Centre of Unitary Subgroup of Modular Group Algebra’s

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Abstract.: We establish the structure of the centre of \(V^*(F_2^{k}(M_{2^n+1}))\), \(V^*(F_2^{k}(M_{2^n+1} \times C_2))\) and \(V^*(F_2^{k}((M_{2^n+1} \times C_2) \times C_2))\) over a finite field of characteristics 2 where \(M_{2^n+1} = \langle \psi, \lambda \mid \psi^{2^n} = \lambda^2 = 1, \lambda \psi = \psi^{2^n+1} \lambda \rangle\) is the Modular group having order \(2^{n+1}\) and \(C_2\) is a cyclic group of order 2.

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1. Introduction

Let \(FG\) denote the group algebra of group \(G\) over the field \(F\). The homomorphism \(\Gamma : FG \rightarrow F\) given by \(\Gamma(\sum_{h \in G} a_h h) = \sum_{h \in G} a_h\) is called the augmentation mapping of \(FG\). Write \(U(FG)\) be the unit group of all invertable elements in \(FG\) and the normalizes unit group denoted by \(V(FG)\) consists of all the invertible elements of \(FG\) of augmentation 1. It is well known that \(U(FG) = U(F) \times V(FG)\). Let \(G\) be a finite \(p\)-group and \(F\) a finite field of characteristic \(p\), then the order of \(V(FG)\) is \(|F|^{[G]} - 1\) and \(V(FG)\) is a finite \(p\)-group. For further details on it see [7]. In 1984 Sandling [10] studied the invertible elements in modular group algebra. This group algebra is of finite abelian \(p\)-group and this work contributes a lot in an area namely presentation of group of units. In 1992 Sandling [11] worked on the presentation for unit groups of a modular group algebras of groups of order 16. Bovdi and Sakah [3] studied unitary subgroups of the multiplicative group of a modular group algebra of a finite abelian \(p\)– group. This paper gave the solution of problem, posed by S.P Novikov, on the structure of group \(V(FG)\) of group algebra over a finite field having characteristic \(p\).

The anti-automorphism of \(FG\) is the mapping \(* : FG \rightarrow FG\) which is defined below
\[ (*)(\sum_{h \in G} a_h h) = \sum_{h \in G} a_h h^{-1} \]
An element $\eta$ which satisfy $\eta^{-1} = \eta^*$, where $\eta$ is an element of normalised unit group, is called unitary element. So unitary unit group of $FG$ is the set of all normalised unit elements that satisfy $\eta^{-1} = \eta^*$ and is denoted by $V_*(FG)$. In 1994 Bovdi and Kovacs [1] established that $V_*(F_{2^m}G)$ is normal in $V_*(F_{2^m}G)$ if $G$ is extraspecial, and studied unitary units of modular group algebra.

In [8] structure of centre of $Z(V_*(F_{2^m}M_{16}))$ unitary unit subgroup $V_*(F_{2^m}M_{16})$ of group algebra $(F_{2^m}M_{16})$ is given where

$$M_{16} = \langle \psi, \lambda \psi \rangle = \lambda^2 = 1, \lambda \psi = \psi^5 \lambda >$$

is modular group of order 16 and $F_{2^m}$ is any finite field of characteristic 2 with $2^m$ elements. They also described the structure of unitary unit subgroup $V_*(F_{2^m}M_{16})$ of group algebra $(F_{2^m}M_{16})$.

In [9], Raza and Ahmad constructed structure of $Z(V_*(F_{2^m}(QD)_{16}))$ where $(QD)_{16}$ is known as quasi dihedral group having order 16. They also described that $Z(V_*(F_{2^m}(QD)_{16})) \cong C_2^{4n}$. We are interested in the structure of the center of unitary unit subgroup of group algebra $(F_{2^m}((M_{2^n+1} \times C_2) \times C_2))$.

2. NOTATIONS AND PRELIMINARIES

This section contains some definitions and results which are very important in our task.

