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Certain Analogous for Generalized Geometrically Convex Functions with Applications

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Abstract. The main objective of this paper is to communicate the concept of a class of generalized functions, known as generalized geometrically convex functions. We derive numerous consequences associated to generalized geometrically convex functions. Additionally, we provide the improved class of Hermite-Hadamard inequality via generalized geometrically convex functions. Our outcomes provide some results which are already existing in the literature.

AMS (MOS) Subject Classification Codes: 26D15; 26D10, 90C23
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1. INTRODUCTION

Few decades ago, the speculation of convexity has transformed into one of the most intriguing and valuable fields to investigate a broad class of issues emerging in mathematics and engineering sciences. Creative strategies and computations have capitulated various fields for the investigation of convex analysis. Lately, different variants for convex functions and their refinements been created by utilizing innovative strategies; see [1, 2, 4, 5, 6, 7, 8, 10, 14, 16, 20, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 35]. The concept of convex functions is firmly identified to principle of variants. It is well regarded that a function is convex, if and only an integral inequality is satisfied, which is referred to as the Hermite-Hadamard inequality; see [11].

$$G\Big(\frac{u+w}{2}\Big) \leq \frac{1}{u-w}\int\limits_{w}^{u} G(x)dx \leq \frac{G(u)+G(w)}{2}.$$

Such sorts of integral inequalities are valuable in computing the estimates. For ongoing improvements and applications, see [9, 11, 12, 13, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. The convex sets and convex functions have been broadened and amplified in various ways utilizing modern techniques to deal with distinct problems in a modified form; see [15, 17, 18, 47, 48, 49, 50, 51, 52, 53, 54]. One of the latest momentous speculations of convex functions is the ϕ-convex function, explored by Gordji et al. [9], one of the prominent feature of this class is the non-convex functions. For modern developments, see [18] and the references therein.The principle purpose of this class of generalized convex functions, that are referred to as generalized geometrically convex functions. We build up some new consequences by utilizing the fundamental inequalities. We develop new Hermite-Hadamard integral inequalities for the generalized geometrically convex functions. Various special cases are thought of. Our results are a major and necessary modification of well-known results for inequalities.

2. Preliminaries

Let I be an interval in real line $\mathbb{R}$ and let $G : I = [x, y] \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on the interior I° of I; moreover, let $w(\cdot,\cdot)\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous bifunction. In the sequel, we will symbolize:

$$\mathbb{R} = (-\infty, +\infty), \ \ \mathbb{R}_+ = (0, \infty) \ \text{ and } \ \mathbb{R}_- = (-\infty, 0).$$

definition 2.1. *(*[9]*) Let $I \subseteq \mathbb{R}$. We say that a function $G\colon I = [x, y] \to \mathbb{R}$ is generalized convex with respect to an arbitrary bifunction $\eta(\cdot,\cdot)\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, if*

$$G(\varepsilon x + (1-\varepsilon)y) \leq G(y) + \varepsilon\eta(G(x), G(y)), \quad \forall x, y \in I, \varepsilon \in [0, 1].$$

If $\eta(x, y) = x - y$, then the generalized convex function becomes a convex function. Every convex function is a generalized convex function, but the converse is not true; see, for example, Examples 2.2 and 2.3.

Example 2.2. *(*[8]*) For convexity of G, we can obtain a function η different from the function $\eta(x, y) = x - y$ such that G is generalized convex. Assume $G(x) = x^2$ and $\eta(x, y) = 2x + y$. Then*

$$\begin{aligned} G(\varepsilon x + (1-\varepsilon)y) &= (\varepsilon x + (1-\varepsilon)y)^2 \\ &\leq \varepsilon x^2 + y^2 + \varepsilon(1-\varepsilon)2xy \\ &\leq \varepsilon x^2 + y^2 + \varepsilon(1-\varepsilon)(x^2 + y^2) \\ &\leq y^2 + \varepsilon(x^2 + x^2 + y^2) \\ &= y^2 + \varepsilon(2x^2 + y^2) \\ &= G(y) + \varepsilon\eta(G(x), G(y)) \end{aligned}$$

for all $x, y \in \mathbb{R}$ *and* $\varepsilon \in (0,1)$. *Additionally we have* $x^2 \leq y^2 + (2x^2 + y^2)$ *and* $y^2 \leq y^2$, $\forall\ x, y \in \mathbb{R}$ *show the correctness of inequality for* $\varepsilon = 1$ *and* $\varepsilon = 0$, *respectively. This means that* G *is generalized convex function. observe that function* $G(x) = x^2$ *is generalized convex with respect to all* $w(x,y) = ax + by$ *with* $a \geq 1, b \geq -1$ *and* $x, y \in \mathbb{R}$.

Example 2.3. *(*[8]*) Consider* $G : \mathbb{R} \to \mathbb{R}$ *as*

$$G(x) = \begin{cases} -x & \text{if } x \geq 0, \\ x & \text{if } x < 0 \end{cases}$$

and define a bifunction $\eta = -x - y$ *for all* $x, y \in \mathbb{R}^- = (-\infty, 0)$. *Then* G *is* η*-convex, but the converse is not true.*

definition 2.4. *(*[17]*) Let* $I \subset \mathbb{R}_+$, *is said to be a geometrically convex set, if*

$$x^\varepsilon y^{1-\varepsilon} \in I, \ \ \forall x, y \in I, \ \ \varepsilon \in [0,1].$$

definition 2.5. *(*[17]*) Let* $I \subset \mathbb{R}_+$ *and we say that a function* $G \colon I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ *is geometrically convex on* I, *if*

$$G(y^{1-\varepsilon} x^\varepsilon) \leq (1-\varepsilon)G(y) + \varepsilon(G(x)) \ \ \forall x, y \in I, \ \ \varepsilon \in [0,1],$$

where $(y^{1-\varepsilon} x^\varepsilon)$ and $(1-\varepsilon)G(y) + \varepsilon(G(x))$ presents as weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $G(x)$ and $G(y)$, respectively.

We now mention the class of generalized convex functions on the geometrically convex set with respect to an arbitrary bifunction $w(\cdot,\cdot)$, which is called the generalized geometrically convex functions is mainly due to [18].

definition 2.6. *(*[18]*) A function* $G : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ *is said to be generalized geometrically with respect to a bifunction* $\eta(\cdot,\cdot) \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *if*

$$G(y^{1-\varepsilon} x^\varepsilon) \leq (1-\varepsilon)G(y) + \varepsilon(G(y) + \eta(G(x), G(y))), \ \ \forall x, y \in I, \ \ \varepsilon \in [0,1]. \quad (2.\ 1)$$

If $\eta(x, y) = x - y$, then the Definition 2.6 reduce to geometrically convex functions given in Definition 2.5.

