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On the Shape Factor of Emden-Fowler Equation of Higher Order and its Numerical Solution by Successive Differentiation Method

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Abstract. The main objective of this work is to numerically solve three types of Fourth order Emden-Fowler equations by using Successive Differentiation Method (SDM) and to inspect its shape factor 'k' in detail. Numerical solutions and graphical illustrations give a detailed insight into the variation of numerical solution due to shape factor and evince the reason why 'k' is always kept positive. The singularity term in Emden-Fowler equations makes it difficult to find its analytical solution. It is usually arduous to obtain the numerical solutions due to its singularity. Successive Differentiation Method is easy to understand and give accurate results as compared to its contemporary methods. Graphical illustrations have been shown to establish our argument.

AMS Subject Classification Codes: 65D25; 60G30; 34-XX; 35B30; 34Axx
Key Words: Successive Differentiation Method, Shape Factor, Singularity, Fourth Order Emden-Fowler Equation.

1. Introduction

Study of Universe always fascinated mankind. There is plethora of secrets in this universe to be revealed. Mathematicians have paved ways in studying these phenomena via developing some very famous equations. One of these equations is Emden-Fowler differential equation. It has a vast variety of applications in different fields of science. The general form of this equation is written as

$$x^{-k}\frac{d^m}{dx^m}\Big(x^k\frac{d^n}{dx^n}\Big)y+f(x)g(y)=0;\ where\, k>0 \tag{1.1}$$

Eq.(1.1) developed by Wazwaz [21] has many interesting facts about it. For fourth order Emden-Fowler equation $n+m=4$, $m,n \geq 1$, three cases arise as

(i) $m=3, n=1$ for which Eq.(1.1) becomes

$$x^{-k}\frac{d^3}{dx^3}\Big(x^k\frac{d}{dx}\Big)y + f(x)g(y) = 0;\ where\, k>0 \tag{1.2}$$

(ii) For $m=2, n=2$, Eq.(1.1) can be written as

$$x^{-k}\frac{d^2}{dx^2}\Big(x^k\frac{d^2}{dx^2}\Big)y + f(x)g(y) = 0;\ where\, k>0 \tag{1.3}$$

(iii) And the last possible case i.e. $m=1, n=3$, Eq.(1.1) becomes

$$x^{-k}\frac{d}{dx}\Big(x^k\frac{d^3}{dx^3}\Big)y + f(x)g(y) = 0;\ where\, k>0 \tag{1.4}$$

The term x^{-k} describes the singularity of this equation at $x=0$. Here k is the shape factor of this equation and it shows the variations in the solution of the equation. Since $m+n$ decides the order of differential equation, therefore, a specific case of Eq.(1.1) with order two is the famous Lane-Emden equation, that has applications in astrophysics, chemistry, physics etc. Further more for second order singular Lane-Emden equation if $g(y)$ is an exponential function, then in stellar structures the density distribution of isothermal gas sphere is represented by this Lane-Emden equation like in thermodynamics, it is considered that the structure of the stars is the gaseous sphere. Another case of Eq.(1.1) in thermodynamics can be if $g(y)=y^m$, where m is known as the index and hence Eq.(1.1) describes the thermal behavior of gases in spherical clouds. Another famous equation is derived from Eq.(1.1), known as white dwarf equation, by $g(y)=(y^2-C)^{3/2}$. This equation becomes Lane-Emden equation with index 3 for $C=0$.

In [10] for the first time in history Fowler presented the idea of Emden-Fowler Equations as an application of astronomical problem. Later in [11] and [12] Fowler discussed its transformation into simpler forms and its special cases. In 1975 James [23] discussed the other properties of this equation such as oscillation, continuability, stability, boundedness, boundary value problem and asymptotic growth. Domoshnitsky et al. [8] discussed the asymptotic behavior of solutions obtained for delay type Emden-Fowler equations. Classification of proper solutions were discussed by Kamo see [14]. Lie symmetry analysis of Emden-Fowler equation was studied by Khalique [16]. In [1] Aslanov obtained the exact solutions of the first kind of Emden-Fowler equation. In [3] Berkovich discussed the generalized Emden-Fowler equation by autonomization technique. Mohamed El-Gamel used B-Spline method to analyze the numerical solution of Emden-Fowler Equation (see [9]). Recently, Wazwaz [21] derived the fourth order singular Emden-Fowler differential equations and solved them numerically by using Adomian Decomposition Method.

To decipher the secrets in such equations, various methods have been developed and applied to analyze their behaviors in certain situations. Emden-Fowler have always been the center of attention among numerous mathematicians due to its singularity property. Since in most cases its analytical solution can not be obtained so numerical solutions are likely to be acquired.

In [13] Freire et al. studied fourth order Emden-Fowler equation by using Lie and Noether symmetries. In [17] Najeeb et al. solved various examples of fourth order Emden-Fowler

