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Permuting Tri-Multiderivation on Incline Algebra

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Abstract.: In this paper, the concept of permuting tri-multiderivation on incline algebra is initiated and some results are proved by using this idea.

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1. Introduction

In 1984, Cao introduced the idea of incline algebra and explored certain properties of this notion [4]. Incline algebra characterized the bolean and fuzzy algebras and it is a specialized category of semi-rings. Many researcher discussed this structure and provided new results in the theory of incline algebra [1, 2, 3, 4, 7, 10, 11, 14]. In 2010, after the idea of derivation in incline algebras started by Alshehry, several researchers added very useful results to this theory by utilizing derivations such as symmetric bi-derivations and permuting tri-derivations [7, 10, 11]. Recently in 2015, the notion of set valued derivations on lattices and symmetric bi-multiderivation on incline algebras is proposed by Rezapour and Sami [13, 14]. In this paper, we have generalized the idea of symmetric bi-multiderivation and investigated related properties.

2. Preliminaries

Definition 2.1. [14] *Let J is a nonempty set then $(J, \vee, \wedge)$ is said to be an incline algebra if following conditions are satisfied:*
$J_1 : (\zeta \vee \xi) \vee \rho = \zeta \vee (\xi \vee \rho)$
$J_2 : \zeta \wedge (\xi \vee \rho) = (\zeta \wedge \xi) \vee (\zeta \wedge \rho)$
$J_3 : (\xi \vee \rho) \wedge \zeta = (\xi \wedge \zeta) \vee (\rho \wedge \zeta)$

$J_4 : (\zeta \wedge \xi) \vee \rho = (\zeta \vee \rho) \wedge (\xi \vee \rho)$
$J_5 : \zeta \wedge (\zeta \vee \xi) = \zeta$
$J_6 : \zeta \vee (\zeta \wedge \xi) = \zeta$
$J_7 : \zeta \wedge (\zeta \vee \xi) = \zeta$
$J_8 : (\zeta \wedge \zeta) = \zeta,$
$J_9 : \zeta \vee (\xi \vee \rho) = (\zeta \vee \xi) \vee \rho$
$J_{10} : \zeta \wedge (\xi \wedge \rho) = (\zeta \wedge \xi) \wedge \rho$
$J_{11} : \zeta \vee \xi = \xi \vee \zeta$ *for all* $\zeta, \xi, \rho \in J$.

Definition 2.2. [14] *Let* $(J, \vee, \wedge)$ *is an incline algebra then it is said to be commutative if* $\zeta \wedge \xi = \xi \wedge \zeta$ *for all* $\zeta, \xi \in J$.

Definition 2.3. [14] *Let* $\wp \neq 0$ *and* $\wp \subset J$, *then* $\wp$ *is called a subincline algebra if* $\wp$ *is closed under* $\vee$ *and* $\wedge$.

Definition 2.4. [14] *A subincline algebra on an incline algebra is said to be an Ideal if* $\zeta \in \wp$ *and* $\zeta \leq \xi$ *implies* $\xi \in \wp$.

Definition 2.5. [14] *A nonzero element 1 in* J *is said to be multiplicative Identity if* $\zeta \wedge 1 = \zeta \ \forall \ \zeta \in J$.

Definition 2.6. [14] *An element* $0 \in J$ *is said to be a zero element of* J *if* $0 \wedge \zeta = \zeta \wedge 0 = 0$ $\forall \zeta \in J$.

Definition 2.7. [14] *Let* J *is incline algebra and* $\zeta \neq 0$, *an element of* J *is called a left or right zero divisor if there exist a nonzero element* $\xi \neq 0$ *in* J *such that* $\zeta \wedge \xi = 0$ *or respectively* $\xi \wedge \zeta = 0$.

3. Permuting tri-multiderivation on Incline Algebra

In this section, we proved some results by using the notion of permuting tri multi-derivation on Incline Algebra.

