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The Power Burr Type X Distribution: Properties, Regression Modeling and Applications

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Abstract.: In probability theory, researchers always prefer a model having simple structure with small estimation cost and higher adequacy for real life data applications. Therefore, in this study we have developed a simple power Burr X (PBX) distribution with an additional shape parameter. We have studied the shapes of the developed distribution with respect to subfamilies depends on the additional parameter. This study reveals some structural properties of this new model such as moments, stochastic ordering, quantile function and Rnyi entropy. We have also developed a location-scale regression model for log power Burr X (LPBX) distribution to enhance its application in survival analysis. To observe the behavior of estimated parameters, we have conducted a Monte Carlo simulation study under maximum likelihood (ML) estimation and observed efficiencies by means of bias and mean square errors. Three life-time applications from different industries justified the adequacy, flexibility and potentiality of PBX distribution as compared to other higher parametric complex generalizations.

AMS (MOS) Subject Classification Codes: 14H10; 46N30; 47N30; 62-09; 62G05
Key Words: Burr X; Power Burr X; Regression Model; Maximum likelihood estimation;

Log Power Burr X.

1. INTRODUCTION

Probabilistic distributions have vast role in statistical modeling to explain the behavior of failure and survival rates of different products. Therefore, new probability models are typically developed and studied to explain the possible outcomes of real life experiments as well as for the natural phenomenon. Naturally existed data are not always symmetric. Therefore, researchers have tried to develop heavy tailed distributions (e.g.[17], [22],[39] and [50]). The exponential and the Weibull distribution are generally used to model real world problems such as curing time of a certain disease, survival time with the dose of any medicine, failure of any electrical appliance, rate of floods, strengths of the laboratory equipment, time to repair objects, wind speed, wind shield strength, solar energy, water draining and the rain-fall patterns (see e.g. [3], [11], [14], [40] and [44]).

A system of twelve lifetime distributions was proposed and studied by [10] making use of Pearson differential equations. Among these 12 distributions, Burr type III (BIII), Burr type X (BX) and Burr type XII (BXII) got maximum attention in literature as it effectively models the failure data due to its flexible failure rate [34]. Moreover, many families of distribution have also contributed in literature for the extension in Burr family; see example ([4] $-$ [6], [12], [15], [24] $-$ [30], [49]). BX is an important continuous probability distribution. It significantly contributes in medical, hydrology and reliability analysis [18]. The probability density function (pdf) and the cumulative distribution function (cdf) given by [47] of two parameter BX distribution are,

$$f(y;\alpha,\beta) = 2\alpha\beta^2 y exp\Big[-\big(\beta y\big)^2\Big]\Big(1-exp\Big[-\big(\beta y\big)^2\Big]\Big)^{\alpha-1} \quad (1.\ 1)$$

$$F(y;\alpha,\beta) = \Big(1-exp\Big[-\big(\beta y\big)^2\Big]\Big)^{\alpha} \quad (1.\ 2)$$

where, $\alpha > 0$ is shape parameter while $\beta > 0$ is a scale parameter.

The motivation to explore BX distribution is the availability of its non-monotone failure rate [18] such as decreasing hazard rate for $\alpha > 1/2$ and bath-tub failure rate for $\alpha \leq 1/2$. It can effectively deal with big data problems [33]. Moreover, it is very effective and adaptable for demonstrating the reliability strength and lifetime data sets [46]. They also observed that the two parameter BX distribution has a close relation with the Weibull (WD) as well as the Rayleigh (RD) distribution and is often named as generalized Rayleigh (GR) or exponentiated Rayleigh (ER) distribution. Therefore, the BX model and its generalized forms provide alternatives for these distributions. For α = 1, BX distribution is equivalent to the Rayleigh distribution (see Eq.1) and for β = 1 single parameter Burr X (BXI) distribution can be obtained from exponentiated Weibull distribution [38].

Numerous studies have been conducted from last few years to develop modified or generalized BX distribution for the enhancement of potentiality of BX model. Recently, [48] have developed three parameter odd log-logistic Burr X (OLLBX) distribution, [43] proposed the three parameter type I half logistic Burr X (TIHLBX) distribution, [23] proposed three parameter transmuted Burr X (TBX) distribution, [22] have developed the four parameter exponentiated generalized Burr X (EGBX) distribution. These are the recent generalizations of the BX distribution. Other prominent studies are two parameter inverse Burr X distribution [20], four parameter beta Burr X (BBX) distribution [34], three parameter

Marshall-Olkin extended Burr X (MOBX) distribution [2], Weibull Burr X (WBX) distribution [18] and six parametric beta Kumaraswamy Burr X distribution [33]. These generalizations have complex structure and have higher estimation cost due to large number of parameters.

The present study is organized as follows: Power Burr X distribution is proposed in Section 2. Section 3 explains some structural properties of PBX distribution. In section 4, we develop the log power Burr X (LPBX) distribution and LPBX regression model under location-scale method. Section 5 illustrates the statistical inference of PBX model by means of ML estimation. Four real life applications are given in section 6 to justify the adequacy and flexibility of model as compared to existing distributions. Section 7 concludes the study.

2. Power Burr X (PBX) Distribution

The power transformation or Box and Cox power transformation [8] is typically used to increase the flexibility of the observed model. If Y follows the baseline distribution G, the transformation $T = Y^{\frac{1}{\theta}}$ will develop the new distribution function and it is named as power G distribution. For example, power Lindley distribution [13] and power Lomax distribution [37] are derived making use of power transformations. Using similar approach, power Burr X (PBX) is proposed in this section.