equation by Haar based wavelet collocation method. Bildik et al. applied Optimal Perturbation Iteration Method (OPIM) and Modified Adomian Decomposition Method and to obtain the numerical solution of Emden-Fowler equation [4], [5]. Details about the Optimal Perturbation Iteration Method is given in detail in [7]. Bildik et al. also solved the delay differential type equation of Emden-Fowler and compared their results with contemporary methods which were proven to be less accurate [6]. In [18] Khoury et al. used Adaptive Collocation method for finding the numerical solution of Emden-Fowler singular differential equation and claimed to be the accurate technique . In [2] Astashove described the existence of oscillatory and non-oscillatory quasi-periodic solutions of higher order Emden-Fowler equation. Randolph et al. [19] also studied two dimensional higher order Emden-Fowler differential equation by Adomian Decomposition method. Recently, Rogachev [20] theoretically proved the existence of solutions of generalized Emden-Fowler equation by Lipschitz continuity and concepts of boundedness.
Among these methods is the Successive Differentiation Method. This method has yet been proved to be the uncomplicated and effortless method for attaining numerical solutions of singular differential equations. In [15], authors solved a variety of singular differential equations by using SDM. In this paper, a fourth order Emden-Fowler differential equation is solved numerically by Successive Differentiation Method (SDM). Three cases have been solved and analyzed in this work. Importance of shape factor k in feasibility of solution is also discussed in detail.
This paper is divided into following parts. Section I contains the brief literature review of this work. Section II is the mathematical formulation of Successive Differentiation Method on fourth order Emden-Fowler differential equation. Section III has the major contribution in this paper as it specifically discusses all three possible cases for fourth order Emden-Fowler differential equation and its numerical approximations for different $f(x)$ and $g(y)$ in each case along with graphical illustrations and detailed discussion of shape factor. Section IV is the last section that concludes the main results and observations of this paper.

2. Mathematical Formulation

Consider the nth order singular differential equation as

$$u^k(x) = xR_\circ(x), \;\; R_\circ(x) = L(x) + N(x) \tag{2.5}$$

which implies

$$u^k(x) = xL\Big(u(x), u^{'}(x), \cdots, u^{(k-1)}(x)\Big) + xN\Big(u(x), u^{'}(x), \cdots, u^{(k-1)}(x)\Big) \tag{2.6}$$

with initial conditions $u(0) = \alpha_1$, $u^{'}(0) = \alpha_2$, $u^{''}(0) = \alpha_3$, ..., $u^{(k-1)}(0) = \alpha_k$. On differentiating Eq.(2.6) successively it becomes

$$\begin{aligned} u^{k+1}(x) &= xL\Big(u^{'}(x), u^{''}(x), \cdots, u^{(k)}(x)\Big) + xN\Big(u^{'}(x), u^{''}(x), \cdots, u^{(k)}(x)\Big) + R_\circ(x) \\ &= xR_1(x) + R_\circ(x) \end{aligned} \tag{2.7}$$

where $R_1(x)$ denote the first derivative of $R_0(x)$.

$$\begin{aligned}
u^{k+2}(x) =& xL\Big(u^{''}(x), u^{'''}(x), \cdots, u^{(k+1)}(x)\Big) + xN\Big(u^{''}(x), u^{'''}(x), \cdots, u^{(k+1)}(x)\Big) + \\
& R_{\circ}(x) \\
=& xR_2(x) + 2R_1(x) \\
u^{k+3}(x) =& xL\Big(u^{'''}(x), u^{''''}(x), \cdots, u^{(k+2)}(x)\Big) + xN\Big(u^{'''}(x), u^{''''}(x), \cdots, u^{(k+2)}(x)\Big) + \\
& R_{\circ}(x) \\
=& xR_3(x) + 3R_2(x) \\
& . \\
& . \\
& . \\
u^{k+n}(x) =& xR_n(x) + nR_{n-1}(x)
\end{aligned} \tag{2.8}$$

where $R_2(x)$, $R_3(x), ..., R_n(x)$ are the second, third ,..., upto nth order derivatives. Now apply the initial conditions on Eq.(2.8) to find the values of higher order derivative terms i.e. $u^{(k)}(0) = \beta_{\circ}$, $u^{(k+1)}(0) = \beta_1$, $u^{(k+2)}(0) = \beta_2$, $u^{(k+3)}(0) = \beta_3$,...etc. By expanding the Taylor series of $u(x)$ it becomes,

$$\sum_{n=0}^{\infty} \frac{u^n(x)}{n!} x^n = \alpha_{\circ} + \alpha_1 x + \alpha_2 \frac{x^2}{2} + \alpha_3 \frac{x^3}{3} + \cdots + \beta_{\circ} \frac{x^k}{k!} + \beta_1 \frac{x^{k+1}}{(k+1)!} + \beta_2 \frac{x^{k+2}}{(k+2)!} + \cdots \tag{2.9}$$

Therefore series given in Eq.(2.9) is said to be the numerical solution of nth order singular differential equation in Eq.(2.5).

3. Analysis of SDM on Fourth Order Emden-Fowler Equations

Fourth order Emden-Fowler equation is the best example for proving the accuracy of any numerical method that claims to solve the singular ordinary differential equations. Therefore in this section two examples of each case of fourth order Emden-Fowler equation have been considered with different $f(x)$ & $g(y)$ and solved numerically by SDM. Also the changes occurring in numerical solution due to shape factor k have been discussed for each example.

3.1. **Case-1.** For $n = 3$, $m = 1$, expanding Eq.(1.2) it becomes

$$y^{iv}(x) + \frac{3k}{x} y^{'''}(x) + \frac{3k(k-1)}{x^2} y^{''}(x) + \frac{3k(k-1)(k-2)}{x^3} y^{'}(x) + f(x)g(y) = 0 \tag{3.10}$$

with initial condition $y(0) = y_\circ$, $y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. Taking successive derivatives of Eq.(3.10), it eventually becomes

$$\begin{aligned}
&g(y)f'(x) - \frac{3(k-2)(k-1)ky'(x)}{x^4} + \frac{(k-2)(k-1)ky''(x)}{x^3} - \frac{6(k-1)ky''(x)}{x^3} \\
&+ \frac{3(k-1)ky^{(3)}(x)}{x^2} - \frac{3ky^{(3)}(x)}{x^2} + \frac{3ky^{(4)}(x)}{x} + y^{(5)}(x) = 0 \\
&g(y)f''(x) + \frac{12(k-2)(k-1)ky'(x)}{x^5} + \frac{18(k-1)ky''(x)}{x^4} - \frac{6(k-2)(k-1)ky''(x)}{x^4} + \\
&\frac{(k-2)(k-1)ky^{(3)}(x)}{x^3} + \frac{6ky^{(3)}(x)}{x^3} - \frac{12(k-1)ky^{(3)}(x)}{x^3} + \frac{3(k-1)ky^{(4)}(x)}{x^2} - \\
&\frac{6ky^{(4)}(x)}{x^2} + \frac{3ky^{(5)}(x)}{x} + y^{(6)}(x) = 0 \\
&. \\
&. \\
&.
\end{aligned} \tag{3.11}$$