Definition 3.1. *Let* $(J, \wedge, \vee)$ *be an incline algebra. A permuting map* $\Im : J \times J \times J \longrightarrow 2^J$ *is called a permuting tri-multimap. If*
$\Im(\zeta \wedge \imath, \xi, \rho) = [\Im(\zeta, \xi, \rho) \wedge \imath] \vee [\zeta \wedge \Im(\imath, \xi, \rho)]$, *for all* $\zeta, \xi, \imath, \rho \in J$. *Then* $\Im$ *is called a permuting tri-multiderivation on* J. *Here* $\Im(\zeta, \xi, \rho) \wedge \imath$ *means* $\Im(\zeta, \xi, \rho) \wedge \{\imath\}$.

Example 3.2. *Let* $(J, \wedge, \vee)$ *be a commutative incline algebra and* $\wp$ *a subincline algebra of* J. *Let* $\Im : J \times J \times J \longrightarrow 2^J$ *be a set valued map defined by* $\Im(\zeta, \xi, \rho) = \zeta \wedge \xi \wedge \rho \wedge \wp$, *for all* $\zeta\xi, \rho \in J$. *Then* $\Im$ *is an isotone permuting tri-multiderivation on* J.

Example 3.3. *Let* J *be a set of non-negative real numbers,* $\zeta \wedge \xi$ *is the greatest lower bound of* ζ *and* ξ *and* $\zeta \vee \xi$ *is the least upper bound of* ζ *and* ξ. *Let* $\Im : J \times J \times J \longrightarrow 2^J$ *be a set valued map defined by* $\Im(\zeta, \xi, \rho) = \gamma \in J : \gamma \leq \zeta \wedge (\xi \wedge \rho) = (\zeta \wedge \xi) \wedge \rho$, *for all* $\zeta\xi, \rho \in J$. *Then* $\Im$ *is called permuting tri-multiderivation on* J.

Proposition 3.4. *Let* $(J, \vee, \wedge)$ *is an incline algebra and* $\Im$ *be a permuting tri-multiderivation on* J. *Then following axioms hold:*
(i) $\Im(\zeta \wedge \imath, \xi, \rho) \preceq \Im(\zeta, \xi, \rho) \vee \Im(\imath, \xi, \rho)$.

(ii) $\Im(\zeta \wedge \imath, \xi, \rho) \preceq \imath$ *whenever* $\zeta \leq \imath$ *and* $\Im(\imath, \xi, \rho) \leq \Im(\zeta, \xi, \rho)$.
(iii) *Moreover* $\Im(\zeta, \xi, \rho) \preceq \zeta$, $\Im(\zeta, \xi, \rho) \preceq \xi$, $\Im(\zeta, \xi, \rho) \preceq \rho$.
(iv) $\Im(\zeta, \xi, \rho) \wedge \Im(\imath, \xi, \rho) \preceq \Im(\zeta \wedge \imath, \xi, \rho)$.