Let a random variable Y follows the Burr X distribution with parameters α and β. Then, the power transformation $T = Y^{\frac{1}{\theta}}$ generates the power Burr X (PBX) distribution. The random variable T follows the PBX distribution with parameters α, β and θ. Symbolically, it is denoted as $T \sim PBX(\alpha, \beta, \theta)$. The cdf of PBX distribution is

$$F(t; \alpha, \beta, \theta) = (1 - exp[-(\beta t^{\theta})^2])^{\alpha} \quad for \quad t > 0 \tag{2. 3}$$

And the corresponding pdf is

$$f(t; \alpha, \beta, \theta) = 2\alpha\beta^2\theta t^{2\theta-1} exp[-(\beta t^{\theta})^2](1 - exp[-(\beta t^{\theta})^2])^{\alpha-1} \tag{2. 4}$$

where, θ is an additional shape parameter. The survival rate function (srf) and the hazard rate function (hrf) of PBX distribution are given respectively,

$$S(t; \alpha, \beta, \theta) = 1 - F(t) = 1 - (1 - exp[-(\beta t^{\theta})^2])^{\alpha} \quad for \quad t > 0 \tag{2. 5}$$

$$h(t; \alpha, \beta, \theta) = \frac{f(t)}{S(t)} = \frac{2\alpha\beta^2\theta t^{2\theta-1} exp[-(\beta t^{\theta})^2](1 - exp[-(\beta t^{\theta})^2])^{\alpha-1}}{1 - (1 - exp[-(\beta t^{\theta})^2])^{\alpha}} \tag{2. 6}$$

2.1. **Limiting behavior for pdf and hrf of PBX distribution.**

This subsection provides the limiting behavior of pdf as well as the failure rate for PBX distribution. The behavior of the model is observed for extreme positions i.e. $t \to 0$ and $t \to \infty$.

Lemma 1: If T follows the $PBX(\alpha, \beta, \theta)$ distribution then the limits for pdf when t approaches to zero are,

$$\lim_{t \to 0} f(t) = \begin{cases} 0 & \text{if} \quad \alpha\theta < 1/2 \\ 2\alpha\beta^4\theta & \text{if} \quad \alpha\theta = 1/2 \\ \infty & \text{if} \quad \alpha\theta > 1/2 \end{cases}$$

It is evident from $Lemma1$ that the behavior for the pdf of PBX distribution depends on the combination of both shape parameters α and θ. It is also observed that the pdf of PBX distribution approaches to zero as $t \to \infty$.

Lemma 2: Since random variable T follows the $PBX(\alpha, \beta, \theta)$ distribution then the limits for hrf when t approaches to infinity are depends on the values of θ. If $\theta < 1/2$, $\lim_{t\to 0} h(t)$ approaches to zero while if $\theta < 1/2$, $\lim_{t\to\infty} h(t)$ approaches to infinity. When $\theta = 1/2$, $\lim_{t\to\infty} h(t)$ becomes equal to $2\beta^2\theta$. As $F(0) = 0$ for all probabilistic models, therefore, limiting behavior of $h(t)$ when $t \to 0$ is same as that of the behavior of pdf.

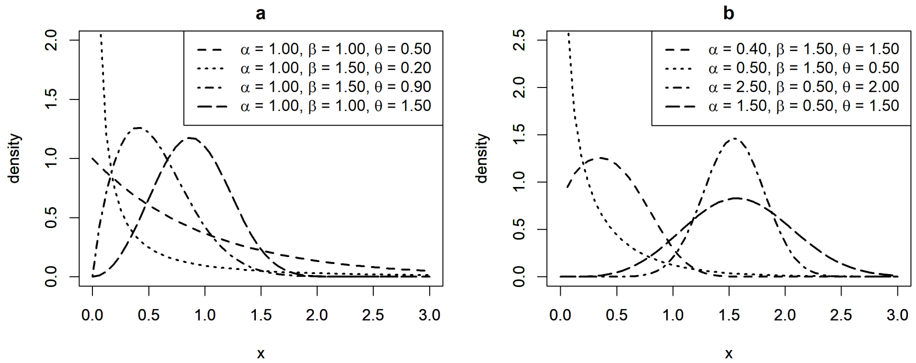

FIGURE 1. pdf curves of PBX distribution at various parameter combinations

2.2. **Shapes of the Distribution.** The shape of PBX distribution is explored with respect to its subfamilies in Figure 1. It can be observed that the shape of PBX distribution Figure 1 is mostly positively skewed that depends on the shape parameters α and θ. Moreover, Figure 1(a) represents the behavior of PBX at fixed α while Figure 1(b) shows the behavior of pdf at varying combinations of α and θ. It is observed from the Figure that the model has exponentially decreasing behavior for $\alpha\theta < 1/2$, exponentially decreasing starting from y-axis for $\alpha\theta = 1/2$, and uni-modal behavior for $\alpha\theta > 1/2$ that is also stated in Lemma 1.

The hrf curves (Figure 2) have monotonically increasing, decreasing, bathtub behavior and partially upside-down bathtub behavior. This primarily depends on the combination of α and θ. Figure (2a) has curves with bathtub and upside down bathtub shapes while Figure (2b) has decreasing curves and increasing curves.

2.3. **Weighted Representation of PBX distribution.** This section provides the weighted representation of PBX distribution using simple binomial expansion. This representation simplifies the evaluation of the properties of PBX distribution. Consider the pdf of PBX distribution given in Eq. (2. 4),

$$f(t; \alpha, \beta, \theta) = 2\alpha\beta^2\theta t^{2\theta-1} exp[-(\beta t^\theta)^2](1 - exp[-(\beta t^\theta)^2])^{\alpha-1}$$

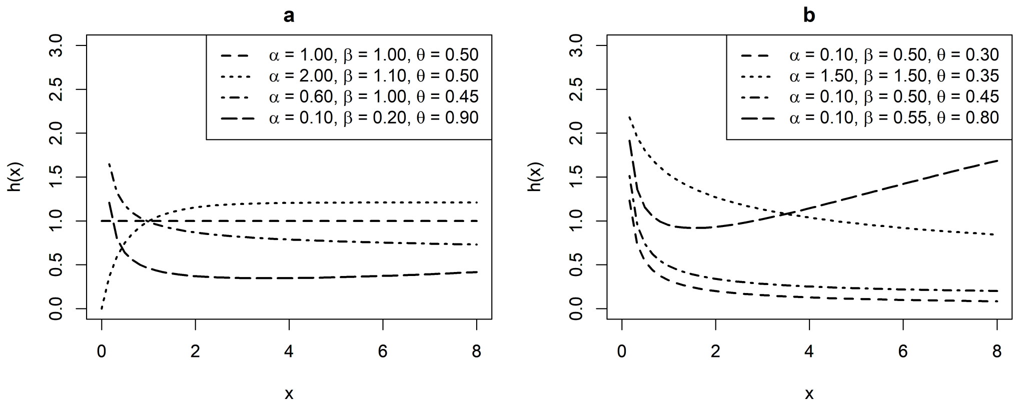

FIGURE 2. hrf curves of PBX distribution at various parameter combinations

By using the binomial expansion $(1-t)^m = \sum_{k=0}^{\infty}(-1)^k\binom{m}{k}t^k \;\; for \;\; |t|<1$, we get

