, (M.P.), India.
Email: gmeena@jecjabalpur.ac.in

J. K. Maitra
Department of Mathematics, RDVV, Jabalpur , (M.P.), India.
Email: jkmrdvv@rediffmail.com

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Abstract.: In this research, we have settled some coupled fixed point results with rational contraction in partially ordered b-metric spaces (PObMS). The findings introduced here are generalizations and extensions of some of the established results in the literature. The applications of our result have been discussed in the final section of the paper.

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1. Introduction and Preliminaries

The study of fixed point is a well-established theory in mathematics which is applied to solve a wide area of problems. Contraction mappings play a key role for resolving existing problems in a myriad of mathematical disciplines. Wolk [20] and Monjardet [14] investigated the extension of the Banach contraction principle to partially ordered sets (poset) in order to obtain fixed points under certain conditions. In 2004, Ran and Reurings [16] established fixed points in partially ordered metric spaces (POMS) with some applications to matrix equations. Later on, many researchers [1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 18, 19] settled fixed point and coupled fixed point results in POMS.

Definition 1.1. [7] *Suppose* $(V, \leq)$ *is a poset. A mapping* $S : V \to V$ *is called strictly increasing if* $S(p) < S(\imath)$, *for all* $p, \imath \in V$ *with* $p < \imath$, *and if* $S(p) > S(\imath)$, *for all* $p, \imath \in V$ *with* $p < \imath$, *is called strictly decreasing .*

Definition 1.2. [7] *Suppose $(V, \leq)$ is a poset. Then $S : V \times V \to V$ has the strict mixed monotone property if $S(p, \imath)$ is strictly increasing in p, and also strictly decreasing in $\imath$. i.e.*

$$p_1 < p_2 \Rightarrow S(p_1, \imath) < S(p_2, \imath) \quad \text{for } any \;\; p_1, p_2 \in V$$
$$also, \quad \imath_1 < \imath_2 \Rightarrow S(p, \imath_1) > S(p, \imath_2) \;\; \text{for } any \;\; \imath_1, \imath_2 \in V.$$

Definition 1.3. [7] *Suppose $(V, \leq)$ is a poset and $S : V \times V \to V$ be a mapping. A point $(p, \imath) \in V \times V$ is known as coupled fixed point of S if $S(p, \imath) = p$ and $S(\imath, p) = \imath$.*

Definition 1.4. [17] *A POMS $(V, \wp, \leq)$ is called ordered complete if for every convergent sequences $\{p_n\}_0^{+\infty}$,$\{\imath_n\}_0^{+\infty} \subset V$, the consequent conditions hold:*

- *If sequence $\{p_n\}$ is nondecreasing in V such that $p_n \to p$ implies that $p_n \leq p$, for all $n \in N$ that is $p = \sup\{p_n\}$.*
- *if sequence $\{\imath_n\}$ is nonincreasing in V such that $\imath_n \to \imath$ implies that $\imath_n \geq \imath$, for all $n \in N$ that is $\imath = \inf\{\imath_n\}$.*

Definition 1.5. [3] *Let $V \neq \phi$ is a set and $s \geq 1$ any real number. A mapping $\wp : V \times V \to R^+$, is called b - metric on V if and only if it fulfill the following conditions*

(i): $\wp(\mu, \nu) = 0 \;\; \Leftrightarrow \mu = \nu$;
(ii): $\wp(\mu, \nu) = \wp(\nu, \mu)$;
(iii): $\wp(\mu, \nu) \leq s[\wp(\mu, w) + \wp(w, \nu)]$ *for all* $\mu, \nu, w \in V$.

Then $(V, \wp)$ is called b - metric space. A b - metric space $(V, \wp)$ is complete if every Cauchy sequence in V converges in V.

In this article, coupled fixed points results have been settled with the monotone property of several contractions. The final section is fully devoted to the applications of the results. The result given here is an extension of results in Arab and Zare [3] in the direction of coupled fixed point.