If $\varepsilon = \frac{1}{2}$ in (2. 1), then we have generalized Jensen geometrically convex function.

$$G(\sqrt{xy}) \leq G(y) + \frac{1}{2}\eta(G(x), G(y)), \ \ \forall x, y \in I, \ \ \varepsilon \in [0,1]. \quad (2.\ 2)$$

We will use the subsequent symbols in the paper:

(i) The arithmetic mean:

$$\mathcal{A}(a, b) = \frac{a+b}{2} \ \ \forall a, b \in \mathbb{R}_+, a \neq b,$$

(ii) The logarithmic mean:

$$\mathcal{L}(a,b)=\frac{b-a}{logb-loga}\quad \forall a,b\in\mathbb{R}_{+},a\neq b,$$

(iii) The generalized logarithmic mean:

$$\mathcal{L}(a,b)=\Big[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\Big]^{\frac{1}{p}},\quad p\neq -1.$$

We will use the following lemma (see Ardic et al. [3]), which plays a key role for proving our coming results.

Lemma 2.7. *([3]) Let $G\colon I\subseteq R_{+}=(0,\infty)\to R$ be differentiable function on I° where $w,u\in I^{\circ}$ with $w<u$.If $G'\in L([w,u])$ for all $v\in[w,u]$ then the following inequality holds :*

$$\begin{aligned}&u^{2}G(u)-w^{2}G(w)-2\int_{w}^{u}\theta G(\theta)d\theta\\&=(lnu-lnv)\int_{0}^{1}(u^{3\varepsilon}\ v^{3(1-\varepsilon)})G'(u^{\varepsilon}v^{1-\varepsilon})d\varepsilon+(lnv-lnw)\int_{0}^{1}(v^{3\varepsilon}\ w^{3(1-\varepsilon)})G'(v^{\varepsilon}w^{1-\varepsilon})d\epsilon.\end{aligned}$$

3. MAIN RESULTS

In this section, we derive some new Hermite-Hadamard type inequalities for generalized geometrically convex functions. We denote $I=[a,b]$, unless otherwise specified.

This segment devoted to some new variants related to the Hermite-Hadamard type for generalized geometrically convex functions.

Theorem 3.1. *Let $G\colon I\subseteq R_{+}=(0,\infty)\to R$ be differentiable function on I° where $w,u\in I^{\circ}$ with $w<u$ and $G'\in L([w,u])$. If $|G'|$ is generalized GA-convex on $[w,u]$, for all $v\in[w,u]$ then the following inequality holds :*

$$\begin{aligned}&u^{2}G(u)-w^{2}G(w)-2\int_{w}^{u}\theta G(\theta)d\theta\\&\leq(lnu-lnv)\ \ |G'(v)|L(u^{3},v^{3})+\eta(|G'(u)|,|G'(v)|)\ \ \frac{3u^{3}(lnu-lnv)-u^{3}+v^{3}}{9(lnu-lnv)^{2}}\\&\quad+(lnv-lnw)\ \ |G'(w)|L(v^{3},w^{3})+\eta(|G'(v)|,|G'(w)|)\ \ \frac{3v^{3}(lnv-lnw)-v^{3}+w^{3}}{9(lnv-lnw)^{2}}\quad. \quad (3.\ 3)\end{aligned}$$

Proof. Utilizing Lemma 2.7, the property of modulus and generalized GA-convexity of $|\, G'|$, we have

$$\begin{aligned}
&\left| u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \right| \\
&\leq (lnu - lnv)\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})|\, G'(u^{\varepsilon}v^{(1-\varepsilon)})|d\varepsilon \\
&\quad + (lnv - lnw)\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})|\, G'(v^{\varepsilon}w^{(1-\varepsilon)})|d\varepsilon \\
&\leq (lnu - lnv)\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})\{|G'(v)| + \varepsilon\eta|(G'(u)|, |G'(v))|\}d\varepsilon \\
&\quad + (lnv - lnw)\int_0^1 (3v^{\varepsilon}w^{3(1-\varepsilon)})\{|G'(w)| + \varepsilon\eta(|G'(v)|, |G'(w)|)\}d\varepsilon \\
&\leq (lnu - lnv)\left[v^3|G'(v)|\int_0^1 \left(\frac{u}{v}\right)^{3\varepsilon} d\varepsilon + v^3\eta(|G'(u)|, |G'(v)|)\int_0^1 \varepsilon\left(\frac{u}{v}\right)^{3\varepsilon} d\varepsilon\right] \\
&\quad + (lnv - lnw)\left[w^3|G'(w)|\int_0^1 \left(\frac{v}{w}\right)^{3\varepsilon} d\varepsilon + w^3\eta(|G'(v)|, |G'(w)|)\int_0^1 \varepsilon\left(\frac{v}{w}\right)^{3\varepsilon} d\varepsilon\right] \\
&\leq (lnu - lnv)\left[|G'(v)|L(u^3, v^3) + \eta(|G'(u)|, |G'(v)|)\left(\frac{3u^3(lnu - lnv) - u^3 + v^3}{9(lnu - lnv)^2}\right)\right] \\
&\quad + (lnv - lnw)\left[|G'(w)|L(v^3, w^3) + \eta(|G'(v)|, |G'(w)|)\left(\frac{3v^3(lnv - lnw) - v^3 + w^3}{9(lnv - lnw)^2}\right)\right].
\end{aligned}$$

After suitable arrangment we immideately get the desired inequality. □

Remark 3.2. *If we choose* $\eta(|\, G'(u)|, |\, G'(v)|) = |\, G'(u)| - |\, G'(v)|$ *and* $\eta(|\, G'(v)|, |\, G'(w)|) = |\, G'(v)| - |\, G'(w)|$, *then Theorem 3.1 becomes Theorem 3.1 in* [3].

Theorem 3.3. *Let* $G\colon I \subseteq R_+ = (0,\infty) \to R$ *be differentiable function on* I° *where* $w, u \in I^\circ$ *with* $w < u$ *and* $G' \in L([w,u])$ *. If* $|\, G'|^q$ *is generalized* GA*-convex on* $[w,u]$*, for all* $v \in [w,u]$ *then the following inequality holds :*

$$\begin{aligned}
&\left| u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \right| \\
&\leq (lnu - lnv)\big(L(u^{3p}, v^{3p})\big)^{\frac{1}{p}}\left(|\, G'(v)|^q + \frac{1}{2}\eta(|\, G'(u)^q|, |\, G'(v)^q)|\right)^{\frac{1}{q}} \\
&\quad + (lnv - lnw)\big(L(v^{3p}, w^{3p})\big)^{\frac{1}{p}}\left(|\, G'(w)|^q + \frac{1}{2}\eta(|\, G'(v)^q|, |\, G'(w)|^q)\right)^{\frac{1}{q}} \quad (3.\ 4)
\end{aligned}$$