$$f(t) = 2\alpha\beta^2\theta t^{2\theta-1}exp\big[-(\beta t^{\theta})^2\big]\sum_{k=0}^{\infty}(-1)^k\binom{\alpha-1}{k}exp\big[-k(\beta t^{\theta})^2\big]$$

$$= 2\alpha\beta^2\theta t^{2\theta-1}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha-1}{k}exp\big[-(k+1)(\beta t^{\theta})^2\big]$$

$$f(t;\alpha,\beta,\theta) = 2\alpha\beta^2\theta t^{2\theta-1}\sum_{k=0}^{\infty}\eta_k exp\big[-(k+1)(\beta t^{\theta})^2\big] \quad (2.\ 7)$$

where, $\eta_k = (-1)^k\binom{\alpha-1}{k}$. We will use Eq. (2. 7) for further evaluation of the properties of PBX distribution.

3. STRUCTURAL PROPERTIES OF PBX DISTRIBUTION

This section explores some important features of the PBX distribution with the help of structural properties. It contains the moments and the related measures, stochastic ordering, quantile function and entropies of PBX distribution.

3.1. **Moments and Related Measures. Theorem 1:** Let the random variable T follows the PBX distribution given in Eq. (2. 7) then the moment of the distribution is derived as

$$\mu_r^{'} = \frac{\alpha}{\beta^{r/\theta}}\sum_{k=0}^{\infty}\frac{\eta_k}{(k+1)^{1+r/2\theta}}\Gamma\big(1+\frac{r}{2\theta}\big) \quad (3.\ 8)$$

As $\Gamma(\pi) = (\pi-1)!$

Proof: By definition, $\mu_r^{'} = E(T^r) = \int_{-\infty}^{\infty} t^r f(t)dt$

$$\mu_r^{'} = 2\alpha\beta^2\theta\sum_{k=0}^{\infty}\eta_k\int_0^{\infty} t^{2\theta+r-1}exp\big[-(k+1)(\beta t^{\theta})^2\big]dt$$

Let,

$(k+1)(\beta t^{\theta})^2 = y$ and $t = \left(\frac{y}{\beta^2(k+1)}\right)^{1/2\theta}$

$dt = \frac{y^{\frac{1}{2\theta}-1}}{2\theta\left(\beta^2(k+1)\right)^{1/2\theta}}dy$ As $t \to 0, y \to 0$ and $t \to \infty, y \to \infty$

$$\mu_r^{'} = 2\alpha\beta^2\theta\sum_{k=0}^{\infty}\eta_k\int_0^{\infty}\left(\frac{y}{\beta 2(k+1)}\right)^{\frac{1}{2\theta}(2\theta+r-1)}exp(-y)\frac{y^{\frac{1}{2\theta}}-1}{2\theta\left(\beta^2(k+1)\right)^{1/2\theta}}dy$$

After simplification, we get

$$\mu_r^{'} = \frac{\alpha}{\beta^{\frac{r}{\theta}}}\sum_{k=0}^{\infty}\frac{\eta_k}{(k+1)^{1+\frac{r}{2\theta}}}\int_0^{\infty}y^{1+\frac{r}{2\theta}-1}exp(-y)dy$$

This completes the proof. The raw moments of PBX distribution and further moments about mean, skewness and kurtosis of the distribution are straightforward.

3.2. **Quantile Function.** Let p follows the uniform distribution with parameters (0, 1), then the quantile function of the PBX distribution is evaluated as

$$t_p = \left[\frac{1}{\beta}\left(log\left(\frac{1}{1-p^{1/\alpha}}\right)\right)^{\frac{1}{2}}\right]^{\frac{1}{\theta}} \tag{3. 9}$$

By inserting the value of $p = 0.25, 0.50, 0.75$, we can compute the real solution for the quartiles of PBX distribution. Moreover, we can easily compute all the percentiles by using given quantile function. This measurement can also be used to compute the skewness and kurtosis of the distribution by using Bowley [7] and Moors [35] approach respectively. The coefficient of Bowley skewness s_k based on quartiles is given as

$$s_k = \frac{Q_{3/4} + Q_{1/4} - 2Q_{1/2}}{Q_{3/4} - Q_{1/4}} \tag{3. 10}$$

Therefore, the coefficient of Moors kurtosis (k_r) based on octiles is given as

$$k_r = \frac{Q_{3/8} - Q_{1/8} + Q_{7/8} - Q_{5/8}}{Q_{3/4} - Q_{1/4}} \tag{3. 11}$$

Table 1 represent the quartiles, inter quartile range, the Bowley skewness and Moors kurtosis of the PBX distribution for $\alpha = 3.0$ and $\beta = 2.0$. It can be observed that all the quartiles decreases by increasing the value of θ. Moreover, the skewness and kurtosis of the distribution decrease by increasing θ. It is also an interesting fact that when θ approaches to α, the distribution becomes approximately symmetrical.

Figure 3 represents skewness and kurtosis of PBX distribution for the combination of shape parameters keeping scale parameter fixed at $\beta = 0.5$. It can be observed that the skewness (Figure 3a) and kurtosis (Figure 3b) decrease for increasing values of shape parameters α and θ.