2. Main Results

In this part, coupled fixed point results in PObMS have been incepted

Theorem 2.1. *Assume $(V, \wp, \leq)$ is a complete PObMS and $S : V \times V \to V$ is a continuous mapping satisfying the strictly monotone property on V, with the condition*

$$\begin{aligned}\wp(S(p, \imath), S(\mu, v)) \leq\; & c_1 \wp(p, \mu) + c_2[\wp(\mu, S(\mu, v)) + \wp(p, S(p, \imath))] \\ & + c_3 \frac{\wp(\mu, S(p, \imath) + \wp(p, S(\mu, v)}{s} \\ & + c_4[\wp(\mu, S(\mu, v))\varphi(\wp(p, \mu), \wp(p, S(p, \imath)))] \\ & + c_5[\wp(\mu, S(p, \imath))\varphi(\wp(p, \mu), \wp(p, S(\mu, v)))] \\ & + c_6 \wp(p, \mu)\varphi(\wp(p, \mu), \wp(p, S(p, \imath)) + \wp(\mu, S(p, \imath))) + c_7 \wp(\mu, S(p, \imath)),\end{aligned} \tag{2. 1}$$

where c_i $(i = 1, 2, \ldots, 7)$ are non negative constants such that

$$\sum_{i=1}^{7} c_i < \frac{1}{s+1}$$

and $\varphi : R^+ \times R^+ \to R^+$ *is a function such that*

$$\varphi(t,t) = 1, \quad for\ all \quad t \in R^+.$$

If there exists two points $p_0, \imath_0 \in V$ *with* $p_0 < S(p_0, \imath_0)$ *and* $\imath_0 > S(\imath_0, p_0)$*, then* S *possesses a coupled fixed point* $(p, \imath) \in V \times V$*.*

Proof. Let $p_0, \imath_0 \in V$ such that $p_0 < S(p_0, \imath_0)$ and $\imath_0 > S(\imath_0, p_0)$. Now, construct sequence $\{p_n\}$ and $\{\imath_n\}$ in V by

$$p_{n+1} = S(p_n, \imath_n)\ and\ \imath_{n+1} = S(\imath_n, p_n). \tag{2. 2}$$

Now, we have to show that $p_n < p_{n+1},\ \imath_n > \imath_{n+1}$.
Assume $n = 0$. Since $p_0 < S(p_0, \imath_0)$ and $\imath_0 > S(\imath_0, p_0)$ by equation (2. 2) we get $p_0 < p_1$ and $\imath_0 > \imath_1$ so, the inequality holds for $n = 0$. By the property of S, we get

$$\begin{aligned} p_{n+1} &= S(p_n, \imath_n) < S(p_{n+1}, \imath_n) \\ &< S(p_{n+1}, \imath_{n+1}) = p_{n+2}, \end{aligned}$$

$$p_{n+1} < p_{n+2}, \tag{2. 3}$$

and

$$\imath_{n+1} = S(\imath_n, p_n) > S(\imath_{n+1}, p_n)$$

$$> S(\imath_{n+1}, p_{n+1}) = \imath_{n+2}$$

$$\imath_{n+1} > \imath_{n+2}, \tag{2. 4}$$

and the above inequalities (2. 3) and (2. 4) imply

$$p_0 < p_1 < p_2 < ... < p_n < p_{n+1} < ...$$

$$\imath_0 > \imath_1 > \imath_2 > ... > \imath_n > \imath_{n+1} > \dots \quad .$$

From hypotheses, we have $p_n < p_{n+1}, \imath_n > \imath_{n+1}$, and from (2. 2)

$$\begin{aligned} \wp(p_{n+1}, p_n) &= \wp(S(p_n, \imath_n), S(p_{n-1}, \imath_{n-1})) \\ &= \wp(S(p_{n-1}, \imath_{n-1}), S(p_n, \imath_n)) \\ &\leq c_1 \wp(p_{n-1}, p_n) + c_2(\wp(p_n, S(p_n, \imath_n)) + \wp(p_{n-1}, S(p_{n-1}, \imath_{n-1}))) \\ &+ c_3 \frac{\wp(p_n, S(p_{n-1}, \imath_{n-1}) + \wp(p_{n-1}, S(p_n, \imath_n)}{s} \\ &+ c_4[\wp(p_n, S(p_n, \imath_n))\varphi(\wp(p_{n-1}, p_n), \wp(p_{n-1}, S(p_{n-1}, \imath_{n-1}))) \\ &+ c_5[\wp(p_n, S(p_{n-1}, \imath_{n-1}))\varphi(\wp(p_{n-1}, p_n), \wp(p_{n-1}, S(p_n, \imath_n))) \\ &+ c_6 \wp(p_{n-1}, p_n)\varphi(\wp(p_{n-1}, p_n), \wp(p_{n-1}, S(p_{n-1}, p_{n-1})) \\ &+ \wp(p_n, S(p_{n-1}, \imath_{n-1}))) + c_7 \wp(p_n, S(p_{n-1}, \imath_{n-1})) \end{aligned}$$