Proof. Utilizing Lemma 2.7, the property of modulus, generalized GA-convexity of $|G'|^q$ and Hölder inequality we have

$$\begin{aligned}
&\left| u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \right| \\
&\leq (lnu - lnv)\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})|\, G'(u^{\varepsilon}v^{(1-\varepsilon)})|d\varepsilon \\
&\quad +(lnv - lnw)\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})|\, G'(v^{\varepsilon}w^{(1-\varepsilon)})|d\varepsilon \\
&\leq (lnu - lnv)\left(\int_0^1 u^{3\varepsilon p}v^{3(1-\varepsilon)p}d\varepsilon\right)^{\frac{1}{p}}\left(\int_0^1 |\, G'(u^{\varepsilon}v^{(1-\varepsilon)})|^q d\varepsilon\right)^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left(\int_0^1 v^{3\varepsilon p}w^{3(1-\varepsilon)p}d\varepsilon\right)^{\frac{1}{p}}\left(\int_0^1 |\, G'(v^{\varepsilon}w^{(1-\varepsilon)})|^q d\varepsilon\right)^{\frac{1}{q}} \\
&\leq (lnu - lnv)\left(v^{3p}\int_0^1 \left(\frac{u}{v}\right)^{3\varepsilon p} d\varepsilon\right)^{\frac{1}{p}}\left(\int_0^1 (|\, G'(v)|^q + \varepsilon\eta(|\, G'(u)|^q, |\, G'(v)|^q)d\varepsilon)\right)^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left(w^{3p}\int_0^1 \left(\frac{v}{w}\right)^{3\varepsilon p} d\varepsilon\right)^{\frac{1}{p}}\left(\int_0^1 \big(|\, G'(w)|^q + \varepsilon\eta(|\, G'(v)|^q, |\, G'(w)|^q\big)d\varepsilon\right)^{\frac{1}{q}} \\
&\leq (lnu - lnv)\left(\frac{u^{3p} - v^{3p}}{3p(lnu - lnv)}\right)^{\frac{1}{p}}\left(|\, G'(v)|^q + \frac{1}{2}\eta\big(|\, G'(u)|^q, |\, G'(v)|^q\big)\right)^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left(\frac{v^{3p} - w^{3p}}{3p(lnv - lnw)}\right)^{\frac{1}{p}}\left(|\, G'(w)|^q + \frac{1}{2}\eta\big(|\, G'(v)|^q, |\, G'(w)|^q\big)\right)^{\frac{1}{q}}.
\end{aligned}$$

After suitable arrangment we immediately get the desired inequality (4. 8). □

Remark 3.4. *If we choose* $\eta(|\, G'(u)|^q, |\, G'(v)|^q) = |\, G'(u)|^q - |\, G'(v)|^q$ *and* $\eta(|\, G'(v)|^q, |\, G'(w)|^q) = |\, G'(v)|^q - |\, G'(w)|^q$, *then Theorem 3.3 becomes Theorem 3.4 in* [3].

Theorem 3.5. *Under the assumption of Theorem 3.3 , the following inequality holds* :

$$\begin{aligned}
&u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \\
&\leq (lnu - lnv)\;\; L(u^{3q}, v^{3q})\;\; |\, G'(v)|^q + \frac{3q(lnu - lnv) - 1}{3q(lnu - lnv)}\eta\;\; |\, G'(u)|^q, |\, G'(v)|^q \qquad \frac{1}{q} \\
&\quad +(lnv - lnw)\;\; L(v^{3q}, w^{3q})\;\; |\, G'(w)|^q + \frac{3q(lnv - lnw) - 1}{3q(lnv - lnw)}\eta\;\; |\, G'(v)|^q, |\, G'(w)|^q \qquad \frac{1}{q}, \quad (3.\ 5)
\end{aligned}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Utilizing Lemma 2.7, the property of modulus, generalized GA-convexity of $\mid G'\mid^q$ and Hölder inequality we have

$$\begin{aligned}
&\left| u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \right| \\
&\leq (lnu - lnv)\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})\mid G'(u^{\varepsilon}v^{(1-\varepsilon)})|d\varepsilon \\
&\quad +(lnv - lnw)\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})\mid G'(v^{\varepsilon}w^{(1-\varepsilon)})|d\varepsilon \\
&\leq (lnu - lnv)\left(\int_0^1 d\varepsilon\right)^{\frac{1}{p}}\left[v^{3q}\int_0^1 \left(\frac{u}{v}\right)^{3\varepsilon q}\big(\mid G'(v)|^q + \varepsilon\eta(\mid G'(u)|^q, \mid G'(v)|^q)\big)d\varepsilon\right]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left(\int_0^1 d\varepsilon\right)^{\frac{1}{p}}\left[w^{3q}\int_0^1 \left(\frac{v}{w}\right)^{3\varepsilon q}\big(\mid G'(w)|^q + \varepsilon\eta(\mid G'(v)|^q, \mid G'(w)|^q)\big)d\varepsilon\right]^{\frac{1}{q}} \\
&\leq (lnu - lnv)\left(\int_0^1 d\varepsilon\right)^{\frac{1}{p}}\left[v^{3q}\mid G'(v)|^q\int_0^1 \left(\frac{u}{v}\right)^{3\varepsilon q}d\varepsilon\right. \\
&\quad \left. +v^{3q}\eta(\mid G'(u)|^q, \mid G'(v)|^q)\int_0^1 \varepsilon\left(\frac{u}{v}\right)^{(3\varepsilon q)}d\varepsilon\right]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left(\int_0^1 d\varepsilon\right)^{\frac{1}{p}}\left[w^{3q}\mid G'(w)|^q\int_0^1 \left(\frac{v}{w}\right)^{3\varepsilon q}d\varepsilon\right. \\
&\quad \left. +w^{3q}\eta(\mid G'(v)|^q, \mid G'(w)|^q)\int_0^1 \varepsilon\left(\frac{v}{w}\right)^{(3\varepsilon q)}d\varepsilon\right]^{\frac{1}{q}} \\
&\leq (lnu - lnv)\left[\frac{u^{3q} - v^{3q}}{3q(lnu - lnv)}\left(\mid G'(v)|^q + \frac{3q(lnu - lnv) - 1}{3q(lnu - lnv)}\eta(\mid G'(u)|^q, \mid G'(v)|^q)\right)\right]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left[\frac{v^{3q} - w^{3q}}{3q(lnv - lnw)}\left(\mid G'(w)|^q + \frac{3q(lnv - lnw) - 1}{3q(lnv - lnw)}\eta(\mid G'(v)|^q, \mid G'(w)|^q)\right)\right]^{\frac{1}{q}} \\
&\leq (lnu - lnv)\left[L(u^{3q}, v^{3q})\left(\mid G'(v)|^q + \frac{3q(lnu - lnv) - 1}{3q(lnu - lnv)}\eta(\mid G'(u)|^q, \mid G'(v)|^q)\right)\right]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left[L(v^{3q}, w^{3q})\left(\mid G'(w)|^q + \frac{3q(lnv - lnw) - 1}{3q(lnv - lnw)}\eta(\mid G'(v)|^q, \mid G'(w)|^q)\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

This is the required result. □

Remark 3.6. *If we choose* $\eta(\mid G'(u)|^q, \mid G'(v)|^q) = \mid G'(u)|^q - \mid G'(v)|^q$ *and* $\eta(\mid G'(v)|^q, \mid G'(w)|^q) = \mid G'(v)|^q - \mid G'(w)|^q$*, then Theorem 3.5 becomes Theorem 3.5 in* [3].