Estimating the higher order derivative values at $x = 0$ from Eq.(3.11) and expanding them by the Taylor series of $y(x)$ it turns out to be

$$\begin{aligned}
&y_\circ - \frac{g(y)f(0)}{4(k^3+6k^2+11k+6)}x^4 - \frac{g(y)f^{'}(0)}{5(k^3+9k^2+26k)}x^5 - \\
&\frac{g(y)f^{''}(0)}{12(k^3+12k^2+47k+60)}x^6 + \cdots
\end{aligned} \tag{3.12}$$

Series in Eq.(3.12) is known to be the numerical solution for case I. It gives an insight about the existence of solution of Eq.(3.10). As it can be observed that the polynomial term of the shape factor in the denominator shows that the solution for this problem does not exist $\forall k \in Z^-$. To be more specific, by observing the role of shape factor and check the accuracy of this method more precisely, two numerical examples for this case have been solved.

3.1.1. *Example 1.* For the first case, let $g(y) = e^{-4y(x)}$, $f(x) = 150x(4x^{10} - 17x^5 + 4)$ and initial conditions be as $y_\circ = 0$,$y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. After applying initial conditions the obtained Taylor series from Eq.(3.12) becomes

$$y(x) = \frac{30}{k^2+5k+6}x^5 - \frac{5(17k^2+85k+582)}{3(k^2+5k+6)(k^2+15k+56)}x^{10} + \cdots \tag{3.13}$$

which is the general numerical solution of case I for arbitrary k. Now to verify the accuracy of numerical solution let $k = 2$, then the series turns out to be

$$y(x) = x^5 - \frac{1}{2}x^{10} + \frac{1}{3}x^{10} - \frac{1}{4}x^{10} + \cdots \tag{3.14}$$

This is exactly the series obtained from its exact solution $y(x) = \ln(1+x^5)$ see [22]. Fig.1 clearly seconds this argument. If the polynomials given in the denominator of Eq.(3.13) are solved, they suggests that the existence of this numerical solution is only possible if

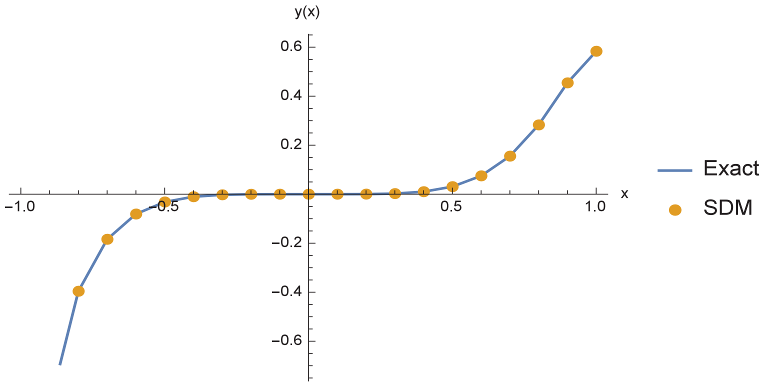

FIGURE 1. Numerical solution obtained from SDM of example 3.1.1 and its comparison with exact solution to verify the accuracy of SDM.

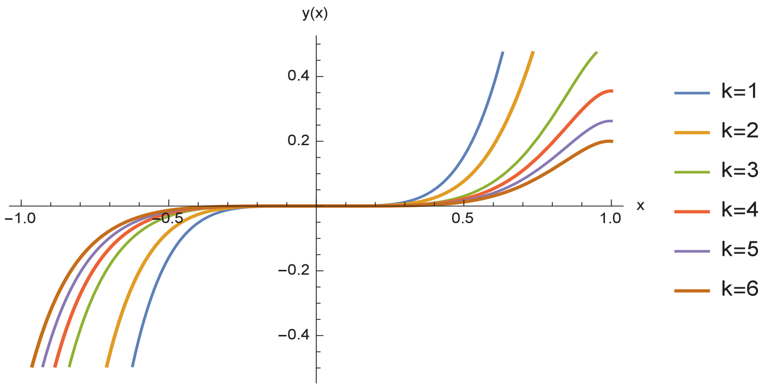

FIGURE 2. Graphical illustration of numerical solution of example 3.1.1 for different values of k. Higher deviation can be observed in this plot due to greater number of terms of shape factor in the Emden-Fowler equation.

$k \notin \{-2, -3, -7, -8\}$. Fig.2 contains the graphical illustration of Eq.(3.13) for different positive values of k. Since in Eq.(3.10) there are three terms of shape factor involved i.e. $k(k-1)(k-2)$, therefore, the variational change in the solution is much higher and graph deviates from its actual place as shown in Fig.2.