$$\begin{aligned}
\wp(p_{n+1}, p_n) &\leq c_1\wp(p_{n-1}, p_n) + c_2(\wp(p_n, p_{n+1}) + \wp(p_{n-1}, p_n)) \\
&+ c_3[\wp(p_n, p_n) + \wp(p_{n-1}, p_{n+1})] \\
&+ c_4[\wp(p_n, p_{n+1}))\varphi(\wp(p_{n-1}, p_n), \wp(p_{n-1}, p_n)) \\
&+ c_5[\wp(p_n, p_n)\varphi(\wp(p_{n-1}, p_n), \wp(p_{n-1}, p_{n+1}))] \\
&+ c_6\wp(p_{n-1}, p_n)\varphi(\wp(p_{n-1}, p_n), \wp(p_{n-1}, p_n)) + c_7\wp(p_n, p_n) \\
&\leq c_1\wp(p_{n-1}, p_n) + c_2\wp(p_{n-1}, p_n) + c_2\wp(p_n, p_{n+1}) + c_3\wp(p_{n-1}, p_n) \\
&+ c_3\wp(p_n, p_{n+1}) + c_4[\wp(p_n, p_{n+1})) + c_5(0) + c_6\wp(p_{n-1}, p_n) + c_7.(0) \\
&\leq (c_1 + c_2 + c_3 + c_6)\wp(p_{n-1}, p_n) + (c_2 + c_3 + c_4)\wp(p_n, p_{n+1})
\end{aligned}$$

$$\wp(p_n, p_{n+1}) \leq \frac{c_1 + c_2 + c_3 + c_6}{1 - (c_2 + c_3 + c_4)}\wp(p_{n-1}, p_n). \tag{2. 5}$$

Likewise, we get

$$\wp(\imath_n, \imath_{n+1}) \leq \frac{c_1 + c_2 + c_3 + c_6}{1 - (c_2 + c_3 + c_4)}\wp(\imath_{n-1}, \imath_n). \tag{2. 6}$$

By relations (2. 5) and (2. 6) we get

$$\wp(p_n, p_{n+1}) + \wp(\imath_n, \imath_{n+1}) \leq \frac{c_1 + c_2 + c_3 + c_6}{1 - (c_2 + c_3 + c_4)}(\wp(p_{n-1}, p_n) + \wp(\imath_{n-1}, \imath_n)).$$

Let's establish a sequence

$$\{S_n\} = \wp(p_{n+1}, p_n) + \wp(\imath_{n+1}, \imath_n).$$

By induction we get

$$0 \leq \ S_n \leq \kappa S_{n-1} \leq \kappa^2 S_{n-2} \leq ... \leq \kappa^n S_0,$$

where $\kappa = \frac{c_1+c_2+c_3+c_6}{1-(c_2+c_3+c_4)} < 1$, and so

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}[\wp(p_n, p_{n-1}) + \wp(\imath_n, \imath_{n-1})] = 0.$$

From this we get,

$$\lim_{n\to+\infty} \wp(p_n, p_{n-1}) = 0,$$

$$\lim_{n\to+\infty} \wp(\imath_n, \imath_{n-1}) = 0.$$

For every $m \geq n$, we have

$$\wp(p_m, p_n) \leq \wp(p_m, p_{m-1}) + \wp(p_{m-1}, p_{m-2}) + ... + \wp(p_{n+1}, p_n),$$

and

$$\wp(\imath_m, \imath_n) \leq \wp(\imath_m, \imath_{m-1}) + \wp(\imath_{m-1}, \imath_{m-2}) + ... + \wp(\imath_{n+1}, \imath_n).$$