Proof. Let $\zeta, \xi, \rho \in J$ then,
(i) Since $\Im(\zeta, \xi, \rho) \wedge \imath \preceq \Im(\zeta, \xi, \rho)$. Also $\zeta \wedge \Im(\imath, \xi, \rho) \preceq \Im(\imath, \xi, \rho)$. This implies $(\Im(\zeta, \xi, \rho) \wedge \imath) \vee (\zeta \wedge \Im(\imath, \xi, \rho)) \preceq \Im(\zeta, \xi, \rho) \vee \Im(\imath, \xi, \rho)$. This gives $\Im(\zeta \wedge \imath, \xi, \rho) \preceq \Im(\zeta, \xi, \rho) \vee \Im(\imath, \xi, \rho)$.
(ii) Let $\zeta \leq \imath$ then,
$\zeta \wedge \Im(\imath, \xi, \rho) \preceq \imath \wedge \Im(\zeta, \xi, \rho) \preceq \imath$. Also $\Im(\zeta, \xi, \rho) \wedge \imath \preceq \imath$. This implies $\Im(\zeta \wedge \imath, \xi, \rho) = (\Im(\zeta, \xi, \rho) \wedge \imath) \vee (\zeta \wedge \Im(\imath, \xi, \rho)) \preceq \imath \vee \imath$. This gives $\Im(\zeta \wedge \imath, \xi, \rho) \preceq \imath$.
(iii) Let J is a lattice and $\Im$ be a permuting tri-multiderivation on J, then
$\Im(\zeta, \xi, \rho) = \Im(\zeta \wedge \zeta, \xi, \rho)$=$[\Im(\zeta, \xi, \rho) \wedge \zeta] \wedge [\zeta \wedge \Im(\zeta, \xi, \rho)]$. Also $\Im(\zeta, \xi, \rho) \vee \zeta = [\Im(\zeta, \xi, \rho) \wedge \zeta] \vee [\zeta \wedge \Im(\zeta, \xi, \rho)] \vee \zeta$=$(\Im(\zeta, \xi, \rho) \wedge \zeta) \vee \zeta \vee (\zeta \wedge \Im(\zeta, \xi, \rho))$=$\zeta$. This implies $\Im(\zeta, \xi, \rho) \preceq \zeta$.
(iv) Let J be a lattice and $\imath, \xi, \rho \in J$, then
$\Im(\zeta, \xi, \rho) \wedge \Im(\imath, \xi, \rho) \subset [\Im(\zeta, \xi, \rho) \wedge \Im(\imath, \xi, \rho)] \vee [\Im(\zeta, \xi, \rho) \wedge \Im(\imath, \xi, \rho)] \preceq (\Im(\zeta, \xi, \rho) \wedge \imath) \vee \zeta \wedge \Im(\imath, \xi, \rho)$=$\Im(\zeta \wedge \imath, \xi, \rho)$. Which implies $\Im(\zeta, \xi, \rho) \wedge \Im(\imath, \xi, \rho) \subset \Im(\zeta \wedge \imath, \xi, \rho)$. □

Proposition 3.5. *Let* $(J, \wedge, \vee)$ *be an incline algebra with a zero element and* $\Im$ *be a permuting tri-multiderivation on* J *with trace* $\hbar$. *Then* $\hbar(0) = 0$.

Proof. Since $\hbar(0) = \Im(0, 0, 0)$=$\Im(\zeta \wedge 0, 0, 0)$=$(\Im(\zeta, 0, 0) \wedge 0) \vee (\zeta \wedge \Im(0, 0, 0))$= $0 \vee (\zeta \wedge \Im(0, 0, 0))$= $(\zeta \wedge \Im(0, 0, 0))$. Taking $\zeta = 0$ we get, $\hbar(0) = 0$. □

Proposition 3.6. *Let* J *be an incline algebra with multiplicative identity 1 and* $\hbar$ *be a trace of* $\Im$*, Then following hold:*
(i) $\zeta \wedge \Im(1, \xi, \rho) \preceq \Im(\zeta, \xi, \rho)$.
(ii) *If* $\hbar(1) = 1$ *then* $\zeta \preceq \Im(\zeta, 1, 1)$.
(iii) *Moreover* $\Im(1, \xi, \xi) \preceq \Im(\zeta, \xi, \xi)$ *whenever* $\Im(1, \xi, \xi) \preceq \zeta$.
(iv) *If* $\zeta \preceq \Im(1, \xi, \xi)$ *then* $\zeta \in \Im(\zeta, \xi, \xi)$.