TABLE 1. Quartiles, inter quartile range, skewness and kurtosis of PBX distribution

Measures θ	Q_1	Q_2	Q_3	$I.Q.R$	s_k	k_r
0.1	0.0009	0.0095	0.0764	0.0755	0.7717	3.6048
0.3	0.0982	0.2122	0.4244	0.3262	0.3008	1.4227
0.5	0.2485	0.3946	0.5980	0.3494	0.1640	1.2784
0.7	0.3699	0.5146	0.6926	0.3227	0.1029	1.2462
1.0	0.4985	0.6281	0.7733	0.2747	0.0563	1.2342
1.5	0.6287	0.7334	0.8425	0.2137	0.0199	1.2321
2.0	0.7060	0.7925	0.8793	0.1733	0.0017	1.2315

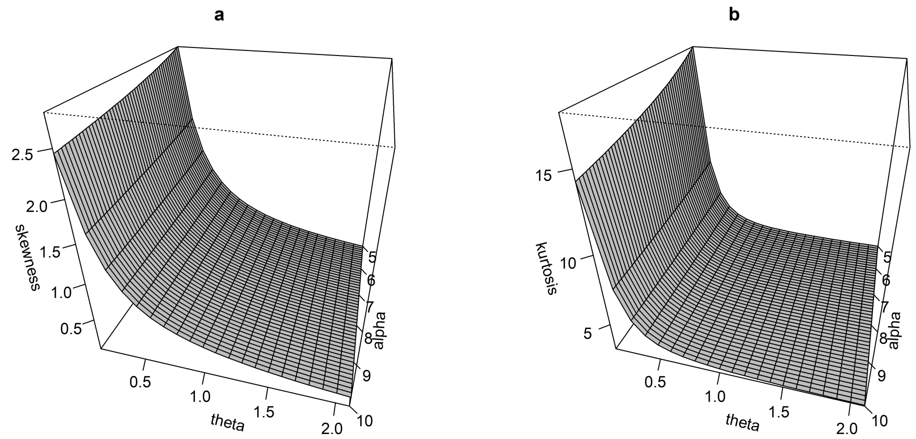

FIGURE 3. (a) Skewness and (b) Kurtosis of PBX distribution.

3.3. **Stochastic Ordering.** The uni-variate stochastic ordering is defined as the partial order relation between the probabilities of two random variables of a probability distribution linked with a commutable space (e.g. [32],[42]). The likelihood ordering of two random variables following $PBX(\alpha, \beta, \theta)$ distribution is considered below.

Theorem 2: Let T_1 follows $PBX(\alpha_1, \beta, \theta_1)$ and T_2 follows $PBX(\alpha_2, \beta, \theta_2)$ with their respective distribution functions $f(t)$ and $g(t)$. Keeping the scale parameter (β) constant, the ratio between the pdfs is,

$$\frac{f(t)}{g(t)} = \frac{\alpha_1\theta_1 t^{2(\theta_1-\theta_2)} exp\Big((\beta t^{\theta_2})^2 - (\beta t^{\theta_1})^2\Big)\Big(1 - exp\big(-(\beta t^{\theta_1})^2\big)\Big)^{\alpha_1-1}}{\alpha_2\theta_2\Big(1 - exp\big(-(\beta t^{\theta_2})^2\big)\Big)^{\alpha_2-1}} \qquad (3.\ 12)$$

Assume that $\alpha_1 = \alpha_2 = \alpha$, we get

$$\frac{f(t)}{g(t)} = \frac{\theta_1 t^{2(\theta_1-\theta_2)} exp\Big((\beta t^{\theta_2})^2 - (\beta t^{\theta_1})^2\Big)\Big(1 - exp\big(-(\beta t^{\theta_1})^2\big)\Big)^{\alpha-1}}{\theta_2\Big(1 - exp\big(-(\beta t^{\theta_2})^2\big)\Big)^{\alpha-1}}$$

By taking derivatives, it is found $d\frac{f(t)}{g(t)} > 0$ for $\theta_1 > \theta_2$ and $d\frac{f(t)}{g(t)} < 0$ for $\theta_1 < \theta_2$. This implies the existence of likelihood ratio that increases when $\theta_1 > \theta_2$ and decreases when $\theta_1 < \theta_2$ (for $\alpha_1 = \alpha_2 = \alpha$). This leads to the following theorems.

Theorem 3: If T_1 follows $PBX(\alpha, \beta, \theta_1)$ and T_2 follows $PBX(\alpha, \beta, \theta_2)$, then $T_1 \geq_{st} T_2$ for $\theta_1 > \theta_2$ and $T_1 \leq_{st} T_2$ for $\theta_1 < \theta_2$. From above theorem, we also conclude that $T_1 \geq_{st} T_2$ for $\theta_1 > \theta_2$ and $T_1 \leq_{st} T_2$ for $\theta_1 < \theta_2$. where, $\geq_{st}$ is the ordering of random variables (e.g. [41]). Also, it can be observed that when $\theta_1 < \theta_2$ there is no likelihood ratio. We conclude another theorem under similar assumption of fixed θ.

Theorem 4: If T_1 follows $PBX(\alpha, \beta, \theta_1)$ and T_2 follows $PBX(\alpha, \beta, \theta_2)$, then $T_1 \geq_{lr} T_2$ for $\theta_1 > \theta_2$ and $T_1 \leq_{lr} T_2$ for $\theta_1 < \theta_2$.

3.4. **Rnyi Entropy.** Entropy typically deals with the uncertainty, diversity and randomness of the system. Rnyi entropy is commonly used due to its vast applications in numerous fields such as econometrics, statistical inference and problem identification in statistics [19]. By definition, Rnyi entropy is [9]

$$I_R(\vartheta) = \frac{1}{1-\vartheta} log(I(\vartheta)) = \frac{1}{1-\vartheta} log\Bigg[\int_{-\infty}^{\infty} f^{\vartheta}(t)dt\Bigg], \quad for \vartheta > 0 \ and \ \vartheta \neq 1 \quad (3.\ 13)$$

$$I(\vartheta) = \int_0^{\infty} 2^{\vartheta}\alpha^{\vartheta}\beta^{2\vartheta}\theta^{\vartheta} t^{\vartheta(2\theta-1)} exp\Big(-\vartheta(\beta t^{\theta})^2\Big)\Bigg(1 - exp\Big(-\vartheta(\beta t^{\theta})^2\Big)\Bigg)^{\vartheta(\alpha-1)} dt$$

By using the binomial expansion $(1-t)^m = \sum_{k=0}^{\infty}(-1)^k\binom{m}{k}t^k \quad for \quad |t| < 1$, we get

$$I(\vartheta) = 2^{\vartheta}\alpha^{\vartheta}\beta^{2\vartheta}\theta^{\vartheta}\sum_{k=0}^{\infty}(-1)^k\binom{\vartheta(\alpha-1)}{k}\int_0^{\infty} t^{\vartheta(2\theta-1)} exp\Big(-(k+\vartheta)(\beta t^{\theta})^2\Big)dt$$