Hence,

$$\begin{aligned}
\wp(p_m, p_n) + \wp(\imath_m, \imath_n) &\leq S_m + S_{m-1} + \cdots + S_n \\
&\leq (\kappa^{m-1} + \kappa^{m-2} + ... + \kappa^n)S_0 \\
&\leq \frac{\kappa^n}{1-\kappa}S_0,
\end{aligned}$$

and taking $n \to +\infty$ we get

$$\wp(p_m, p_n) + \wp(\imath_m, \imath_n) \to 0,$$

which shows that both sequences $\{p_n\}$ and $\{\imath_n\}$ are Cauchy in V. So, there exists $(p, \imath) \in V \times V$ such that

$$p_n \to p \ \ and \ \ \imath_n \to \imath.$$

Again, by the continuity of S, we get

$$\begin{aligned} p &= \lim_{n\to+\infty} p_{n+1} \\ &= \lim_{n\to+\infty} S(p_n, \imath_n) \\ &= S(\lim_{n\to+\infty} p_n, \lim_{n\to+\infty} \imath_n) \\ &= S(p, \imath), \end{aligned}$$

$$\begin{aligned} \imath &= \lim_{n\to+\infty} \imath_{n+1} \\ &= \lim_{n\to+\infty} S(\imath_n, p_n) \\ &= S(\lim_{n\to+\infty} \imath_n, \lim_{n\to+\infty} p_n) \\ &= S(\imath, p). \end{aligned}$$

Since, sequence $\{p_n\}$ is increasing and converges to $p \in V$, so

$$p = \sup p_n \quad i.e. \ \ p_n \leq p.$$

By the strict monotone property, we have

$$S(p_n, \imath_n) < S(p, \imath_n). \tag{2. 7}$$

Likewise, sequence $\{\imath_n\}$ is decreasing in V and is converges to $\imath \in V$. Then

$$\imath = \inf \imath_n \quad i.e. \ \ \imath_n \geq \imath, \quad for\ all\ n \in N,$$

and, by the strict monotone property,

$$S(p, \imath_n) < S(p, \imath). \tag{2. 8}$$

By inequalities (2. 7) and (2. 8), we get

$$S(p_n, \imath_n) < S(p, \imath).$$

So

$$p_{n+1} < S(p, \imath).$$

Since $p_n < p_{n+1} < S(p, \imath)$, for all $n \in N$ and $p = \sup\{p_n\}$ then, we have

$$p \leq S(p, \imath).$$

Now, let $z_0 = p$ and $z_{n+1} = S(z_n, \imath_n)$ then $\{z_n\}$ is a non decreasing sequence as $z_0 = S(z_0, \imath_0)$ and converges to z in V. Then $z = \sup\{z_n\}$.
Since, for every $n \in N$,

$$p_n \leq p = z_0 \leq S(z_0, \imath_0) \leq z_n \leq z,$$

from inequality (2. 1), we have

$$\begin{aligned}
\wp(p_{n+1}, z_{n+1}) &= \wp(S(p_n, \imath_n), S(z_n, \imath_n)) \\
&\leq c_1\wp(p_n, z_n) + c_2[\wp(z_n, S(z_n, \imath_n) + \wp(p_n, S(p_n, \imath_n))] \\
&+ c_3\frac{\wp(z_n, S(p_n, \imath_n) + \wp(p_n, S(z_n, \imath_n)}{s} \\
&+ c_4[\wp(z_n, S(z_n, \imath_n))\varphi(\wp(p_n, z_n), \wp(p_n, S(p_n, \imath_n)))] \\
&+ c_5[\wp(z_n, S(p_n, \imath_n))\varphi(\wp(p_n, z_n), \wp(p_n, S(z_n, \imath_n)))] \\
&+ c_6\wp(p_n, z_n)\varphi(\wp(p_n, z_n), \wp(p_n, S(p_n, \imath_n)) + \wp(z_n, S(p_n, \imath_n))) \\
&+ c_7\wp(z_n, S(p_n, \imath_n)) \\
&\leq c_1\wp(p_n, z_n) + c_2[\wp(z_n, z_{n+1}) + \wp(p_n, p_{n+1})] \\
&+ c_3\frac{\wp(z_n, p_{n+1}) + \wp(p_n, z_{n+1})}{s} \\
&+ c_4[\wp(z_n, z_{n+1})\varphi(\wp(p_n, z_n), \wp(p_n, p_{n+1}))] \\
&+ c_5[\wp(z_n, p_{n+1})\varphi(\wp(p_n, z_n), \wp(p_n, z_{n+1}))] \\
&+ c_6\wp(p_n, z_n)\varphi(\wp(p_n, z_n), \wp(p_n, p_{n+1}) + \wp(z_n, p_{n+1})) + c_7\wp(z_n, p_{n+1}).
\end{aligned}$$