Proof. (i) Since $\Im(\zeta, \xi, \xi)$=$\Im(\zeta \wedge 1, \xi, \xi)$=$(\Im(\zeta, \xi, \xi) \wedge 1) \vee (\zeta \wedge \Im(1, \xi, \xi))$=$\Im(\zeta, \xi, \xi) \vee (\zeta \wedge \Im(1, \xi, \xi))$. This implies $(\zeta \wedge \Im(1, \xi, \xi)) \preceq t(\zeta, \xi, \xi)$.
(ii) let $\hbar(1) = 1$ then by using $\xi = 1$ in above result we have, $\zeta \wedge \Im(1, 1, 1) \preceq \Im(\zeta, 1, 1)$. Which implies $\zeta \wedge \hbar(1) \preceq \Im(\zeta, 1, 1)$. Hence $\zeta \preceq \Im(\zeta, 1, 1)$.
(iii) Now $\Im(1, \xi, \xi) \preceq \Im(\zeta, \xi, \xi)$ when $\Im(1, \xi, \xi) \preceq p$. As we know $\zeta \wedge \Im(1, \xi, \xi) \preceq \Im(\zeta, \xi, \xi)$. This implies $\Im(1, \xi, \xi) \preceq \Im(\zeta, \xi, \xi)$ $therefore$ by given condition
(iv) Let $\zeta \preceq \Im(1, \xi, \xi)$. Then we have, $\zeta \preceq r$ for some $r \in \Im(1, \xi, \xi)$. Therefore $\zeta = \Im(\zeta, \xi, \xi) \vee p$ forall $\zeta, \xi \in J$ therefore by $2.5(iii)$ we have, $= \Im(\zeta, \xi, \xi) \vee (\zeta \wedge r) \in \Im(\zeta, \xi, \xi) \vee (\zeta \wedge \Im(1, \xi, \xi)) = (\Im(\zeta, \xi, \xi) \wedge 1) \vee (\zeta \wedge \Im(1, \xi, \xi))$. $\Im(\zeta \wedge 1, \xi, \xi) = \Im(\zeta, \xi, \xi)$. This gives $\zeta \in \Im(\zeta, \xi, \xi)$. □

Proposition 3.7. *Let* $(J, \vee, \wedge)$ *be an integral incline algebra,* $\Im$ *be permuting tri-multideviation on* $J \times J \times J$. *If* $\alpha, \zeta, \xi \in J$ *and* $\alpha \wedge \Im(\zeta, \xi, \rho) = 0$. *Then either* $\alpha = 0$ *or* $\Im = 0$.

Proof. Let $\imath \in J$ and $\alpha \wedge \Im(\zeta, \xi, \rho) = 0$. Replacing ζ by $\zeta \wedge \imath$, we have $0 = \alpha \wedge \Im(\zeta \wedge \imath, \xi, \rho) = \alpha \wedge (\Im(\zeta, \xi, \rho) \wedge \imath) \vee (\zeta \wedge \Im(\imath, \xi, \rho)) = (\alpha \wedge \Im(\zeta, \xi, \rho) \wedge \imath) \vee (\alpha \wedge (\zeta \wedge \Im(\imath, \xi, \rho)))$= $\alpha \wedge (\zeta \wedge \Im(\imath, \xi, \rho))$ forall $\zeta, \xi, \imath, \rho \in J$. By using $\zeta = 1$ we get $0 = \alpha \wedge (1 \wedge \Im(\imath, \xi, \rho))$

$= \alpha \wedge \Im(\imath, \xi, \rho)$
Since J has no zero divisor therefore either $\alpha = 0$ or $\Im(\imath, \xi, \rho) = 0$ forall $\zeta, \xi, \imath, \rho \in J$. However we can get $\alpha = 0$ or $\Im = 0$ where $\Im(\zeta, \xi, \rho) \wedge \alpha = 0$. □

Proposition 3.8. *Let $(J, \vee, \wedge)$ be an incline algebra, $\Im$ a superjoinitive permuting tri-multiderivation on J and $\hbar$ be a trace of J, So we get the following:*
(i) $\hbar(\zeta) \vee \hbar(\xi) \preceq \hbar(\zeta \vee \xi)$.
(ii) $\Im(\zeta \wedge \xi, \xi, \xi) \preceq \hbar(\zeta)$.
(iii) *Also $\Im$ is an isotone permuting tri-multideviation on J.*