Let,

$(k+\vartheta)(\beta t^{\theta})^2 = y$ and $t = \left(\frac{y}{\beta^2(k+\vartheta)}\right)^{1/2\theta}$

$dt = \frac{y^{\frac{1}{2\theta}-1}}{2\theta\left(\beta^2(k+\vartheta)\right)^{1/2\theta}}dy$ As $t \to 0, y \to 0$ and $t \to \infty, y \to \infty$

After simplification, we get

$$I(\vartheta) = 2^{\vartheta-1}\alpha^{\vartheta}\beta^{\frac{\vartheta-2}{2\theta}}\theta^{\vartheta-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+\vartheta)^{\vartheta-\frac{\vartheta-1}{2\theta-1}}}\binom{\vartheta(\alpha-1)}{k}\Gamma\Big(\vartheta - \frac{\vartheta-1}{2\theta}\Big)$$

By putting the value of $I(\vartheta)$ in Eq. (3. 13) we get,

$$I_R(\vartheta) = \frac{1}{1-\vartheta} log\Bigg[(2\theta)^{\vartheta-1}\alpha^{\vartheta}\beta^{\frac{\vartheta-2}{2\theta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\vartheta)^{\vartheta-\frac{\vartheta-1}{2\theta-1}}} \binom{\vartheta(\alpha-1)}{k} \Gamma\left(\vartheta - \frac{\vartheta-1}{2\theta}\right)\Bigg] \quad (3.\ 14)$$

Eq. (3. 14) is the explicit expression for Rnyi entropy of PBX distribution.

4. Regression Modeling for Log-PBX Distribution

Let T follows the PBX distribution given in Eq. (2. 4). Then, $Z = logT$ follows the log power Burr X (LPBX) distribution, the pdf of Z after parameterization in terms of $\mu = -log\beta^{1/\theta}$ and $\sigma = (2\theta)^{-1}$ can be expressed as

$$f(z;\alpha,\mu,\sigma) = \frac{\alpha}{\sigma} exp\left[\frac{z-\mu}{\sigma}\right] exp\left[-exp\left[\frac{z-\mu}{\sigma}\right]\right] \left[1 - exp\left[-exp\left[\frac{z-\mu}{\sigma}\right]\right]\right]^{\alpha-1} \quad (4.\ 15)$$

where, $\alpha > 0$ and $\sigma > 0$ are the shape parameter while $\mu \in \Re$ is location parameter. Thus, if $Z \sim LPBX(\alpha,\mu,\sigma)$, then the survival function for corresponding pdf given in Eq. (2. 4) is

$$S(z) = 1 - \left[1 - exp\left[-exp\left[\frac{z-\mu}{\sigma}\right]\right]\right]^{\alpha} \quad (4.\ 16)$$

where, $-\infty \leq Z \leq \infty$ and $-\infty \leq \mu \leq \infty$. The LPBX distribution has more flexible behavior (Figure 4). It can be used to model negatively skewed data sets.

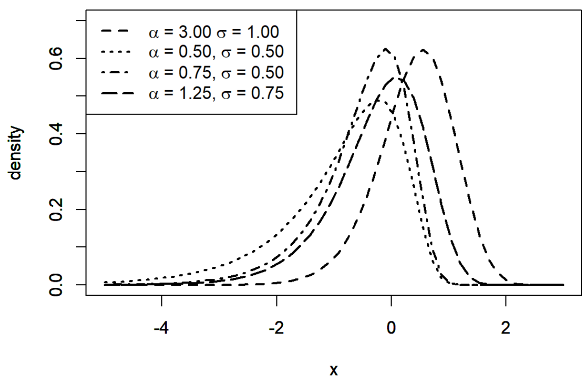

FIGURE 4. Pdf curves for LPBX distribution (for $\mu = 0$)

Practically, lifetime of variable X_j is effected from many variables such as blood pressure, cholesterol level and many others. Therefore, under location and scale modeling technique, we propose a regression model for LPBX distribution. Let $t_j = (t_{j1}, t_{j2}, t_{j3}, \ldots, t_{jn})^T$ are the independent variable vector associated with j^{th} dependent variable Z_j, for $j = 1, 2, \ldots, n$. Consider a sample of n independent observations say $(z_1, x_1), (z_2, x_2), \ldots, (z_n, x_n)$, where each random response is define as $z_j = minlog(t_j), log(c_j)$, where $log(t_j)$ and $log(c_j)$ are the log-lifetime and log-censoring respectively.

Now the regression model for dependent variable Z_j for LPBX distribution is given as

$$z_j = t_j^T \gamma + \sigma x_j \tag{4. 17}$$

where, z_j is the random error with pdf given in Eq. (4. 15), $\gamma = (\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n), \sigma > 0$, is the scale parameter, $\alpha > 0$ is shape parameter and t_j is the vector of explanatory variables that is used for modeling the location parameters $\mu_j = t_j^T \gamma$. The log-likelihood function for the parameters of LPBX distribution $\phi = (\alpha, \sigma, \gamma^T)^T$ can be expressed from Eq. (4. 15) and (4. 16) as

$$\begin{aligned} I(\phi) = q[log\alpha - log\sigma] &+ \sum_{j \in F} \left(\frac{z_j - \mu_j}{\sigma} \right) - \sum_{j \in F} exp\left(\frac{z_j - \mu_j}{\sigma} \right) \\ &+ (\alpha - 1) \sum_{j \in F} log\left[1 - exp\left(- exp\left(\frac{z_j - \mu_j}{\sigma} \right) \right) \right] \\ &+ \sum_{j \in C} log\left\{ 1 - \left[1 - exp\left(- exp\left(\frac{z_j - \mu_j}{\sigma} \right) \right) \right]^{\alpha} \right\} \end{aligned} \tag{4. 18}$$

where, q is the observed number of failures while F and C are the set of individuals for which z_j is the log-lifetime. We can obtain the estimates of LPBX regression model by maximizing the likelihood function given in Eq. (4. 18). The estimated survival function is given as

$$S(z|t) = 1 - \left[1 - exp\left(- exp\left(\frac{z_j - \hat{\mu}_j}{\hat{\sigma}} \right) \right) \right]^{\hat{\alpha}} \tag{4. 19}$$

For parameter estimation of LPBX regression model, SAS (2004) with NLMixed method is used. For $\sigma = 1$, the regression model for LPBX given in Eq. 4. 15 is equivalent to the regression model of BX distribution. Under regularity conditions of vector space $\phi = (\alpha, \sigma, \gamma^T)^T$, the asymptotic distribution of $(\hat{\phi} - \phi)$ is multivariate Normal $N_{p+2}\{(\alpha, \sigma, \gamma^T)^T, K(\phi)^{-1}\}$, where $K(\phi)$ is the observed information matrix of $(p+3) \times (p+3)$ (e.g. [16]).