Taking limit $n \to +\infty$, we get

$$\wp(p, z) \leq (c_1 + c_3 + c_5 + c_6 + c_7)\wp(p, z),$$

which is contradiction. So, $\wp(p, z) = 0$. Hence, $p = z = \sup\{p_n\}$, which implies that

$$p \leq S(p, \imath) \leq p.$$

Hence, $S(p, \imath) = p$. Also, by the above , we get $S(\imath, p) = \imath$. So, S admits a coupled fixed point in $V \times V$.

Remark 2.1. The elements p and z of poset are comparable if $p \leq z$ or $z \leq p$.

Theorem 2.2. *Along the assumptions stated in Theorem 2.1, consider that for each* $(p, \imath), (z, t) \in V \times V$, *there exists* $(\mu, v) \in V \times V$ *such that* $(S(\mu, v), S(v, \mu))$ *is comparable to* $(S(p, \imath), S(\imath, p))$ *and* $(S(z, t), S(t, z))$. *Then,* S *possesses a unique coupled fixed point, i.e., there exists* $(p, \imath) \in V \times V$ *such that* $p = S(p, \imath)$ *and* $\imath = S(\imath, p)$.

Proof. Theorem 2.1 shows that the set of coupled fixed points of S is non empty. Let $(p, \imath)$ and (z, t) be coupled fixed points of S, i.e., $p = S(p, \imath), \imath = S(\imath, p), z = S(z, t)$ and $t = S(t, z)$. We claim that $p = z$ and $\imath = t$. By the hypotheses there exists $(\mu, v) \in V \times V$ such that $(S(\mu, v), S(v, \mu))$ is comparable with $(S(p, \imath), S(\imath, p))$ and $(S(z, t), S(t, z))$. Put $\mu_0 = \mu$, $v_0 = v$ and choose $\mu_1, v_1 \in V$ so that $\mu_1 = S(\mu_0, v_0)$ and $v_1 = S(v_0, \mu_0)$.
Then with the same path as in Theorem 2.1, we can incept sequences $\{\mu_n\}, \{v_n\}$ as $\mu_{n+1} = S(\mu_n, v_n)$ and $v_{n+1} = S(v_n, \mu_n)$, for all n. Further, set $p_0 = p, \imath_0 = \imath, z_0 = z, t_0 = t$ and in the same way define the sequences $\{p_n\}, \{\imath_n\}$, and $\{z_n\}, \{t_n\}$. Then, as in Theorem 2.1, we can verify that $p_n \to p = S(p, \imath), \imath_n \to \imath = S(\imath, p), z_n \to z = S(z, t), t_n \to t = S(t, z)$, for all $n \geq 1$. Since, $(S(p, \imath), S(\imath, p)) = (p, \imath)$ and $(S(\mu_0, v_0), S(v_0, \mu_0)) = (\mu_1, v_1)$ are comparable, then $p \geq \mu_1$ and $\imath \leq v_1$. Now, we shall verify that $(p, \imath)$ and (μ_n, v_n) are comparable, i.e., $p \geq \mu_n$ and $\imath \leq v_n$, for all natural numbers n. Consider it holds for some $n \geq 0$, then by property of S, we have

$\mu_{n+1} = S(\mu_n, v_n) \leq S(p, \imath) = p$ and $v_{n+1} = S(v_n, \mu_n) \geq S(\imath, p) = \imath$. Hence $p \geq \mu_n$ and $\imath \leq v_n$ hold for all natural numbers n. So, by theorem (2.1), we have