Proof. (i) Since $\Im$ is superjoinitive so we get $\hbar(\zeta \vee \xi)$=$\Im(\zeta \vee \xi, \zeta \vee \xi, \zeta \vee \xi) \supseteq \Im(\zeta, \zeta \vee \xi, \zeta \vee \xi) \vee \Im(\xi, \zeta \vee \xi, \zeta \vee \xi) \supseteq \Im(\zeta, \zeta, \zeta \vee \xi) \vee \Im(\zeta, \xi, \zeta \vee \xi) \vee \Im(\xi, \zeta, \zeta \vee \xi) \vee \Im(\xi, \xi, \zeta \vee \xi) \supseteq \Im(\zeta, \zeta, \zeta) \vee \Im(\zeta, \zeta, \xi) \vee \Im(\zeta, \xi, \zeta) \vee \Im(\zeta, \xi, \xi) \vee \Im(\xi, \zeta, \zeta) \vee \Im(\xi, \zeta, \xi) \vee \Im(\xi, \xi, \zeta) \vee \Im(\xi, \xi, \xi)$. Which gives $\hbar(\zeta \vee \xi) \supseteq \hbar(\zeta) \vee \hbar(\xi) \vee \Im(\zeta, \zeta, \xi) \vee \Im(\zeta, \xi, \xi)$. Hence we get $\hbar(\zeta) \vee \hbar(\xi) \preceq \hbar(\zeta \vee \xi)$ forall $\zeta, \xi \in J$.
(ii) $\hbar(\zeta) = \Im(\zeta, \zeta, \zeta) = \Im(\zeta \vee (\zeta \wedge \xi), \zeta, \zeta) \supseteq \Im(\zeta, \zeta, \zeta) \vee \Im(\zeta \wedge \xi, \zeta, \zeta) = \hbar(\zeta) \vee \Im(\zeta \wedge \xi, \zeta, \zeta)$. This implies $\hbar(\zeta) \succeq \Im(\zeta \wedge \xi, \zeta, \zeta)$.
(iii) Let $(\zeta, \xi, \rho) \preceq (\aleph, \beta, \gamma)$ that is $\zeta \leq \aleph\, \xi \leq \beta$ and $\rho \leq \gamma$. Then we get $\Im(\aleph, \beta, \gamma)$=$\Im((\aleph, \beta, \gamma) \vee (\aleph, \beta, \gamma)) \supseteq \Im(\aleph, \beta, \gamma) \vee \Im(\zeta, \xi, \rho) \vee \Im(\zeta, \beta, \gamma) \vee \Im(\aleph, \xi, \rho)$. This gives $\Im(\zeta, \xi, \rho) \preceq \Im(\aleph, \beta, \gamma)$. □

Definition 3.9. *Let $(J, \vee, \wedge)$ be an incline algebra, $\Im$ be a permuting tri-multideviation on $J \times J \times J$ and $\hbar$ be the trace of $\Im$. Then the set of fixed point of $\Im$ is $Fix_{\Im}(J \times J \times J)$=$\{(\zeta, \xi, \rho) \in J \times J \times J | \zeta, \xi, \rho \in \Im(\zeta, \xi, \rho)\}$. Since $\Im$ is permuting $(\zeta, \xi, \rho) \in Fix_{\Im}(J \times J \times J)$ iff $(\rho, \xi, \zeta) = (\xi, \rho, \zeta) = (\xi, \zeta, \rho) = (\rho, \zeta, \xi) \in Fix_{\Im}(J \times J \times J)$. The set of fixed point of $\hbar$ is denoted by $Fix_{\hbar}(J) = \{\zeta \in J, | \zeta \in \hbar(\zeta)\}$.*

Remark 3.10. *An Inline algebra J is a Distributive Lattice iff $\zeta \wedge \zeta = \zeta$ for all $\zeta \in J$.*

Theorem 3.11. *Let J is a Distributive Lattice, $\Im$ a superjoinitive permuting tri-multideviation on $J \times J \times J$ and $\hbar$ be the trace of $\Im$. Then following hold: (i)$\hbar(\zeta \wedge \xi) \preceq (\hbar(\zeta) \wedge \xi) \vee (\zeta \wedge \hbar(\xi))$.*
(ii)$\hbar(\zeta \wedge \xi) \succeq (\hbar(\zeta) \wedge \xi) \vee (\zeta \wedge \hbar(\xi))$ forall $\zeta, \xi \in J$.