Theorem 3.7. *Let* $G\colon I \subseteq R_+ = (0, \infty) \to R$ *be differentiable function on* I° *where* $w, u \in I^{\circ}$ *with* $w < u$ *and* $G' \in L([w, u])$ *. If* $\mid G'\mid^q$ *is generalized GA-convex on* $[w, u]$*,* *for all* $v \in [w,u]$ *and* $\forall\, q \geq 1$, *then the following inequality holds* :

$$\begin{aligned}
&\left| u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \right| \\
&\leq (lnu - lnv)(L(u^3,v^3))^{1-\frac{1}{q}}\Bigg[| G'(v)|^q L(u^3,v^3) \\
&\quad +\eta(| G'(u)|^q, | G'(v)|^q)\left(\frac{3u^3(lnu-lnv) - u^3 + v^3}{9(lnu-lnv)^2}\right)\Bigg]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)(L(v^3,w^3))^{1-\frac{1}{q}}\Bigg[| G'(w)|^q L(v^3,w^3) \\
&\quad +\eta(| G'(v)|^q, | G'(w)|^q)\left(\frac{3v^3(lnv-lnw) - v^3 + w^3}{9(lnv-lnw)^2}\right)\Bigg]^{\frac{1}{q}}. \qquad (3.\ 6)
\end{aligned}$$

Proof. Utilizing Lemma 2.7, the property of modulus, generalized GA-convexity of $| G'|^q$ and power mean integral inequality, we have

$$\begin{aligned}
&\left| u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \right| \\
&\leq (lnu - lnv)\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})| G'(u^{\varepsilon}v^{(1-\varepsilon)})|d\varepsilon \\
&\quad +(lnv - lnw)\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})| G'(v^{\varepsilon}w^{(1-\varepsilon)})|d\varepsilon \\
&\leq (lnu - lnv)\left(\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})d\varepsilon\right)^{1-\frac{1}{q}}\left[\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})| G'(u^{\varepsilon}v^{1-\varepsilon})|^q d\varepsilon\right]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left(\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})d\varepsilon\right)^{1-\frac{1}{q}}\left[\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})| G'(v^{\varepsilon}w^{1-\varepsilon})|^q d\varepsilon\right]^{\frac{1}{q}} \\
&\leq (lnu - lnv)\left(v^3\int_0^1 \left(\frac{u}{v}\right)^{3\varepsilon}d\varepsilon\right)^{1-\frac{1}{q}}\left[v^3\int_0^1 \left(\frac{u}{v}\right)^{3\varepsilon}| G'(u^{\varepsilon}v^{1-\varepsilon})|^q d\varepsilon\right]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\left(w^3\int_0^1 \left(\frac{v}{w}\right)^{3\varepsilon}d\varepsilon\right)^{1-\frac{1}{q}}\left[w^3\int_0^1 \left(\frac{v}{w}\right)^{3\varepsilon}| G'(v^{\varepsilon}w^{1-\varepsilon})|^q d\varepsilon\right]^{\frac{1}{q}}
\end{aligned}$$

$$\leq (lnu-lnv)\bigg(v^3\int_0^1\left(\frac{u}{v}\right)^{3\varepsilon}d\varepsilon\bigg)^{1-\frac{1}{q}}\left[v^3\int_0^1\left(\frac{u}{v}\right)^{3\varepsilon}\big(|\,G'(v)|^q+\varepsilon\eta(|\,G'(u)|^q,|\,G'(v)|^q)\big)d\varepsilon\right]^{\frac{1}{q}}$$
$$+(lnv-lnw)\bigg(w^3\int_0^1\left(\frac{v}{w}\right)^{3\varepsilon}d\varepsilon\bigg)^{1-\frac{1}{q}}\left[w^3\int_0^1\left(\frac{v}{w}\right)^{3\varepsilon}\big(|\,G'(w)|^q+\varepsilon\eta(|\,G'(v)|^q,|\,G'(w)|^q)\big)d\varepsilon\right]^{\frac{1}{q}}$$
$$\leq (lnu-lnv)\bigg(v^3\int_0^1\left(\frac{u}{v}\right)^{3\varepsilon}d\varepsilon\bigg)^{1-\frac{1}{q}}$$
$$\times\left[v^3|\,G'(v)|^q\int_0^1\left(\frac{u}{v}\right)^{3\varepsilon}d\varepsilon+v^3\eta(|\,G'(u)|^q,|\,G'(v)|^q)\int_0^1\varepsilon\left(\frac{u}{v}\right)^{3\varepsilon}d\varepsilon\right]^{\frac{1}{q}}$$
$$+(lnv-lnw)\bigg(w^3\int_0^1\left(\frac{v}{w}\right)^{3\varepsilon}d\varepsilon\bigg)^{1-\frac{1}{q}}$$
$$\times\left[w^3|\,G'(w)|^q\int_0^1\left(\frac{v}{w}\right)^{3\varepsilon}d\varepsilon+w^3\eta(|\,G'(v)|^q,|\,G'(w)|^q)\int_0^1\varepsilon\left(\frac{v}{w}\right)^{3\varepsilon}d\varepsilon\right]^{\frac{1}{q}}$$
$$\leq (lnu-lnv)\left(\frac{u^3-v^3}{3(lnu-lnv)}\right)^{1-\frac{1}{q}}\bigg[|\,G'(v)|^q\frac{u^3-v^3}{3(lnu-lnv)}$$
$$+\eta(|\,G'(u)|^q,|\,G'(v)|^q)\left(\frac{3u^3(lnu-lnv)-u^3+v^3}{9(lnu-lnv)^2}\right)\bigg]^{\frac{1}{q}}$$
$$+(lnv-lnw)\left(\frac{v^3-w^3}{3(lnv-lnw)}\right)^{1-\frac{1}{q}}\bigg[|\,G'(w)|^q\frac{v^3-w^3}{3(lnv-lnw)}$$
$$+\eta(|\,G'(v)|^q,|\,G'(w)|^q)\left(\frac{3v^3(lnv-lnw)-v^3+w^3}{9(lnv-lnw)^2}\right)\bigg]^{\frac{1}{q}}.$$