$$\begin{aligned}
\wp(p, \mu_{n+1}) &= \wp(S(p,\imath), S(\mu_n, v_n)) \\
&\leq c_1(p,\mu_n)\wp(p,\mu_n) + c_2(p,\mu_n)[\wp(\mu_n, S(\mu_n, v_n) + \wp(p, S(p,\imath)] \\
&+ c_3(p,\mu_n)\frac{\wp(\mu_n, S(p,\imath) + \wp(p, S(\mu_n, v_n)}{s} \\
&+ c_4(p,\mu_n)[\wp(\mu_n, S(\mu_n, v_n))\varphi(\wp(p,\mu_n), \wp(p, S(p,\imath)))] \\
&+ c_5(p,\mu_n)[\wp(\mu_n, S(p,\imath))\varphi(\wp(p,\mu_n), \wp(p, S(\mu_n, v_n)))] \\
&+ c_6(p,\mu_n)\wp(p,\mu_n)\varphi(\wp(p,\mu_n), \wp(p, S(p,\imath)) + \wp(\mu_n, S(p,\imath))) + c_7\wp(\mu_n, S(p,\imath)),
\end{aligned}$$

which implies that

$$\wp(p, \mu_{n+1}) \leq \frac{c_1 + c_2 + c_3 + c_6}{1 - (c_2 + c_3 + c_4)}\wp(p, \mu_n).$$

Likewise, we get

$$\wp(\imath, v_{n+1}) \leq \frac{c_1 + c_2 + c_3 + c_6}{1 - (c_2 + c_3 + c_4)}\wp(\imath, v_n)$$

$$\wp(\imath, v_{n+1}) \leq k\,\wp(\imath, v_n),$$

where $k = \frac{c_1+c_2+c_3+c_6}{1-(c_2+c_3+c_4)} < 1$. Then, from the above relations, we get

$$\begin{aligned}
\wp(p, \mu_{n+1}) + \wp(\imath, v_{n+1}) &\leq k(\wp(p,\mu_n) + \wp(\imath, v_n)) \\
&\leq k^2(\wp(p,\mu_{n-1}) + \wp(\imath, v_{n-1})) \\
&\ldots \quad \ldots \\
&\leq k^n(\wp(p,\mu_0) + \wp(\imath, v_0)).
\end{aligned}$$

Taking $n \to +\infty$ in above equation, we get

$$\lim_{n\to+\infty} \wp(p, \mu_{n+1}) + \wp(\imath, v_{n+1}) = 0,$$

therefore

$$\lim_{n\to+\infty} \wp(p, \mu_{n+1}) = 0 \;\; and \quad \lim_{n\to+\infty} \wp(\imath, v_{n+1}) = 0.$$

Likewise, we can prove that

$$\lim_{n\to+\infty} \wp(z, \mu_{n+1}) = 0 \;\; and \quad \lim_{n\to+\infty} \wp(t, v_{n+1}) = 0.$$

Finally, we arrive at

$$\wp(p, z) \leq \wp(p, \mu_n) + \wp(\mu_n, z) \quad and \;\; \wp(\imath, t) \leq \wp(\imath, v_n) + \wp(v_n, t).$$

Taking $n \to +\infty$ in the above inequalities, we have

$$\wp(p, z) = 0 \;\; and \;\; \wp(\imath, t) = 0,$$

that is $p = z \; and \;\; \imath = t$. So, S has a unique coupled fixed point.

Example 2.3. *Let $V = [0,1]$ with the usual relation $\leq$, and $\wp(p,\imath) = |p-\imath|^2$. Then, clearly $(V,\wp)$ is a complete b-metric space. Here $s = 2$ also take $c_1 = \frac{1}{10000}$ and $S : V^2 \to V$ defined by $S(p,\imath) = \frac{p+\imath}{10000}$, for all $p,\imath \in V$. Thus, it follows that*

$$\begin{aligned}
\wp(S(p,\imath),S(\mu,v)) &= \wp(\frac{p+\imath}{10000},\frac{\mu+v}{10000}) \\
&= |\frac{p+\imath}{10000}-\frac{\mu+v}{10000}|^2 \\
&= \frac{1}{10000}|\frac{(p-\mu)+(\imath-v)}{100}|^2 \\
&\leq \frac{1}{10000}|p-\mu|^2 \\
&\leq c_1\,\wp(p,\mu),
\end{aligned}$$

for all $p,\imath,\mu,v \in V$. Thus, by Theorem 2.1, the coupled fixed point of S is $(0,0)$, which is unique .

3. Applications

Here, we apply our result to a mapping with a contraction of integral type.
Let us assume the collection of all functions η defined on $[0,+\infty)$ satisfy the following conditions:

(1) Each η is Lebesque integrable mapping on every compact subset of $[0,+\infty)$.
(2) For any $\epsilon > 0$,we have $\int_0^\epsilon \eta(t) > 0$.