Proof. Let $\zeta, \xi \in J$. Since $\hbar(\zeta) = \Im(\zeta, \zeta, \zeta) = \Im(\zeta \wedge \zeta, \zeta, \zeta) = (\Im(\zeta, \zeta, \zeta) \wedge \zeta) \vee (\zeta \wedge \Im(\zeta, \zeta, \zeta)) = (\hbar(\zeta) \wedge \zeta) \vee (\zeta \wedge \hbar(\zeta)) \preceq \zeta$. Also $\hbar(\zeta \wedge \xi) = \Im(\zeta \wedge \xi, \zeta \wedge \xi, \zeta \wedge \xi) = (\Im(\zeta, \zeta \wedge \xi, \zeta \wedge \xi) \wedge \xi) \vee (\zeta \wedge \Im(\xi, \zeta \wedge \xi, \zeta \wedge \xi) \wedge \xi)$. So by using proposition 3.8 we get $\hbar(\zeta \wedge \xi) \preceq (\hbar(\zeta) \wedge \xi) \vee (\zeta \wedge \hbar(\xi))$.
(ii) On the other hand $\hbar(\zeta \wedge \xi) = \Im(\zeta \wedge \xi, \zeta \wedge \xi, \zeta \wedge \xi) = (\Im(\zeta, \zeta \wedge \xi, \zeta \wedge \xi) \wedge \xi) \vee (\zeta \wedge \Im(\xi, \zeta \wedge \xi, \zeta \wedge \xi)) = \{(\Im(\zeta, \zeta, \zeta \wedge \xi) \wedge \xi) \vee (\zeta \wedge \Im(\zeta, \xi, \zeta \wedge \xi) \wedge \xi)\} \vee \{(\zeta \wedge (\Im(\xi, \zeta, \zeta \wedge \xi) \wedge \xi) \vee (\zeta \wedge \Im(\xi, \xi, \zeta \wedge \xi)))\}$. Since J be a distributive lattice so we have $= (\Im(\zeta, \zeta, \zeta \wedge \xi) \wedge \xi) \vee \Im(\zeta, \xi, \zeta \wedge \xi) \vee (\Im(\xi, \zeta, \zeta \wedge \xi)) \vee \zeta \wedge \Im(\xi, \xi, \zeta \wedge \xi)$.
$= [\{(\Im(\zeta, \zeta, \zeta) \wedge \xi) \vee (\zeta \wedge \Im(\zeta, \zeta, \xi))\} \wedge \xi] \vee [(\zeta \wedge \Im(\zeta, \xi, \xi)) \vee (\Im(\zeta, \xi, \zeta) \wedge \xi)] \vee [(\zeta \wedge \Im(\xi, \zeta, \xi)) \vee (\Im(\xi, \zeta, \zeta) \wedge \xi)] \vee [(\zeta \wedge \{\Im(\xi, \xi, \zeta) \wedge \xi) \vee (\zeta \wedge \Im(\xi, \xi, \xi)\}]$. Since J be a distributive lattice and $\Im$ is permuting so we get $= (\hbar(\zeta) \wedge \xi) \vee (\zeta \wedge \hbar(\xi)) \vee \Im(\zeta, \zeta, \xi) \vee \Im(\zeta, \xi, \xi)$. This implies $(\hbar(\zeta) \wedge \xi) \vee (\zeta \wedge \hbar(\xi)) \preceq \hbar(\zeta \wedge \xi)$. □

Theorem 3.12. *Let J is a Distributive Lattice, $\Im$ a superjoinitive permuting tri-multideviation on $J \times J \times J$ and $\hbar$ be the trace of $\Im$. If $\xi \leq \zeta$ and $\zeta \in \hbar(\zeta)$ then $\xi \in \hbar(\xi)$.*

Proof. Let $\xi \leq \zeta$ and $\zeta \in \hbar(\zeta)$ then by using theorem 3.11(i) we have $\hbar(\xi) \preceq \xi \leq \zeta$. Also $\hbar(\xi) \vee (\xi \wedge \hbar(\zeta)) = (\hbar(\xi) \wedge \zeta) \vee (\xi \wedge \hbar(\zeta))$. Also by using theorem 3.11(ii) we get $\preceq h(\zeta \wedge \xi) = h(\xi)$ and $\hbar(\xi) \preceq (\hbar(\xi) \wedge \zeta) \vee (\xi \wedge \hbar(\zeta)) = \hbar(\xi) \vee (\xi \wedge \hbar(\zeta)$. Thus $\xi \wedge \hbar(\zeta) \preceq \hbar(\xi) \leq \xi$. On the other hand $\xi = \xi \wedge \zeta \in \xi \wedge \hbar(\zeta) \preceq \hbar(\xi)$. Hence we get $\xi \in \hbar(\xi)$. □