After simple calculations we get the desired inequality (3. 6). □

Remark 3.8. *If we choose* $\eta(|\,G'(u)|^q,|\,G'(v)|^q)=|\,G'(u)|^q-|\,G'(v)|^q$ *and* $\eta(|\,G'(v)|^q,|\,G'(w)|^q)=|\,G'(v)|^q-|\,G'(w)|^q$, *then Theorem 3.7 becomes,*

$$\left|u^2G(u)-w^2G(w)-2\int_w^u\theta G(\theta)d\theta\right|$$
$$\leq (lnu-lnv)\big(L(u^3,v^3)\big)^{1-\frac{1}{q}}\bigg[\frac{3u^3(lnu-lnv)-u^3+v^3}{9(lnu-lnv)^2}|\,G'(u)|^q$$
$$+\left(L(u^3,v^3)-\frac{3u^3(lnu-lnv)-u^3+v^3}{9(lnu-lnv)^2}\right)|\,G'(v)|^q\bigg]^{\frac{1}{q}}$$
$$+(lnv-lnw)\big(L(v^3,w^3)\big)^{1-\frac{1}{q}}\bigg[\frac{3v^3(lnv-lnw)-v^3+w^3}{9(lnv-lnw)^2}|\,G'(v)|^q$$
$$+\left(L(v^3,w^3)-\frac{3v^3(lnv-lnw)-v^3+w^3}{9(lnv-lnw)^2}\right)|\,G'(w)|^q\bigg]^{\frac{1}{q}}.$$

Theorem 3.9. *Let* $G\colon I\subseteq R_+=(0,\infty)\to R$ *be differentiable function on* I° *where* $w,u\in I^\circ$ *with* $w<u$ *and* $G'\in L([w,u])$ *. If* $|\,G'|^q$ *is generalized* GA*-convex on* $[w,u]$*,*

for all $v \in [w,u]$ *and* $\forall\ q \geq 1$, *then the following inequality holds:*

$$\begin{aligned}
&u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \\
&\leq (lnu - lnv)\ \ L(u^{\frac{3(q-p)}{q-1}}, v^{\frac{3(q-p)}{q-1}})^{\frac{q-1}{q}}\ C_q(u,v)^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\ \ L(v^{\frac{3(q-p)}{q-1}}, w^{\frac{3(q-p)}{q-1}})^{\frac{q-1}{q}}\ C_q(v,w)^{\frac{1}{q}}, \qquad (3.\ 7)
\end{aligned}$$

where

$C_q(u,v) = |\, G'(v)|^q L(u^{3p}, v^{3p}) + \frac{\eta(|G'(u)|^q, |G'(v)|^q)}{3p(lnu-lnv)}\big(u^{3p} - L(u^{3p}, v^{3p})\big)$

$C_q(v,w) = |\, G'(w)|^q L(v^{3p}, w^{3p}) + \frac{\eta(|G'(v)|^q, |G'(w)|^q)}{3p(lnv-lnw)}\big(v^{3p} - L(v^{3p}, w^{3p})\big).$

Proof. Utilizing Lemma 2.7, the property of modulus, generalized GA-convexity of $|\, G'|^q$ and power mean integral inequality, we have

$$\begin{aligned}
&\left| u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \right| \\
&\leq (lnu - lnv)\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})|\, G'(u^{\varepsilon}v^{(1-\varepsilon)})|d\varepsilon \\
&\quad +(lnv - lnw)\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})|\, G'(v^{\varepsilon}w^{(1-\varepsilon)})|d\varepsilon \\
&\leq (lnu - lnv)\bigg(\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})^{\frac{q-p}{q-1}}d\varepsilon\bigg)^{\frac{q-1}{q}} \\
&\quad \times\bigg[\int_0^1 (u^{3\varepsilon}v^{3(1-\varepsilon)})^p\big(|\, G'(v)|^q + \varepsilon\eta(|\, G'(u)|^q, |\, G'(v)|^q)\big)d\varepsilon\bigg]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\bigg(\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})^{\frac{q-p}{q-1}}d\varepsilon\bigg)^{\frac{q-1}{q}} \\
&\quad \times\bigg[\int_0^1 (v^{3\varepsilon}w^{3(1-\varepsilon)})^p\big(|\, G'(w)|^q + \varepsilon\eta(|\, G'(v)|^q, |\, G'(w)|^q)\big)d\varepsilon\bigg]^{\frac{1}{q}} \\
&\leq (lnu - lnv)\bigg(v^{\frac{3(q-p)}{q-1}}\int_0^1 \Big(\frac{u}{v}\Big)^{\frac{3\varepsilon(q-p)}{q-1}}d\varepsilon\bigg)^{\frac{q-1}{q}} \\
&\quad \times\bigg[v^{3p}\int_0^1 \Big(\frac{u}{v}\Big)^{3\varepsilon p}\big(|\, G'(v)|^q + \varepsilon\eta(|\, G'(u)^q, |\, G'(v)|^q)\big)d\varepsilon\bigg]^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\bigg(w^{\frac{3(q-p)}{q-1}}\int_0^1 \Big(\frac{v}{w}\Big)^{\frac{3\varepsilon(q-p)}{q-1}}d\varepsilon\bigg)^{\frac{q-1}{q}} \\
&\quad \times\bigg[w^{3p}\int_0^1 \Big(\frac{v}{w}\Big)^{3\varepsilon p}\big(|\, G'(w)|^q + \varepsilon\eta(|\, G'(v)|^q, |\, G'(w)|^q)\big)d\varepsilon\bigg]^{\frac{1}{q}}.
\end{aligned}$$

After solving above integrals we immediately get the desired inequality (3. 7). □

Remark 3.10. *If we choose* $\eta(\mid G'(u)|^q, \mid G'(v)|^q) = \mid G'(u)|^q - \mid G'(v)|^q$ *and* $\eta(\mid G'(v)|^q, \mid G'(w)|^q) = \mid G'(v)|^q - \mid G'(w)|^q$, *then Theorem 3.9 becomes,*

$$\begin{aligned}
&\left| u^2G(u) - w^2G(w) - 2\int_w^u \theta G(\theta)d\theta \right| \\
&\leq (lnu - lnv)\left(L(u^{\frac{3(q-p)}{q-1}}, v^{\frac{3(q-p)}{q-1}})\right)^{\frac{q-1}{q}} \left[\frac{1}{3p(lnu-lnv)}\big(u^{3p} - L(u^{3p}, v^{3p})\big)\mid G'(u)|^q \right.\\
&\quad \left. + \left(L(u^{3p}, v^{3p}) + \frac{1}{3p(lnu-lnv)}\big(L(u^{3p}, v^{3p}) - u^{3p}\big)\right)\mid G'(v)|^q\right]^{\frac{1}{q}} \\
&\quad + (lnv - lnw)\left(L(v^{\frac{3(q-p)}{q-1}}, w^{\frac{3(q-p)}{q-1}})\right)^{\frac{q-1}{q}} \left[\frac{1}{3p(lnv-lnw)}\big(v^{3p} - L(v^{3p}, w^{3p})\big)\mid G'(v)|^q \right.\\
&\quad \left. + \left(L(v^{3p}, w^{3p}) + \frac{1}{3p(lnv-lnw)}\big(L(v^{3p}, w^{3p}) - v^{3p}\big)\right)\mid G'(w)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