3.1.2. *Example 2.* For the second example for numerical approximation of case I, let $g(y) = y^9(x)$, $f(x) = 60(3x^8 - 18x^4 + 7)$ and initial conditions be as $y(0) = 1$,

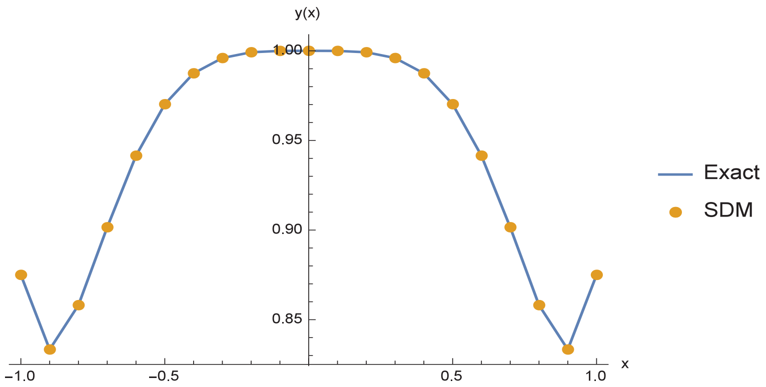

FIGURE 3. Numerical series solution obtained for example 3.1.2 by SDM compared by its exact solution to verify the accuracy of SDM.

$y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. Then Eq.(3.12) transforms into

$$y(x) = 1 - \frac{105}{k^3+6k^2+11k+6}x^4 + \frac{135(2k^3+12k^2+22k+747)}{2(k^3+6k^2+11k+6)(k^3+18k^2+107k+210)}x^8 + \cdots \tag{3.15}$$

which implies to be the general numerical solution of this example for arbitrary k. For $k = 4$, Eq.(3.15) converts into the series as

$$y(x) = 1 - \frac{1}{2}x^4 + \frac{3}{8}x^8 + \cdots \tag{3.16}$$

This numerical series solution is exactly the same as the series of exact solution of Eq.(3.10) i.e. $y(x) = \dfrac{1}{\sqrt{1+x^4}}$ see ([22]). Also Fig.3 verifies this comparison and confirms the efficiency of SDM. Polynomial term in the denominator of the general solution presented in Eq.(3.15) describes its existence. This polynomial suggests that the integrity of Eq.(3.15) is only possible if $k \notin \{-1,-2,-3,-5,-6,-7\}$. Fig.4 depicts the graphical illustration of Eq.(3.15) for different positive values of k. Since in Eq.(3.11), there are three terms of shape factor involved i.e. $k(k-1)(k-2)$, therefore, graph deviates from its original position and high variation can be observed.

3.2. **Case-2.** For $n = 2$, $m = 2$, Eq.(1.3) is acquired and by extending its derivative terms it develops into

$$y^{iv}(x) + \frac{2k}{x}y^{'''}(x) + \frac{k(k-1)}{x^2}y^{''}(x) + f(x)g(y) = 0 \tag{3.17}$$

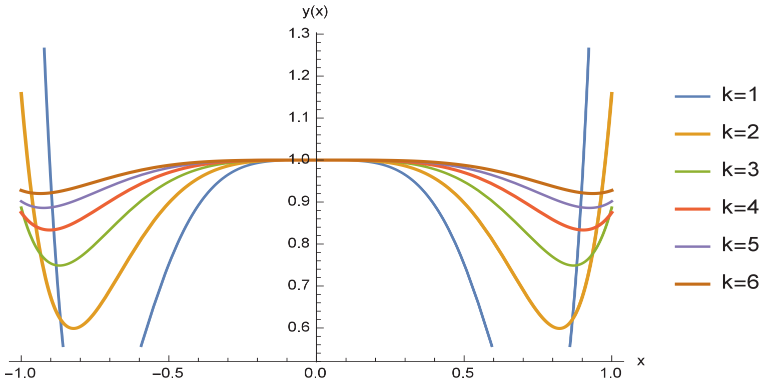

FIGURE 4. Graphical representation of numerical solution of example 3.1.2 for different values of k. Higher deviation can be observed in this plot due to greater number of terms of shape factor in the Emden-Fowler equation.

with initial condition $y(0) = y_\circ$, $y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. Taking successive derivatives of Eq.(3.17) following higher order equations are obtained

$$f(x)g(y) + \frac{(k-1)ky''(x)}{x^2} + \frac{2ky^{(3)}(x)}{x} + y^{(4)}(x) = 0$$
$$g(y)f'(x) - \frac{2(k-1)ky''(x)}{x^3} + \frac{(k-1)ky^{(3)}(x)}{x^2} - \frac{2ky^{(3)}(x)}{x^2} + \frac{2ky^{(4)}(x)}{x} + y^{(5)}(x) = 0$$
$$\vdots \tag{3.18}$$

When initial conditions are employed on Eq.(3.18) and expanded by Taylor series it emerges as

$$y_\circ - \frac{g(y)f(0)}{12(k^2+3k+2)}x^4 - \frac{g(y)f^{'}(0)}{20(k^2+5k+6)}x^5 - \frac{g(y)f^{''}(0)}{60(k^2+7k+12)}x^6 + \cdots \tag{3.19}$$

Upon analyzing this general solution, it is observed that the polynomials of shape factor in denominator suggests that solution for this problem does not exist $\forall k \in Z^-$. Now, let's consider the two numerical examples for this case with different $f(x)$ & $g(y)$ and two terms of shape factor involved in Emden-Fowler equation i.e. k & $(k-1)$ to understand the role of shape factor more precisely and to check the competence of SDM in this case.