Corollary 3.13. *Let J be the distributive lattice with a greatest element 1, $\Im$ a superjoinitive permuting tri multiderivative on $J \times J \times J$ and $\hbar$ be a trace of $\Im$. Then $1 \in \hbar(1)$ iff $Fix_{\hbar}(J) = J$.*

Proof. Suppose that $1 \in \hbar(1)$. Since $\zeta \leq 1$ forall $\zeta \in J$. So by using theorem 3.12 we get $p \in h(p)$ for all $p \in J$. This gives $Fix_{\hbar}(J) = J$. □

Definition 3.14. *Let $(J, \vee, \wedge)$ be an integral incline algebra ,$\Im$ be a permuting tri-multideviation on $J \times J \times J$ and $\hbar$ be the trace of $\Im$. Then we define $\hbar^2(\zeta) = \hbar(\hbar(\zeta)) = \bigsqcup_{\xi \in \hbar^2(\zeta)} \hbar(\xi)$.*

Theorem 3.15. *Let J be a distributive lattice and $\Im$ a superjoinitive permuting tri-multideviation on $J \times J \times J$ and $\hbar$ be the trace of $\Im$. Then $Fix_{\hbar}(J)$ is an Ideal of J.*

Proof. Let $\zeta, \xi \in Fix_{\hbar}(J)$. Then

$$\zeta \vee \xi \in \hbar(\zeta) \vee \hbar(\xi) \tag{3. 1}$$

By using proposition 3.8 we get J is an Isotone permuting tri-multiderivation on $J \times J \times J$. Hence we have $\hbar(\zeta) \preceq \hbar(\zeta \vee \xi)$ and $\hbar(\xi) \preceq \hbar(\zeta \vee \xi)$. This implies

$$\hbar(\zeta) \vee \hbar(\xi) \preceq \hbar(\zeta \vee \xi) \tag{3. 2}$$

By combining equation 3. 1 and 3. 2 we have $\zeta \vee \xi \in \hbar(\zeta) \vee \hbar(\xi) \preceq \hbar(\zeta \vee \xi)$. This gives $\zeta \vee \xi \in \hbar(\zeta \vee \xi)$. This Implies $\zeta \vee \xi \in Fix_{\hbar}(J)$. Moreover $\zeta \vee \xi = (\zeta \vee \xi) \vee (\zeta \vee \xi) \in (\hbar(\zeta) \wedge \xi) \vee (\zeta \wedge \hbar(\xi)) \preceq \zeta \wedge \xi$. Hence, $\zeta \wedge \xi \in Fix_{\hbar}(J)$. Now suppose $\zeta \in Fix_{\hbar}(J)$ and $\xi \in J$ such that $\xi \leq \zeta$. Then by using theorem 3.12 we have $Fix_{\hbar}(J)$ is an ideal of J. □

Theorem 3.16. *Let $(J, \vee, \wedge)$ be an integral incline algebra. Suppose there exist joinitive permuting tri-multiderivations $\Im_1$ and $\Im_2$ such that $\Im_1(\hbar_2(\zeta), \zeta, \zeta) = 0$ forall $\zeta \in J$. Then either $0 \in \hbar_1(\zeta)$ or $0 \in \hbar_2(\zeta)$.*