4. Applications

Proposition 4.1. *For* $0 < w < v < u, \ \rho > 0$, *we obtain:*

$$\begin{aligned}
&3(\ln u - \ln v)(u - w)\big(\mathcal{L}_{\rho+2}(v,\mu)\big)^{\rho+2} \\
&\leq \mathcal{L}(u^3, v^3)\Big[3\rho v^\rho(\ln u - \ln v)^2 - \eta(\rho u^\rho, \rho v^\rho)\Big] + u^3\eta(\rho u^\rho, \rho v^\rho).
\end{aligned}$$

Proof. If w set $G(x) = \frac{x^{\rho+1}}{\rho+1}$ for $x \in \mathcal{R}_+$ and $\rho > 0$. Then, it is clear that $G'(x) = x^\rho$ is a generalized geometrically convex function on $x \in \mathcal{R}_+$ and from the inequality that is given in Theorem 3.1, we can write

$$\begin{aligned}
&\left|\mu^2G(\mu) - \eta^2G(\eta) - 2\int\limits_a^b \psi G(\psi)d\psi\right| \\
&= \left|\frac{\mu^{\rho+3}}{\rho+1} - \frac{\eta^{\rho+3}}{\rho+1} - 2\left(\frac{\mu^{\rho+3} - \eta^{\rho+3}}{(\rho+1)(\rho+3)}\right)\right| \\
&= (\mu - \eta)\big(\mathcal{L}_{\rho+2}(\eta,\mu)\big)^{\rho+2}.
\end{aligned}$$

For the right hand side of the inequality, we get

$$\begin{aligned}
&(lnu - lnv)\left[|G'(v)|L(u^3, v^3) + \eta(\mid G'(u)|, \mid G'(v)|)\left(\frac{3u^3(lnu - lnv) - u^3 + v^3}{9(lnu - lnv)^2}\right)\right] \\
&\quad + (lnv - lnw)\left[|G'(w)|L(v^3, w^3) + \eta(\mid G'(v)|, \mid G'(w)|)\left(\frac{3v^3(lnv - lnw) - v^3 + w^3}{9(lnv - lnw)^2}\right)\right] \\
&= \mathcal{L}(u^3, v^3)\Big[3\rho v^\rho(\ln u - \ln v)^2 - \eta(\rho u^\rho, \rho v^\rho)\Big] + u^3\eta(\rho u^\rho, \rho v^\rho).
\end{aligned}$$

This is the required result. □

Proposition 4.2. *For* $0 < w < v < u,\ \rho > 0$, *we obtain:*

$$\begin{aligned}&(u-w)\big(\mathcal{L}_{\rho+2}(v,\mu)\big)^{\rho+2}\\ &\leq (\ln u-\ln v)\big(L(u^{3p},v^{3p})\big)^{\frac{1}{p}}\left(|v|^{q\rho}+\frac{1}{2}\eta(|\,u|^{q\rho},|\,v|^{q\rho})\right)^{\frac{1}{q}}\\ &\quad+(lnv-lnw)\big(L(v^{3p},w^{3p})\big)^{\frac{1}{p}}\left(|w|^{q\rho}+\frac{1}{2}\eta(|\,v|^{q\rho},|\,w|^{q\rho})\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. If we set $G(x)=\frac{x^{\rho+1}}{\rho+1}$ for $x\in\mathcal{R}_+$ and $\rho>0$, then, it is clear that $G'(x)=x^{\rho}$ is a generalized geometrically convex function on $x\in\mathcal{R}_+$ and from the inequality that is given in Theorem 3.3, we can write

$$\begin{aligned}&(u-w)\big(\mathcal{L}_{\rho+2}(v,\mu)\big)^{\rho+2}\\ &\leq (\ln u-\ln v)\big(L(u^{3p},v^{3p})\big)^{\frac{1}{p}}\left(|v|^{q\rho}+\frac{1}{2}\eta(|\,u|^{q\rho},|\,v|^{q\rho})\right)^{\frac{1}{q}}\\ &\quad+(lnv-lnw)\big(L(v^{3p},w^{3p})\big)^{\frac{1}{p}}\left(|w|^{q\rho}+\frac{1}{2}\eta(|\,v|^{q\rho},|\,w|^{q\rho})\right)^{\frac{1}{q}}.\end{aligned}$$

This is the required result. □

Proposition 4.3. *For* $0 < w < v < u,\ \rho > 0$, *we obtain:*

$$\begin{aligned}&(u-w)\big(\mathcal{L}_{\rho+2}(v,\mu)\big)^{\rho+2}\\ &\leq (\ln u-\ln v)\big(L(u^{3q},v^{3q})\big)\left(|u|^{q\rho}+\frac{3q(\ln u-\ln v)-1}{3q(\ln u-\ln v)}\eta(|\,u|^{q\rho},|\,v|^{q\rho})\right)^{\frac{1}{q}}\\ &\quad+(lnv-lnw)\big(L(v^{3q},w^{3q})\big)\left(|w|^{q\rho}+\frac{3q(\ln v-\ln w)-1}{3q(\ln v-\ln w)}\eta(|\,v|^{q\rho},|\,w|^{q\rho})\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. If w set $G(x)=\frac{x^{\rho+1}}{\rho+1}$ for $x\in\mathcal{R}_+$ and $\rho>0$, then, it is clear that $G'(x)=x^{\rho}$ is a generalized geometrically convex function on $x\in\mathcal{R}_+$ and from the inequality that is given in Theorem 3.5, then we get the immediate consequences. □

Proposition 4.4. *For* $0 < w < v < u,\ \rho > 0$, *we obtain:*

$$\begin{aligned}&(u-w)\big(\mathcal{L}_{\rho+2}(v,\mu)\big)^{\rho+2}\\ &\leq (lnu-lnv)(L(u^3,v^3))^{1-\frac{1}{q}}\Bigg[|\,v|^{q\rho}L(u^3,v^3)\\ &\quad+\eta(|\,u|^{q\rho},|\,v|^{q\rho})\left(\frac{3u^3(lnu-lnv)-u^3+v^3}{9(lnu-lnv)^2}\right)\Bigg]^{\frac{1}{q}}\\ &\quad+(lnv-lnw)(L(v^3,w^3))^{1-\frac{1}{q}}\Bigg[|w|^{q\rho}L(v^3,w^3)\\ &\quad+\eta(|\,v|^{q\rho},|\,w|^{q\rho})\left(\frac{3v^3(lnv-lnw)-v^3+w^3}{9(lnv-lnw)^2}\right)\Bigg]^{\frac{1}{q}}.\end{aligned}$$

Proof. If w set $G(x) = \frac{x^{\rho+1}}{\rho+1}$ for $x \in \mathcal{R}_+$ and $\rho > 0$, then, it is clear that $G'(x) = x^\rho$ is a generalized geometrically convex function on $x \in \mathcal{R}_+$ and from the inequality that is given in Theorem 3.7, then we get the immediate consequences. □