5. Statistical Inference

5.1. **ML Estimation.** Maximum likelihood (ML) estimation is extensively used for the estimation of parameters. The normal approximation of ML estimator contributes analytically and statistically in large sample theory. The parameters of $PBX(\alpha, \beta, \theta)$ distribution are estimated by using this approach. Let $T_1, T_2, T_3, \ldots, T_n$ are n identically independent random variables following the PBX distribution. The log-likelihood function (L) for unknown parameters $\varphi = (\alpha, \beta, \theta)^T$ is given as

$$L = nlog(2\alpha) + nlog\theta + 2nlog\beta + (2\theta - 1)\sum_{j=0}^{n} logt_j - \sum_{j=0}^{n}\left(\beta t_j^{\theta}\right)^2 + (\alpha - 1)\sum_{j=0}^{n} log\Big(1 - exp\Big(-\left(\beta t_j^{\theta}\right)^2\Big)\Big) \quad (5.\ 20)$$

The necessary condition to maximize the log-likelihood function given in Eq. 5. 20 is the existence of its first derivative with respect to estimated parameters and equates it to zero.

Now,

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{j=0}^{n} log\Big(1 - exp\Big(-\left(\beta t_j^{\theta}\right)^2\Big)\Big) \quad (5.\ 21)$$

$$\frac{\partial L}{\partial \beta} = \frac{2n}{\beta} - 2\beta\sum_{j=0}^{n}\left(t_j^{2\theta}\right) + (\alpha - 1)\sum_{j=0}^{n}\frac{2\beta t_j^{2\theta} exp\Big(-\left(\beta t_j^{\theta}\right)^2\Big)}{\Big(1 - exp\Big(-\left(\beta t_j^{\theta}\right)^2\Big)\Big)} \quad (5.\ 22)$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + 2\sum_{j=0}^{n} logt_j - 2\beta^2\sum_{j=0}^{n}\left(t_j^{2\theta}\right)logt_j + (\alpha - 1)\sum_{j=0}^{n}\frac{2\beta^2 t_j^{2\theta} exp\Big(-\left(\beta t_j^{\theta}\right)^2\Big)logt_j}{\Big(1 - exp\Big(-\left(\beta t_j^{\theta}\right)^2\Big)\Big)} \quad (5.\ 23)$$

Equations (5. 21), (5. 22) and (5. 23) can be solved simultaneously to evaluate the unknown parameters but these equations are not linear as well as have no exact solution. Therefore, we use nonlinear iterative procedure such as Newton-Raphson method for the estimation of parameters arithmetically.

Since ML estimators have large sample property, therefore, the ML estimates of $\hat{\varphi}$ are assumed to follow approximate normal distribution. The mean of this approximation is φ and variances and co-variances can be obtained from the inverse of the expected information matrix such as $\sqrt{n}(\hat{\varphi} - \varphi) \sim N(0, V_n)$, where $V_n = (v_{ij}) = I_n^{-1}(\varphi)$ is considered as the expected information matrix and its inverse will provide variances and co-variances. One can easily compute the variance co-variances elements by using second derivative of Eq. (5. 21), (5. 22) and (5. 23).

5.2. **Simulations.** Simulation study enables us to understand the behavior of model parameters by generating the data from pseudo-random sampling. Estimation of parameters from these samples explore the efficiency of parameters on the basis of $bias$ and MSE. We have performed a simulation study for the unknown parameters of PBX distribution. The study contains $10,000$ repetition of experiment and has different sample sizes such as $n = 10, 100, 300$ and 500. We have considered two sets for sim1ulation given as set 1 is $\alpha = 1.50, \beta = 1.00$ and $\theta = 0.50$ and set 2 is $\alpha = 1.50, \beta = 1.00$ and $\theta = 1.50$. The generated results are given in Table 2.

It can be seen that the bias, MSE and C.I for all the parameters of PBX distribution decreases for increasing sample size (Table 2. It is interesting to note that the parameters have less bias and MSE for small value of new additional shape parameter θ.

TABLE 2. Bias, MSE and 95% confidence interval for the parameters of PBX distribution

	n	Parameters	A.E	A.B	Var	MSE	C.I	
							Lower	Upper
Set 1	10	α	1.6598	0.1598	0.3445	0.3701	1.4672	1.8523
		β	1.2139	0.3129	2.7528	2.7986	-0.241	2.6697
		θ	0.6268	0.1268	0.1319	0.1480	0.5458	0.7037
	100	α	1.5168	0.0168	0.0234	0.0236	1.5129	1.5206
		β	1.0134	0.0134	0.0150	0.0151	1.0109	1.0158
		θ	0.5113	0.0113	0.0057	0.0059	0.5103	0.5211
	300	α	1.5050	0.0050	0.0074	0.0075	1.5042	1.5057
		β	1.0046	0.0046	0.0048	0.0049	1.0041	1.0050
		θ	0.5033	0.0033	0.0018	0.0018	0.5031	0.5034
	500	α	1.5039	0.0039	0.0043	0.0044	1.5035	1.5042
		β	1.0017	0.0017	0.0027	0.0028	1.0014	1.0172
		θ	0.5018	0.0018	0.0011	0.0011	0.5017	0.5019
Set 2	10	α	1.6712	0.1711	0.3431	0.3724	1.4774	1.8649
		β	1.1910	0.1910	0.8158	0.8523	0.7473	1.6343
		θ	1.9159	0.4159	1.2843	1.4573	1.1578	2.6739
	100	α	1.5134	0.0134	0.0229	0.0231	1.5096	1.5171
		β	1.0163	0.0163	0.0153	0.0155	1.0137	1.0188
		θ	1.5343	0.0343	0.0510	0.0522	1.5257	1.5428
	300	α	1.5041	0.0041	0.0075	0.0075	1.5033	1.5048
		β	1.0044	0.0044	0.0047	0.0048	1.0039	1.0048
		θ	1.5094	0.0093	0.0167	0.0167	1.5078	1.5109
	500	α	1.5036	0.0036	0.0045	0.0045	1.5032	1.5039
		β	1.0023	0.0023	0.0028	0.0028	1.0020	1.0025
		θ	1.5062	0.0062	0.0099	0.0099	1.5054	1.5069