Proof. Since $\hbar_2(\zeta) \subset \hbar_2(\zeta) \vee (\hbar_2(\zeta) \wedge \zeta)$. Then $0 = \Im_1(\hbar_2(\zeta), \zeta, \zeta) \subset \Im_1(\hbar_2(\zeta) \vee (\hbar_2(\zeta) \wedge \zeta), \zeta, \zeta) = \Im_1(\hbar_2(\zeta), \zeta, \zeta) \vee \Im_1(\hbar_2(\zeta) \wedge \zeta), \zeta, \zeta) = 0 \vee \Im_1(\hbar_2(\zeta) \wedge \zeta), \zeta, \zeta) = \Im_1((\hbar_2(\zeta), \zeta, \zeta)) \wedge \zeta \vee (\hbar_2(\zeta) \wedge \Im_1(\zeta, \zeta, \zeta)) = 0 \vee (\hbar_2(\zeta) \vee \hbar_1(\zeta))= \hbar_2(\zeta) \vee \hbar_1(\zeta)$. Hence there exist $\aleph \in \hbar_2(\zeta)$ and $\alpha \in \hbar_1(\zeta)$ such that 0= $\aleph \wedge \alpha$. Since J is integral Incline Algebra so either $\aleph = 0$ or $\alpha = 0$. Therefore either $0 \in \hbar_1(\zeta)$ or $0 \in \hbar_2(\zeta)$. □

Conclusion: Keeping in view the importance of genralization of derivations which are appearing more useful and convenient tool in the field of abstract algebra, we have generalized symmetric bi-multderivation on incline algebra with permuting tri-multiderivation on incline algebra. By using this notion we have proved some useful results.

REFERENCES

1. Ahn S. S. , Jun Y. B. and Kim H. S. , *Ideal and Qoutients of incline algebras*, commun. Korean Math. Soc. , **(16)**(2001), 573-583.
2. Ahn S. S. , Jun Y. B. and Kim H. S. , *On r-ideals in incline algebra*, commun. Korean Math. Soc. , **17**(2002) 229-235.
3. Alshehri N. O. , *On derivations of Incline algebras*, Sci. Math Jpn. , **71**(2010) 199-205.
4. Cao Z. Q. , Kim K. H. and Roush F. W. , *Incline algebras and applications*, Ellis Horward Series: Mathematics and Its Applications, Ellis Horward, Chichester, UK. , 1984.
5. Chaudhry M. A. and Khan A. R., *On symmetric f-triderivations of lattices*, Quaestiones Mathematcae. , *35*(No.2(2012) 203-207.
6. Jun Y. B. and Ahn S. S. and Kim H. K. , *Fuzzy subincline (ideals) of incline algebras*, fuzzy Sets and System. , 123(2001) 217-255.
7. Khan A. R. and Javaid I. Chaudhary M. A. , *Permuting tri-derivation in incline algebras*, World Applied Sciences Journal. , **19,**No.11(2012) 1649-1652.
8. Kim H. M., *Symmetric Bi-f-derivations in lattices*, Int. J. Math. Archive, **3,**No.10(2012) 3676-3683.
9. Khan A. R. and Chaudhary M. A. , *Permuting f-triderivation on lattice*s, Int. J. Algeb. , **5,**No.10(2011) 471-481.
10. Ozbal S. and Firat A. , *On f-derivation on incline algebras* , Int. J. Pure Appl. Math. , (2011).
11. Ozbal S. and Firat A. , *On Symmetric bi-derivation of incline algebras*, Int. Math. Forum. , **6**(2011) 2031-2036.
12. Posner E. *Derivations in prime rings*, Proc. Amer. Math. Soc. , **8**(1957) 1093-1100.
13. Rezapour S. and Sami S., *On some properties on isotone and joinitive multiderivations of lattices*, Filomat, 30**(30,)**No.10 (2016) 2743-2748.
14. Rezapour S. and Sami S., *Symmetric bi-multiderivations on incline algebras*, J. Adv. Math. Stud. , **8,** No.1(2015) 53-59.
15. Xin X. L. and Li T. Y., *Fixed Set of derivations in Lattices*, Fixed Point Theory Appl. , (2012) 212-218.
16. xin X. L. li. T. Y. and Lu. J. H., *On Derivation of Lattices*, Int. Sci. , (178)(2008) 307-316.
17. Yazarli H., and Ozturk M. A., *Permuting Tri-f-derivations of lattices*, Common Korean Math. Soc., **45,**No.4(2008) 701-704.