Proposition 4.5. *For* $0 < w < v < u,\ \rho > 0$, *we obtain:*

$$\begin{aligned}
&(u-w)\big(\mathcal{L}_{\rho+2}(v,\mu)\big)^{\rho+2} \\
&\leq (lnu - lnv)\bigg(L(u^{\frac{3(q-p)}{q-1}}, v^{\frac{3(q-p)}{q-1}})\bigg)^{\frac{q-1}{q}} \big(C_q(u,v)\big)^{\frac{1}{q}} \\
&\quad +(lnv - lnw)\bigg(L(v^{\frac{3(q-p)}{q-1}}, w^{\frac{3(q-p)}{q-1}})\bigg)^{\frac{q-1}{q}} \big(C_q(v,w)\big)^{\frac{1}{q}},
\end{aligned}$$

where
$C_q(u,v) = |v|^{q\rho} L(u^{3p}, v^{3p}) + \frac{|\ \eta(u^\rho, v^\rho)|^q}{3p(lnu - lnv)}\big(u^{3p} - L(u^{3p}, v^{3p})\big)$
$C_q(v,w) = |w|^{q\rho} L(v^{3p}, w^{3p}) + \frac{|\ \eta(v^\rho, w^\rho)|^q}{3p(lnv - lnw)}\big(v^{3p} - L(v^{3p}, w^{3p})\big).$

Proof. If w set $G(x) = \frac{x^{\rho+1}}{\rho+1}$ for $x \in \mathcal{R}_+$ and $\rho > 0$, then, it is clear that $G'(x) = x^\rho$ is a generalized geometrically convex function on $x \in \mathcal{R}_+$ and from the inequality that is given in Theorem 3.9, then we get the immediate consequences. □

CONCLUSION

In this paper, we have introduced and studied a new class of generalized geometrically convex functions. Several new integral inequalities for these generalized functions have been derived, which have important applications in physics and material sciences. These estimates also useful in numerical analysis for finding the error bounds for the approximate solution. We have also discussed important several special cases, which can be obtained from our results.

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REFERENCES

[1] A. O. Akdemir, M. E. Özdemir and F. Sevinc, *Some inequalities for GG-convex functions*, RGMIA, **18**(2015).
[2] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl. **335**(2007).
[3] M. A. Ardıç, A. O. Akdemir, and K. Yıldız, On some new inequalities via GG-Convexity and GA-Convexity, Filomat **32,** No.16(2018) 5707-5717.

[4] H. -H. Chu, S. Rashid, Z. Hammouch, Y.-M. Chu.: *New fractional estimates for Hermite-Hadamard- Mercer's type inequalities*, Alexandria Eng. J. (2020) https://doi. org/10.1016/j.aej.2020.06.040.

[5] S. S. Dragomir, *Inequalities of Hermite-Hadamard type for GG-convex functions*, RGMIA, **18**(2015).

[6] S. S. Dragomir, *Inequalities of Hermite-Hadamard type for GA-convex functions*, RGMIA, **18**(2015).

[7] S. S. Dragomir, *Jensen type inequalities for GA-convex functions*, RGMIA, **18**(2015).

[8] S. S. Dragomir, E. E. M, Pearce: *Selected Topics on Hermite-Hadamard Inequalities and Applications*, Victoria University, Australia (2000)

[9] M. E. Gordji, M. R. Delavar, M. D. Sen,: *On φconvex functions*, J. Math. Inequal. **10,** No.1(2016) 173-183.

[10] M. Gürbüz, A. O. Akdemir, S. Rashid, E. Set, *Hermite-Hadamard inequality for fractional integrals of Caputo-Fabrizio type and related inequalities*, J. Inequal. Appl. **2020** (2020) 172.

[11] J. Hadamard, *Etude sur les proprietes des fonctions entieres et en particulier dune fonction consideree par Riemann*, J. Math. Pures Appl. **58**(1893) 171-215.

[12] İ. İşcan, *Some generalized Hermite-Hadamard type inequalities for some quasi-geometrically convex functions*, Amer.J.Math.Anal. **1,**No.3(2013) 48-52.

[13] İİşcan, Hermite-Hadamard type inequalities for GA-s-convex functions, arXiv:1306.1960v2.

[14] F. Jarad, T. Abdeljawad, S. Rashid, Zakia Hammouch, *More properties of the proportional fractional integrals and derivatives of a function with respect to another function*, Adv. Diff. Eqs. **2020**(2020) 303.

[15] M. A. Latif, *New Hermite-Hadamard type integral inequalities for GA-convex functions with applications*, Analysis, **34**(2014) 379–389.

[16] M. A. Latif, S. Rashid, S. S. Dragomir, Y. -M. Chu, *Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications*, J. Inequal. Appl, **2019**(2019):317, doi.org/10.1186/s13660-019-2272-7.

[17] C. P. Niculescu, *Convexity according to the geometric mean*, Math. Inequal. Appl. **3**(2000) 155-167.

[18] M. A. Noor, K. I. Noor and F. Safdar, *Generalized geometrically convex functions and inequalities*, J. Inequal. Appl, **202**(2017).

[19] D. Nie, S. Rashid, A. O. Akdemir, D. Baleanu, J. B. Liu, *On some new weighted inequalities for differentiable exponentially convex and exponentially quasi-convex functions with applications*, Mathematics, **727**,No.7(2019) doi:10.3390/math7080727.

[20] J. F. Li, S. Rashid, J. B. Liu, A. O. Akdemir, F. Safdar, *Inequalities involving conformable approach for exponentially convex functions and their applications*, J. Fun. spaces, **2020** Article ID 6517068, 17 pages; doi.org/10.1155/2020/6517068.

[21] S. Rashid, T. Abdeljawed, F. Jarad, M. A. Noor, *Some estimates for generalized Riemann-Liouville fractional integrals of exponentially convex functions and their applications*, Mathematics, **807**, No.7(2019) doi:10.3390/math7090807.

[22] S. Rashid, A. O .Akdemir. F. Jarad, M. A. Noor, K. I. Noor, *Simpson's type integral inequalities for K-fractional integrals and their applications*, AIMS. MATH. **4**, No. 4(2019) 1087-1100, doi:10.3934/math.2019.4.1087.

[23] S. Rashid, A. O. Akdemir, K. S. Nisar, T. Abdeljawad, G. Rahman, *New generalized reverse Minkowski and related integral inequalities involving generalized fractional conformable integrals*, J. Inequal. Appl. **2020**(2020) 177. https://doi.org/10.1186/s13660-020-02445-2.

[24] S. Rashid, A. O. Akdemir, M. A. Noor, K. I. Noor, *Generalization of inequalities analogous to preinvex functions via extended generalized Mittag-Leffler functions, In Proceedings of the International Conference on Applied and Engineering Mathematics-Second International Conference*, ICAEM 2018, Hitec Taxila, Pakistan, 27-29 August 2018.

[25] S. Rashid, Z. Hammouch, H. Kalsoom, R. Ashraf, Y. -M. Chu, *New Investigation on the Generalized K-Fractional Integral Operators*, Frontier in Physics, **25**, No. 8(2020) doi: 10.3389/fphy.2020.00025.