3.2.1. *Example 1.* Consider $g(y) = e^{-4y(x)}$, $f(x) = 36x^2(35x^{12} + 146x^6 + 35)$ in Eq.(3.17) and initial conditions as $y(0) = y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. Then for this example Eq.(3.19) can be transformed into the general solution for arbitrary k that is

$$y(x) = -\frac{42}{k^2+7k+12}x^6 - \frac{6(73k^2+511k+3816)}{11(k^2+7k+12)(k^2+19k+90)}x^{12} + \cdots \tag{3.20}$$

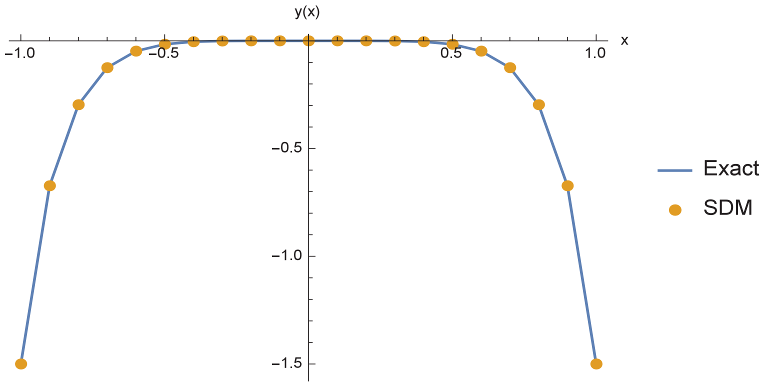

FIGURE 5. Numerical solution obtained from SDM of example 3.2.1 compared by its exact solution to check the accuracy of SDM.

The polynomial term in Eq.(3.20) interprets that the existence of this solution is only possible if $k \notin \{-1,-2,-3,-5,-6,-7\}$. To investigate whether Eq.(3.20) is the accurate numerical solution or not, put $k=3$ in Eq.(3.20) to get

$$y(x) = -x^6 - \frac{1}{2}x^{12} + \cdots \tag{3.21}$$

which is exactly the series of exact solution $y(x) = \ln(1-x^6)$ (see [22]). Hence this shows that SDM is a competent method for handling singular differential equations. Fig.5 shows the comparison of exact solution and numerical solution by SDM and this graphical comparison also supports the argument. Fig.6 sketches the numerical solution in Eq.(3.20) for different values of positive k's. A mild change can be observed in Fig.3.10 from its actual place due to the less number of shape factor terms i.e. $k(k-1)$ in Emden-Fowler equation.

3.2.2. *Example 2.* Now, consider the second example for case 2 to be $g(y) = y^9(x)$, $f(x) = 25x(-49x^{15}+540x^{10}-342x^5+16)$ and initial conditions as $y(0) = y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. Then the obtained Taylor series from Eq.(3.19) is the general solution for this problem for arbitrary k i.e.

$$y(x) = 1 - \frac{20}{k^2+5k+6}x^5 + \frac{5(19k^2+95k+274)}{(k^2+5k+6)(k^2+15k+56)}x^{10} + \cdots \tag{3.22}$$

Eq.(3.22) has polynomial in the denominator that shows that the only possibility of existence of this solution is if $k \notin \{-2,-3,-7,-8\}$. In Fig.8 the graph has been plotted for different values of positive k to show the mild variation occurring in the solution due to the terms involved in Emden-Fowler equation. For $k=2$ the general solution in Eq.(3.22) becomes

$$y(x) = -x^5 - \frac{1}{2}x^{10} + \frac{1}{3}x^{15} - \frac{1}{4}x^{20} + \cdots \tag{3.23}$$

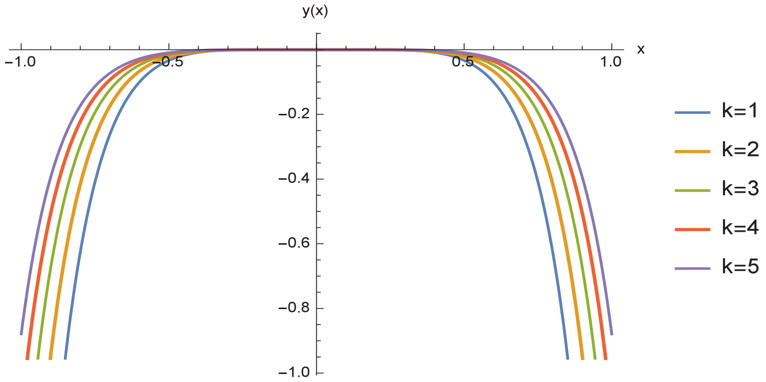

FIGURE 6. Graphical illustration of numerical solution of example 3.2.1 for different values of k. Mild deviation can be observed in this plot due to two terms of shape factor in the Emden-Fowler equation.

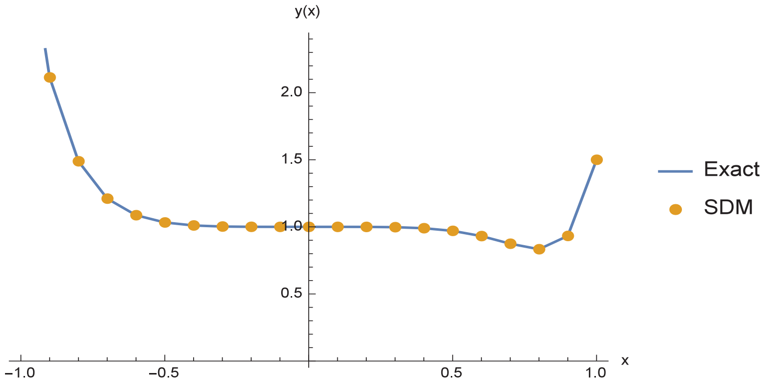

FIGURE 7. Numerical solution of example 3.2.2 obtained from SDM and compared by its exact solution. The plot here clearly verifies the efficiency of SDM.

which is exactly the series of exact solution $y(x) = \dfrac{1}{\sqrt{1+2x^5}}$ (see [22]). In Fig.7 the comparison of exact and numerical solution describes the accuracy of SDM.