6. APPLICATIONS

In this section, the implementation of the PBX distribution is explored. It is compared with other closely related distributions. For this purpose, three real life data sets are used (Table 3). Data set 1 contains 100 observations on breaking stress of carbon fibers (in Gba) [36]. Data set 2 are the gauge lengths of 10 mm ([1],[31]) consisting of 63 observations. Data set 3 is the rainfall data of Peninsular Malaysia ([21], [45], [51]). It is the average maximum daily rainfall for 30 years from 1975-2004 of 35 stations. These data sets are used to compare PBX distribution with other closely related distributions (Table 4, Table 5 and Table 6). These include odd log-logistic Burr X (OLLBX) [48], Marshall-Olkin Burr X (MOBX) [2], type I half logistic Burr X (TIHLBX) [43], exponentiated generalized Burr X

TABLE 3. Descriptive measures for all observed data sets

	Min.	Q_1	Median	Mean	Q_3	Max.
Data Set 1	0.390	1.840	2.700	2.621	3.220	5.560
Data set 2	1.901	2.554	2.996	3.059	3.421	5.020
Data set 3	0.703	0.955	1.048	4.137	1.163	109.2

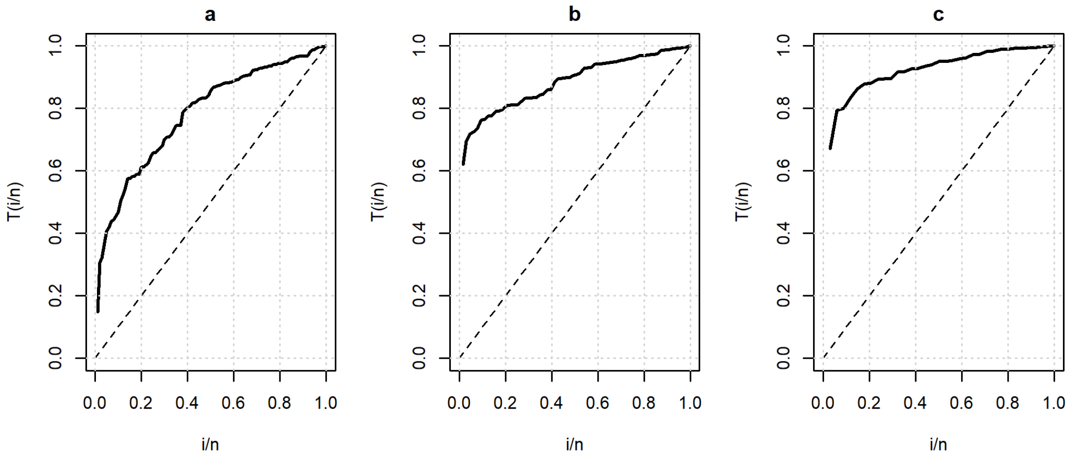

FIGURE 5. TTT plots for (a) data set 1, (b) data set 2, (c) and data set 3

(EGBX) [22], Weibull Burr X (WBX) [18] and beta Kumaraswamy Burr X (BKBX) [33] distribution. Numerous measures are used to observe the goodness of fit such as Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Kolmogorov-Smirnov (K-S), Cramer-Von Mises (W*) and Anderson Darling (A*). The results are presented in Table 4, 5 and 6. While Figure 6 presents the profile likelihood plot for each parameters for all data sets. For all the three data sets, all the goodness of fit measure such as AIC, BIC, K-S, A^* and W^* has minimum value for PBX distribution (Table 4, 5 and 6). Therefore, the PBX distribution fits well to the three data sets as compared to MOBX, OLLBX, TIHLBX, EGBX and BKBX distribution. Figure 7 further validates the fitting of the PBX distribution over the three data sets.

7. CONCLUDING REMARKS

This paper develops an extension of Burr X distribution that is named as power Burr X. This model can deal with the heavy tailed skewed data sets as well as symmetrical data sets under some constraints. Various properties such as moments and related measures, stochastic ordering and Rnyi entropy of PBX distribution are explored. Parameters are being estimated through ML estimation and Monte Carlo simulation study concludes that the bias and MSE of estimated parameters reduces with increase in sample size while for small value of additional shape parameter the bias and MSE are smaller. It is evident from

TABLE 4. Estimated parameters with goodness of fit measures for data set 1

		Data Set 1						
Models		**PBX**	**MOBX**	**OLLBX**	**TIHLBX**	**EGBX**	**WBX**	**BKBX**
Estimated Parameters (S.E)	α	**1.3171 (0.5967)**	1.5584 (0.4171)	0.1189 (1.0488)	1.2125 (0.1061)	1.4626 (0.7716)	3.9909 (0.8175)	2.0476 (0.1265)
	β	**0.3047 (0.1507)**	0.4604 (0.0422)	0.3456 (0.1539)	0.0728 (0.0095)	0.1940 (0.0371)	0.1397 (0.0135)	0.3620 (0.1373)
	θ	**1.2044 (0.3025)**	1.8203 (1.1811)	1.3726 (0.8469)	62.681 (0.0002)	6.4399 (0.0546)	0.3838 (0.7917)	0.2982 (0.6313)
	γ	-	-	-	-	1.0676 (0.7657)	17.774 (3.9732)	4.6364 (3.3977)
	δ	-	-	-	-	-	-	2.6441 (0.5124)
	p	-	-	-	-	-	-	0.3199 (2.4834)
AIC		**288.66**	289.35	288.84	289.38	290.63	290.54	294.48
BIC		**296.47**	297.17	296.66	297.20	301.05	300.96	310.11
K-S		**0.0643**	0.0681	0.0659	0.0688	0.0647	0.0645	0.0644
A*		**0.4076**	0.4106	0.4103	0.4112	0.4079	0.4078	0.4075
W*		**0.0694**	0.0728	0.0716	0.0739	0.0699	0.0697	0.0696