[26] S. Rashid, Z. Hammouch, D. Baleanu, Y.-M. Chu, *New generalizations in the sense of the weighted non-singular fractional integral operator, Fractals*, **28,**(8)(2020) Article ID: 2040003, 11 pages.

[27] S. Rashid, Z. Hammouch, F. Jarad, Y.-M. Chu, *New estimates of integral inequalities via generalized proportional fractional integral operator with respect to another function*, Fractals, **28,**(8)(2020) Article ID: 2040027, (12 pages).

[28] S. Rashid, İ. İşcan, D. Baleanu, Y. -M. Chu, *Generation of new fractional inequalities via n-polynomials s-type convexity with applications*, Adv. Diff. Eqs. **2020,**(2020) 264.

[29] S. Rashid, H. Kalsoom, Z. Hammouch, R. Ashraf, D. Baleanu, Y. -M. Chu, *New multi-parametrized estimates having pth-Order differentiability in fractional calculus for predominating $\hbar$-convex functions in Hilbert Space*, Symmetry, **12,**No.2(2020); doi.org/10.3390/sym12020222.

[30] S. Rashid, F. Jarad, M. A. Noor, *Grüss-type integrals inequalities via generalizedproportional fractional operators*, RACSAM, **114,** No.93(2020); doi.org/10.1007/s13398-020-00823-5.

[31] S. Rashid, F. Jarad, M. A. Noor, K. I. Noor, D. Baleanu, J, -B. Liu, *Grüss type inequalities for generalized K-fractional integral,* Adv. Differ. Eqs, **2020,**(2020) 203.

[32] S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom, Y.-M. Chu, *Inequalities by means of generalized proportional fractional integral operators with respect to another function*, Mathematics, **1225,**No.7(2020) doi:10.3390/math7121225.

[33] S. Rashid, F. Jarad, M. A. Noor, K. I. Noor, F. Safdar, *On Pólya-Szegö Type Inequalities via K-Fractional Conformable Integrals*, Punjab Univ. j. math.**52**, No. 5, (2020) 63–76.

[34] S. Rashid, A. Khalid, G. Rahman, K. S. Nisar, Y. -M. Chu, *On new modifications governed by quantum Hahn's integral operator pertaining to fractional calculus,* **2020**(2020) Article ID: 8262860, 12 pages, https://doi.org/10.1155/2020/8262860

[35] S. Rashid, M. A. Latif, Z. Hammouch, Y. -M. Chu, *Fractional integral inequalities for strongly h-preinvex functions for a kth order differentiable functions*, Symmetry, **1448,**No.11(2019) doi:10.3390/sym11121448.

[36] M. A. Noor, K. I. Noor and S. Rashid, *Some new clases of preinvex functions and inequalities*, Mathematics. **29,**No.7(2019) doi:10.3390/math7010029.

[37] S. Rashid, M. A. Noor and K. I. Noor,*Fractional exponentially m-convex functions and inequalities*, Inter. J. Anal. Appl, **17,**No.3(2019)464-478, doi.org/10.28924/2291-8639.

[38] S. Rashid, M. A. Noor, K. I. Noor, *Inequalities involving new fractional integrals technique via exponentially convex functions*, Ukrainian Math. J. Preprint, (2020).

[39] S. Rashid, M. A. Noor, K. I. Noor, *Some generalize Riemann-Liouville fractional estimates involving functions having exponentially convexity property*, Punjab Univ. j. math **51,**No.11(2019) 01-15.

[40] S. Rashid, M. A. Noor, K. I. Noor,*Inequalities pertaining fractional approach through exponentially convex functions*, Fractal Fract. **37,**No.3(2019) doi:10.3390/fractalfract3030037.

[41] S. Rashid, M. A. Noor, K. I. Noor, *New Estimates for Exponentially Convex Functions via Conformable Fractional Operator*, Fractal Fract. **19,**No.3(2019); doi:10.3390/fractalfract3020019.

[42] S. Rashid, M. A. Noor, K. I. Noor, *Modified exponential convex functions and inequalities,* Open. Access. J. Math. Theor. Phy, **2**, No.2(2019) 45–51.

[43] S. Rashid, M. A. Noor, K. I. Noor, *Some new estimates for exponentially (h,m)-convex functions via extended generalized fractional integral operators*, Korean J. Math. **27,**No.4(2019) 843–860, doi. Org/10.11568/kjm.2019.27.4.843.

[44] S. Rashid, M. A. Noor, K. I. Noor, A. O. Akdemir, *Some new generalizations for exponentially s-convex functions and inequalities via fractional operators*, Fractal Fract. **24,** No.3(2019) doi:10.3390/fractalfract3020024.

[45] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, *Integral inequalities for generalized preinvex functions*, Punjab Univ.j. math. **51,**No.10(2019) 77-91.

[46] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, *New Hermite-Hadamard type inequalities for exponentially GA and GG convex functions*, Punjab Univ. j. math. **52,**No.2(2020) 15-28.

[47] S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, Y. -M. Chu, *Hermite-Hadamard inequalities for the class of convex functions on time scale, Mathematics*, **956**,No.7(2019) doi:10.3390/math7100956.

[48] S. Rashid, M. A. Noor, K. I. Noor, Y. -M. Chu, *Ostrowski type inequalities in the sense of generalized K-fractional integral operator for exponentially convex functions*, AIMS Mathematics, **5,**No.3(2020) 2629–2645 doi:10.3934/math.2020171.

[49] S. Rashid, F. Safdar, A. O. Akdemir, M. A. Noor, K. I. Noor, *Some new fractional integral inequalities for exponentially m-convex functions via extended generalized Mittag-Leffler function*, J. Inequal. Appl, **2019**(2019):299,doi:.org/10.1186/s13660-019-2248-7.

[50] F. Safdar, M. A. Noor, K. I. Noor, S. Rashid, *Some new estimates of generalized (h_1, h_2)-convex functions,* J. Prime Res. Math. **15**(2019) 129-146.

[51] L. Xu, Y. -M. Chu, S. Rashid, A. A. El-Deeb, K. S. Nisar, *On new unified bounds for a family of functions via fractional q-calculus theory*, **2020** (2002) Article ID: 4984612, 9 pages, https://doi.org/10.1155/2020/4984612

[52] T. Y. Zhang, A. P. Ji and F. Qi,
emphSome inequalities of Hermite-Hadamard type for GA-convex functions with applications to means, Le Matematiche **48**(2013) 229-239.

[53] S. -S. Zhou, S. Rashid, F. Jarad, H. Kalsoom, Y. -M. Chu, *New estimates considering the generalized proportional Hadamard fractional integral operators*, Adv. Differ. Equs. **2020**(2020) 275.

[54] S. -S. Zhou, S. Rashid, S. S. Dragomir, M. A. Latif, A. O. Akdemir, J.-B. Liu, *Some new inequalities involving k-fractional integral for certain classes of functions and their applications*, J. Fun. Spaces. **2020**, Article ID 5285147, 14 pageshttps://doi.org/10.1155/2020/5285147