3.3. **Case-3.** Last case of fourth order Emden-Fowler under discussion is from Eq.(1.4) for which $n = 1$, $m = 3$, Eq.(1.4) becomes

$$y^{iv}(x) + \frac{k}{x}y'''(x) + f(x)g(y) = 0 \qquad (3.24)$$

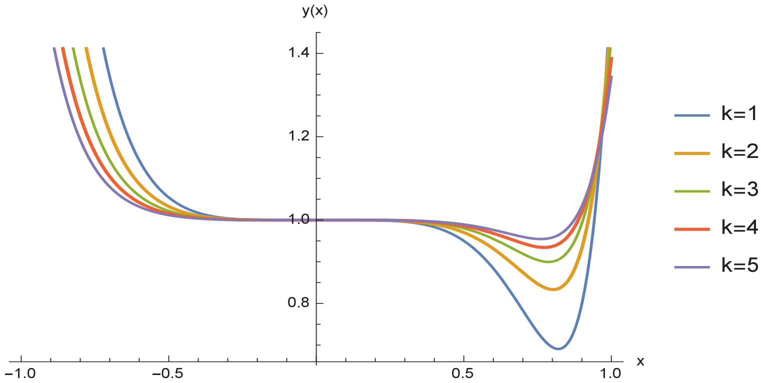

FIGURE 8. Graphical illustration of numerical solution example 3.2.2 for different values of k. Mild deviation can be observed in this plot due to less number of terms of shape factor in the differential equation.

with initial condition $y(0) = y_\circ$, $y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. Taking successive derivatives of Eq.(3.24) it becomes

$$\begin{aligned} &xy^{v}(x) + (k+1)y^{iv}(x) + xg(y)f^{'}(x) + f(x)g(y) = 0 \\ &xy^{vi}(x) + (k+2)y^{v}(x) + xg(y)f^{''}(x) + g(y)f^{'}(x) = 0 \\ &. \\ &. \\ &. \end{aligned} \tag{3.25}$$

Now applying initial conditions and then by expanding via Taylor series its general solution turns out to be

$$y_\circ - \frac{g(y)f(0)}{24(k+1)}x^4 - \frac{g(y)f^{'}(0)}{60(k+2)}x^5 - \frac{g(y)f^{''}(0)}{240(k+3)}x^6 + \cdots \tag{3.26}$$

This clearly depicts that the polynomials of shape factor in denominator suggests that the solution for this problem does not exist $\forall k \in \{-1, -2, -3, \cdots\}$. To establish this argument lets take two numerical examples of this case as well.

3.3.1. *Example 1.* For the first example of numerical approximation of third case of Emden-Fowler, consider $g(y) = e^{y(x)}$, $f(x) = 32(x^{12} + 49x^8 - 129x^4 + 15)$ in Eq.(3.24) and initial conditions as $y(0) = y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. Then the obtained general solution for this problem from the expansion of its Taylor series for arbitrary k is

$$\begin{aligned} y(x) = &-\frac{20}{k+1}x^4 + \frac{2(43k+143)}{7(k+1)(k+5)}x^8 - \\ &\frac{4(343k^3 + 21751k^2 + 138713k + 201305)}{(k+1)^2(k+5)(k+9)}x^{12} + \cdots \end{aligned} \tag{3.27}$$

Upon solving the polynomial in denominator of Eq.(3.27) it can be seen that the existence of numerical solution is only possible if $k \notin \{-1, -5, -9\}$. Fig.10 contains the graphical

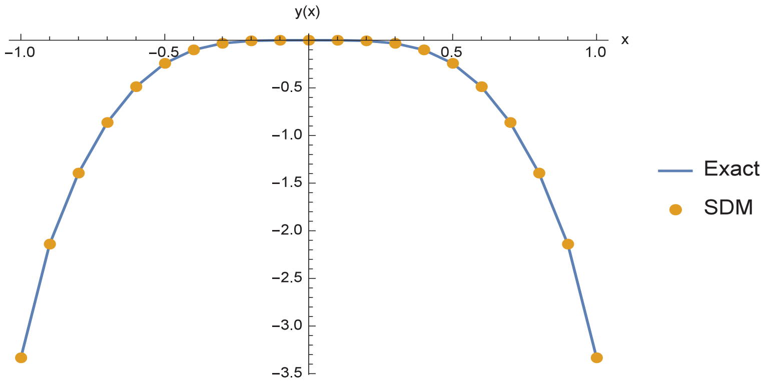

FIGURE 9. Numerical solution obtained from SDM of example 3.3.1 compared by its exact solution to verify the accuracy of SDM.

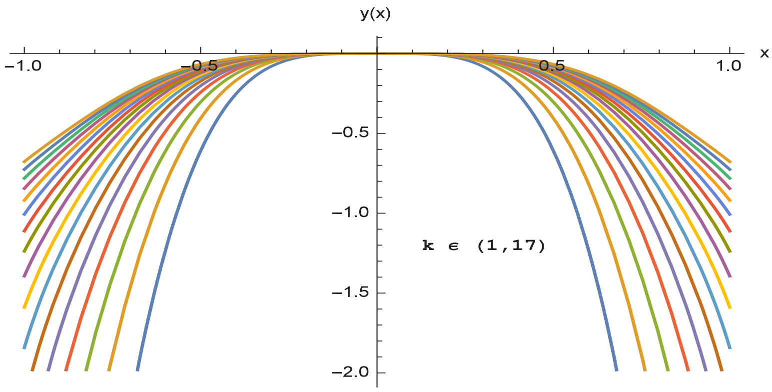

FIGURE 10. Graphical illustration of numerical solution of example 3.3.1 for different values of k. Slight deviation can be observed in this plot due to only one term of shape factor in the Emden-Fowler Equation.

illustration of Eq.(3.27) for different positive values of k. Since in Eq.(3.24) there is only one term of shape factor involved, i.e. k, therefore the variational change in the solution is not much higher and graph deviates from its actual place a slight bit. To verify the accuracy of this solution put $k = 2$ in Eq.(3.27) to obtain

$$y(x) = -4x^4 + 2x^8 - \frac{4}{3}x^{12} + \cdots \tag{3.28}$$

Its exact solution $y(x) = -4\ln(1 + x^4)$ has the same numerical series as in [22]. In Fig.9 the comparison of exact and numerical solution describes the accuracy of this method for example 1 of case 3.