Theorem 3.1. *Let $(V,\wp,\leq)$ be a complete PObMS and $S : V\times V \to V$ be continuous strictly monotone mapping on $V\times V$, which fulfills the condition*

$$\begin{aligned}
\int_0^{\wp(S(p,\imath),S(\mu,v))}\eta(t)dt &\leq c_1\int_0^{\wp(p,\mu)}\eta(t)dt + c_2\int_0^{[\wp(\mu,S(\mu,v)+\wp(p,S(p,\imath))]}\eta(t)dt \\
&+ c_3\int_0^{\frac{\wp(\mu,S(p,\imath))+\wp(p,S(\mu,v))}{s}}\eta(t)dt \\
&+ c_4\int_0^{[\wp(\mu,S(\mu,v))\varphi(\wp(p,\mu),\wp(p,S(p,\imath)))]}\eta(t)dt \\
&+ c_5\int_0^{[\wp(\mu,S(p,\imath))\varphi(\wp(p,\mu),\wp(p,S(\mu,v)))]}\eta(t)dt \\
&+ c_6\int_0^{\wp(p,\mu)\varphi(\wp(p,\mu),\wp(p,S(p,\imath))+\wp(\mu,S(p,\imath)))}\eta(t)dt + c_7\int_0^{\wp(\mu,S(p,\imath))}\eta(t)dt,
\end{aligned}$$

for all $p,\imath,\mu,v \in V$ with $p \geq \mu$ and $\imath \leq v$, $\eta(t)$ fulfills the statements on $[0,+\infty)$ and c_i $(i = 1,2,\ldots,7)$ are non negative constants such that

$$\sum_{i=1}^{7} c_i < \frac{1}{s+1},$$

and $\varphi : R^+\times R^+ \to R^+$ is a function such that

$$\varphi(t,t) = 1, \quad for\ all \quad t \in R^+.$$

If there exists two points $p_0, \imath_0 \in V$ *with* $p_0 < S(p_0, \imath_0)$ *and* $\imath_0 > S(\imath_0, p_0)$. *Then* S *possesses a coupled fixed point* $(p, \imath) \in V \times V$.

Likewise, we have the consequent coupled fixed point result in complete PObMS, by taking $c_2 = 0, c_4 = 0, c_6 = 0$ in the above theorem.

Theorem 3.2. *Let* $(V, \wp, \leq)$ *be a complete PObMS and* $S : V \times V \to V$ *be continuous strictly monotone mapping on* $V \times V$, *which fulfills the condition*

$$\begin{aligned} \int_0^{\wp(S(p,\imath),S(\mu,v))} \eta(t)dt &\leq c_1 \int_0^{\wp(p,\mu)} \eta(t)dt \\ &+ c_3 \int_0^{\frac{\wp(\mu,S(p,\imath))+\wp(p,S(\mu,v))}{s}} \eta(t)dt \\ &+ c_5 \int_0^{[\wp(\mu,S(p,\imath))\varphi(\wp(p,\mu),\wp(p,S(\mu,v)))]} \eta(t)dt \\ &+ c_7 \int_0^{\wp(\mu,S(p,\imath))} \eta(t)dt, \end{aligned}$$

for all $p, \imath, \mu, v \in V$ *with* $p \geq \mu$ *and* $\imath \leq v$, $\eta(t)$ *fulfills the statements on* $[0, +\infty)$ *and the non negative constants* $c_i (i = 1, 3, 5, 7)$ *are such that*

$$c_1 + c_3 + c_5 + c_7 < \frac{1}{s+1},$$

and $\varphi : R^+ \times R^+ \to R^+$ *is a function such that*

$$\varphi(t,t) = 1, \quad for\ all \quad t \in R^+.$$

If there exists two points $p_0, \imath_0 \in V$ *with* $p_0 < S(p_0, \imath_0)$ *and* $\imath_0 > S(\imath_0, p_0)$. *Then* S *possesses a coupled fixed point* $(p, \imath) \in V \times V$.