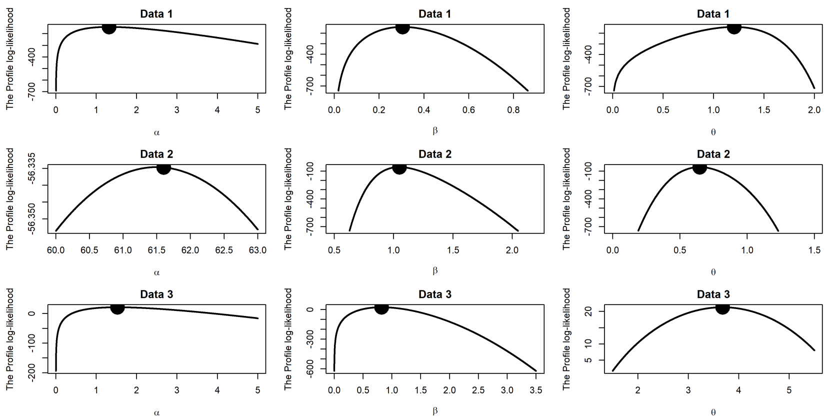

FIGURE 6. The profile likelihood plots for all data sets for observed parameters

the three applications from different fields that the model has simple structure with higher adequacy as that of three parametric models while has low estimation cost and greater adequacy as compared to higher parametric distributions.

TABLE 5. Estimated parameters with goodness of fit measures for data set 2

Data Set 2								
Models		**PBX**	**MOBX**	**OLLBX**	**TIHLBX**	**EGBX**	**WBX**	**BKBX**
Estimated Parameters (S.E)	α	**61.602**	11.652	0.1231	77.283	0.5418	37.915	1.9949
		(0.0004)	(3.3040)	(0.1278)	(1.1954)	(0.2910)	(3.6074)	(0.1892)
	β	**1.0481**	0.5414	0.0199	0.9150	0.2026	0.5369	0.3508
		(0.0657)	(0.0723)	(0.0058)	(0.2083)	(0.3707)	(0.1443)	(0.1961)
	θ	**0.6470**	0.7018	17.723	0.3520	5.8175	0.3269	7.0351
		(0.0581)	(0.6781)	(0.0001)	(0.1928)	(1.7163)	(0.0920)	(1.0427)
	γ	-	-	-	-	53.247	1.8842	0.0924
						(0.7353)	(1.2981)	(0.1164)
	δ	-	-	-	-	-	-	1.5854
								(0.0031)
	p	-	-	-	-	-	-	30.697
								(1.4662)
AIC		**118.66**	118.92	121.96	118.85	120.56	121.13	123.52
BIC		**125.09**	125.35	128.39	125.28	129.13	129.70	136.38
K-S		**0.0647**	0.0856	0.0998	0.0853	0.0741	0.0768	0.0724
A*		**0.3137**	0.3509	0.4912	0.3492	0.3213	0.3212	0.3195
W*		**0.0465**	0.0671	0.0837	0.0669	0.0592	0.0582	0.0517

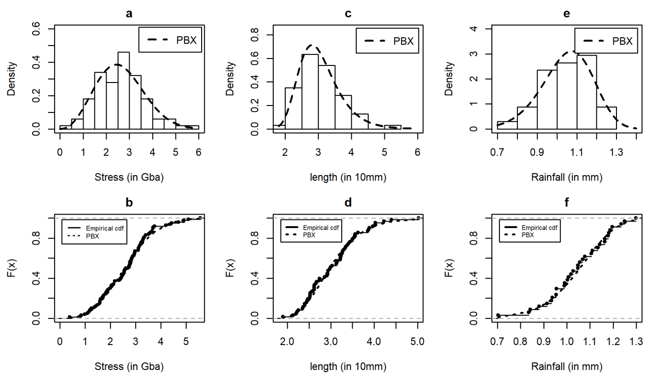

FIGURE 7. Fitted pdf with histogram (1st Row) and empirical and fitted cdf (2nd Row) of PBX model for for all three data sets

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TABLE 6. Estimated parameters with goodness of fit measures for data set 3

		Data Set 3						
Models		PBX	MOBX	OLLBX	TIHLBX	EGBX	WBX	BKBX
Estimated Parameters (S.E)	α	**1.5240**	36.509	0.0897	7.2958	0.1436	12.237	1.1271
		(1.4964)	(4.3462)	(0.0150)	(7.3933)	(0.0246)	(2.5589)	(0.7621)
	β	**0.8221**	2.4515	0.0201	1.0167	0.0008	1.5040	0.0616
		(0.4368)	(0.2141)	(0.0129)	(0.8344)	(0.0010)	(0.9359)	(0.0209)
	θ	**3.6800**	20.619	39.439	16.825	25.107	1.0651	34.621
		(1.9406)	(3.6691)	(0.0008)	(6.1555)	(0.0001)	(0.7548)	(3.1224)
	γ	-	-	-	-	24.842	1.2828	25.107
						(0.00002)	(3.6899)	(1.8945)
	δ	-	-	-	-	-	-	0.6116
								(0.7513)
	p	-	-	-	-	-	-	37.941
								(3.5127)
AIC		**-36.58**	-35.56	-33.99	-36.85	75.09	-34.86	-28.78
BIC		**-32.01**	-30.98	-29.41	-32.37	81.20	-28.75	-19.62
K-S		**0.0659**	0.0821	0.0876	0.0772	0.4249	0.0684	0.0673
A*		**0.1553**	0.1979	0.2670	0.1941	0.9664	0.1617	0.1581
W*		**0.0193**	0.0301	0.0353	0.0333	1.9740	0.0257	0.0231

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