3.3.2. *Example 2.* Now for the second example, consider $g(y) = y^{-7}(x)$, $f(x) = -\frac{1}{9}x(x^2 - 12)$ in Eq.(3.24) and initial conditions as $y(0) = 1$,$y^{'}(0) = y^{''}(0) = y^{'''}(0) = 0$. Then the

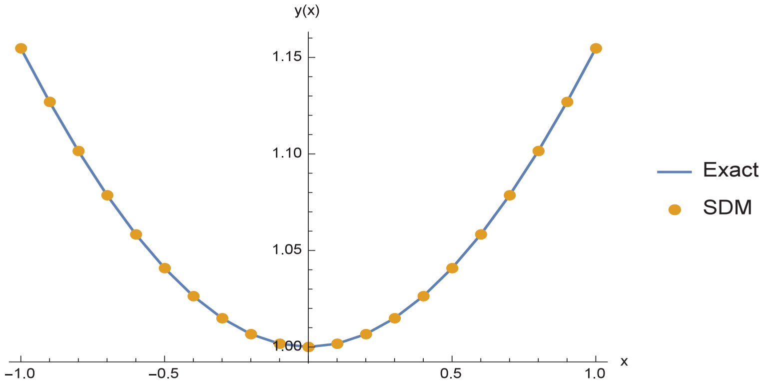

FIGURE 11. Numerical solution of example 3.3.2 obtained from SDM compared by its exact solution to verify the accuracy of SDM.

obtained Taylor series from Eq.(3.26) becomes

$$\begin{aligned} y(x) =& 1 + \frac{1}{6}x^2 - \frac{1}{18(k+1)}x^4 + \frac{1}{72(k+3)}x^6 - \frac{(9k+13)}{2592(k+1)(k+5)}x^8 + \\ & \frac{7(7k^2+38k+48)}{58320(k+1)(k+3)(k+7)}x^{10} + \cdots \end{aligned} \tag{3.29}$$

which is the general solution of this problem for arbitrary k. To prove the preciseness of this general solution put $k = 3$ in Eq.(3.29) to get

$$y(x) = 1 + \frac{1}{6}x^2 - \frac{1}{72}x^4 + \frac{1}{432}x^6 - \frac{5}{10368}x^8 + \frac{7}{62208}x^{10} - \frac{7}{248832}x^{12} + \cdots \tag{3.30}$$

This series is exactly the same as the series of its exact solution, i.e. $y(x) = \sqrt{1 + \frac{x^2}{3}}$ (see [22]). In Fig.11 the comparison of exact and numerical solution of example 2 of case 3 describes the accuracy of this method. While solving the polynomial in denominator of Eq.(3.29) it can be seen that the existence of numerical solution is only possible if $k \notin 1-2n, n \in Z^+$. Fig.12 contains the graphical illustration of Eq.(3.29) for different positive values of k. Since in Eq.(3.24) there is only one term of shape factor involved, i.e. k, therefore the variational change in the graph is not much visible and graph deviates from its actual place a slight bit.
This detailed discussion about the shape factor in solved examples above describes the importance of this parameter. Various numerical solutions with different values of k have been obtained. Usually in previous works this parameter have always kept constant, but by varying this parameter, some interesting observations can be made. Physical situations can be better understood with varying shape factor and use of equations in certain situations can be made more effective accordingly.

4. CONCLUSION

In this work, fourth order Emden-Fowler singular equations have been solved by a numerical method known as Successive Differentiation Method. This numerical method

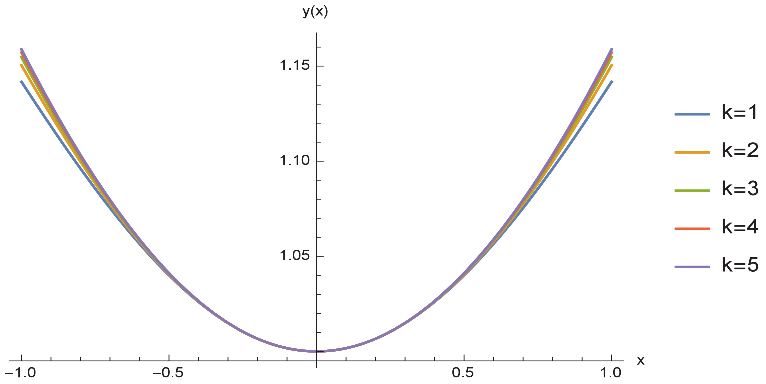

FIGURE 12. Graphical illustration of numerical solution of example 3.3.2 for different values of k. Slight deviation can be observed in this plot due to only one term of shape factor in the Emden-Fowler equation.

utilizes basic concepts of differentiation and expansion of Taylor series. This method is proved to be the easiest and the most accurate method among all other contemporary available methods. No hassle of finding Adomian or Lagrange multipliers and introducing any perturbation parameters. Various examples have been solved numerically and graphical illustrations of its solutions have been provided. Fig.1, Fig.3, Fig.5, Fig.7, Fig.9, Fig.11 represents the accuracy of numerical solutions obtained by SDM for various examples of all three cases. whereas Fig.2, Fig.4, Fig.6, Fig.8, Fig.10, Fig.12 depicts the variation in solution occurring due to the different values of shape factor. Observations made while solving these equations is of shape factor that not only defines the types of equations and reduces the terms in equation of the problem, but the presence of this shape factor in numerical solution also defines the stability and existence of numerical solutions. As explained in this work all these equations have solution for positive shape factors. As soon as the value of k becomes negative the solution ceases to exist. Also, less involvement of shape factor in equations and their numerical solutions gives least variation. Physical variation in graphs is higher when greater number of terms are involved in differential equation otherwise not much significant change is observed in the exact and numerical solution of Emden-Fowler with different values of k.