Theorem 3.3. *Let* $(V, \wp, \leq)$ *be a complete PObMS and* $S : V \times V \to V$ *be continuous strictly monotone mapping on* $V \times V$, *which fulfills the condition*

$$\int_0^{\wp(S(p,\imath),S(\mu,v))} \eta(t)dt \leq c_1 \int_0^{\wp(p,\mu)} \eta(t)dt$$

for all $p, \imath, \mu, v \in V$ *with* $p \geq \mu$ *and* $\imath \leq v$, $\eta(t)$ *fulfill the statements on* $[0, +\infty)$ *and* $c_1 \in [0, 1)$, *and if there exists two points* $p_0, \imath_0 \in V$ *with* $p_0 < S(p_0, \imath_0)$ *and* $\imath_0 > S(\imath_0, p_0)$, *then* S *possesses a coupled fixed point* $(p, \imath) \in V \times V$.

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REFERENCES

[1] R.P. Agarwal , M.A. El-Gebeily and D.O. Regan , *Generalized contractions in partially ordered metric spaces*. Applicable Analysis **87**(2008) 18.
[2] J. Ahmad , M. Arshad and C. Vetro , *On a theorem of Khan in a generalized metric space*.International Journal of Analysis (2013) 852727.

[3] R. Arab and K. Zare , *Fixed point results for rational type contractions in partially ordered b- metric spaces*. International Journal of Analysis and Applications **10**(2016)64-70.

[4] H. Aydi , M. Postolache and W. Shatanawi , *Coupled fixed point results for (psi,phi)-weakly contractive mappings in ordered G-metric spaces*. Comput. Math. Appl. **63**(2012) 298-309.

[5] T.G. Bhaskar and V. Lakshmikantham , *Fixed point theory in partially ordered metric spaces and applications*. Nonlinear Analysis: Theory, Methods and Applications **65**(2006) 13791393.

[6] S. Chandok , Z. Mustafa , M. Postolache, *Coupled common fixed point theorems for mixed g-monotone mappingsin partially ordered G-metric spaces*. U. Politeh. Buch. Ser. A **75**(2013) 13-26.

[7] S. Chandok , T.D. Narang , and M.A. Taoudi , *Some coupled fixed point theorems for mappings satisfying a generalized contractive condition of rational type*. Palestine Journal of Mathematics, **4,** No. 2 (2015)360366.

[8] B.S. Choudhury and A. Kundu , *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*. Nonlinear Analysis: Theory, Methods and Applications **73**(2010)25242531.

[9] B.S. Choudhury, N. Metiya and M. Postolache , *A generalized weak contraction principle with applications to coupled coincidence point problems*. Fixed Point Theory Appl. (2013) 152.

[10] P. Debnath , N. Konwar and S. Radenovic , *Metric Fixed Point Theory, Applications in Science, Engineering and Behavioural Sciences*, Forum for Interdisciplinary Mathematics, Springer (2021).

[11] H. Faraji , D. Savi , S. Radenovi, *Fixed Point Theorems for Geraghty Contraction Type Mappings in b-Metric Spaces and Applications*. Axioms **8**(2019)34.

[12] S. Hong , *Fixed points of multivalued operators in ordered metric spaces with applications*.Nonlinear Analysis: Theory, Methods and Applications **72**(2010)39293942.

[13] V. Lakshmikantham and L.B. Ciric., *Couple fixed point theorems for nonlinear contractions in partially ordered metric spaces*. Nonlinear Analysis: Theory, Methods and Applications **70**(2009)43414349.

[14] B. Monjardet , *Metrics on partially ordered sets: A survey*. Discrete mathematics **35**(1981) 173184.

[15] J. J. Nieto and R.R. Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation*. Acta Mathematica Sinica, English Series, **23,** No. 12(2007)22052212.

[16] A.C.M. Ran and M.C.B. Reurings , *A fixed point theorem in partially ordered sets and some application to matrix equations*. Proceedings of the American Mathematical Society **132**(2004)14351443.

[17] N.S. Rao and K. Kalyani , *Coupled fixed point theorems with rational expressions in partially ordered metric spaces*. J Anal **28**(2020) 10851095 .

[18] W. Shatanawi and A. Pitea , *Omega-distance and coupled fixed point in G-metric spaces*, Fixed Point Theory Appl. (2013) 208.

[19] W. Shatanawi and M. Postolache , *Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces*. Fixed Point Theory Appl. (2013) 60.

[20] E.S. Wolk , *Continuous convergence in partially ordered sets*. General Topology and Its Applications **5**(1975)221234.