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REFERENCES

[1] A. Aslanov, *An elegant exact solutions for the Emden-Fowler equations of the first kind*, Math. Meth. Appl. Sci., **39** (2016) 1039–1042, doi: 10.1002/mma.3541.

[2] I. Astashove, *On quasi-periodic solutions to a higher-order Emden-Fowler type differential equation*, Boundary Value Problems, **1** (2014) 1–8, doi.org/10.1186/s13661-014-0174-7.

[3] L.M. Berkovich, *The generalized Emden-Fowler equation*, Symmetry in Nonlinear Mathematical Physics, **1** (1997) 155–163.

[4] N. Bildik, and S. Deniz, *Modified Adomian decomposition method for solving Riccati differential equations*, Review of the Air Force Academy, **30**, No.3 (2015) 21-26.

[5] N. Bildik and S. Deniz, *A new technique for solving singular IVPs of Emden-Fowler type*, arXiv:1604.05344v1 [math.CA] 18 Apr 2016.

[6] N. Bildik, and S. Deniz, *A new efficient method for solving delay differential equations and a comparison with other methods*, Eur. Phys. J. Plus, **132**, No.1 (2017) 12 pages, doi.org/10.1140/epjp/i2017-11344-9

[7] S. Deniz and N. Bildik, *Applications of optimal perturbation iteration method for solving nonlinear differential equations*, AIP Conference Proceedings, **1798**,No.1 (2017) doi.10.1063/1.4972638

[8] A. Domoshnitsky and R. Koplatadze, *On asymptotic behavior of solutions of generalized Emden-Fowler differential equations with delay argument* , Abstract and Applied Analysis, Volume 2014, (2014) Article ID 168425, doi.org/10.1155/2014/168425

[9] M. El-Gamel, W. El-Bashbashy and A. El-Shenawy, *Numerical solutions for the time-dependent Emden-Fowler type equations by B-Spline method*, Applied Mathematics, **5** (2014) 593-600.

[10] R.H. Fowler, *The form near infinity of real continuous solutions of a certain differential equation of the second order*, Quart. J. of Math., **45** (1914) 289–350.

[11] R.H. Fowler, *Emden's and similar differential equations*, MNRAS, **91**, No.1 (1930) 63–92, doi.org/ 10.1093/ mnras/91.1.63.

[12] R.H. Fowler, *Further studies of Emden's and similar differential equations*, The Quarterly Journal of Mathematics, **2**, No.1 (1931) 259–288, doi.org/ 10.1093/qmath/os-2.1.259

[13] I.L. Freire, P.L. da Silva and M. Torrisi, *Lie and Noether symmetries for a class of fourth-order Emden-Fowler equations*, Journal of Physics A Mathematical and Theoretical, **46**, No.24 (2013) 1–9, doi:10.1088/ 1751-8113/46/24/245206.

[14] K. Kamo, *Classification of proper solutions of some Emden-Fowler equations*, Hiroshima Math. J. **29**, No.3 (1999) 459–477, doi:10.32917/hmj/1206124852.

[15] M. Khalid, M. Sultana and F.S. Khan, *Elementary analysis of Lane-Emden equation by successive differentiation method*, Punjab Univ. j. math., **51**, No.6 (2019) 131–141.

[16] C.M. Khalique, *The Lane-Emden-Fowler equation and its generalizations-Lie Symmetry Analysis*, Astrophysics, Intech Open, Edited by İbrahim Küçük, 2012, doi:10.5772/33136.

[17] N.A. Khan, A. Shaikh and M. Ayaz, *Accurate numerical approximation of nonlinear fourth order Emden-Fowler type equations : A Haar based wavelet-collocation approach*, Waves Wavelets Fractals Adv. Anal., **3** (2017) 75–83.

[18] S. Khoury and A. Sayfy, *A fourth order adaptive collocation approach for the solution of generalized Emden-Fowler type equations*, Numerical Analysis and Applied Mathematics ICNAAM 2012, AIP Conference Proceedings, 1479, 1739-1743.

[19] R. Rach, J.S. Duan and A.M. Wazwaz, *Solving the TwoDimensional Lane-Emden Type Equations by the Adomian Decomposition Method*, Journal of Applied Mathematics and Statistics, **3**, No.1 (2016) 15–26, doi:10.7726/jams.2016.1002.

[20] V.V. Rogachev, *Existence of solutions with a given number of zeros to a higher order regular nonlinear Emden-Fowler equation*, Differential Equations, **54**, No.12 (2018) 1595–1601.

[21] A.M. Wazwaz, R. Rach and J.S. Duan, *Solving new fourth order Emden-Fowler type equations by the Adomian Decomposition method*, International Journal for Computational Methods in Engineering Science and Mechanics, **16**, No.2 (2015) 121–131, doi.org/10.1080/15502287.2015.1009582

[22] A.M. Wazwaz, *The variational iteration method for solving new fourth order EmdenFowler type equations*, Chemical Engineering Communications, **202**, No. 11 (2015) 1425–1437, doi.org/10.1080/ 00986445. 2014.952814

[23] J.S.W. Wong, *On the generalized Emden-Fowler equation*, SIAM Review, **17**, No.2 (1975) 339–360.