**Definition 2.1.** Let $R$ be a associative commutative ring with 1, a circulant matrix over $R$ is a square $n \times n$ matrix of the form

$$c_i(r_1, r_2, ..., r_n) = \begin{pmatrix}
  r_1 & r_2 & r_3 & \cdots & r_n \\
  r_2 & r_3 & r_4 & \cdots & r_1 \\
  r_3 & r_4 & r_5 & \cdots & r_2 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  r_n & r_1 & r_2 & \cdots & r_{n-1}
\end{pmatrix}$$

where $r_i \in R$.

The sum and product of any two circulant matrices is again circulant matrix and is commutable. For further details related to circulant matrices see [4].

Let $G$ be a finite group and $G = \{m_1, m_2, ..., m_n\}$, be the fix listing of elements of $G$ then $MG$ is called matrix of $G$.

$$M(G) = \begin{pmatrix}
  m_1^{-1}m_1 & m_1^{-1}m_2 & m_1^{-1}m_3 & \cdots & m_1^{-1}m_n \\
  m_2^{-1}m_1 & m_2^{-1}m_2 & m_2^{-1}m_3 & \cdots & m_2^{-1}m_n \\
  m_3^{-1}m_1 & m_3^{-1}m_2 & m_3^{-1}m_3 & \cdots & m_3^{-1}m_n \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  m_n^{-1}m_1 & m_n^{-1}m_2 & m_n^{-1}m_3 & \cdots & m_n^{-1}m_n
\end{pmatrix}$$
Take any element of RG let say \( w \in RG, w = \sum_{t \in G} a_t t \) then RG matrix of \( w \) is defined as

\[
M(RG, w) = \begin{pmatrix}
    a_{t_1}^{-1} t_1 & a_{t_1}^{-1} t_2 & \cdots & a_{t_1}^{-1} t_n \\
    a_{t_2}^{-1} t_1 & a_{t_2}^{-1} t_2 & \cdots & a_{t_2}^{-1} t_n \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{t_n}^{-1} t_1 & a_{t_n}^{-1} t_2 & \cdots & a_{t_n}^{-1} t_n
\end{pmatrix}
\]

In [6] Hurley introduced the following theorem:

**Theorem 2.2.** For a given list of elements of group \( G \) which has order \( n \), there is a bijective ring homomorphism exist between \( RG \) and the \( \eta : b \mapsto M(RG, b) \)

Let \( M_{2^n+1} = \langle \psi, \lambda \rangle \lambda^{2^n} = 1, \lambda \psi = \psi^{2^{n+1}} \lambda \rangle \) be modular group of finite order \( 2^{n+1} \) and \( F_{2^n} M_{2^n+1} \) be group algebra having scalars from \( F_{2^n} \), finite field of characteristics \( 2 \). Take arbitrary \( v \in V, v = \sum_{a=0}^{2^n-1} t_a ((\psi^a, 1), 1) + \sum_{a=0}^{2^n-1} u_a ((\psi^a \lambda, 1), 1) + \sum_{a=0}^{2^n-1} v_a ((\psi^a \lambda, 1), 1) + \sum_{a=0}^{2^n-1} w_a ((\psi^a, 1), 1) + \sum_{a=0}^{2^n-1} \psi_a ((\psi^a \lambda, 1), 1) + \sum_{a=0}^{2^n-1} \lambda_a ((\psi^a \lambda, 1), 1) + \sum_{a=0}^{2^n-1} s_a ((\psi^a \lambda, 1), 1) \)

Then we have RG-matrix representation of \( v \)

\[
\sigma(v) = \begin{pmatrix}
    \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{2^n-1} \\
    \nu_0 & \nu_1 & \nu_2 & \cdots & \nu_{2^n-1} \\
    \psi_0 & \psi_1 & \psi_2 & \cdots & \psi_{2^n-1} \\
    \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{2^n-1} \\
    \sigma_0 & \sigma_1 & \sigma_2 & \cdots & \sigma_{2^n-1}
\end{pmatrix}
\]

where, \( \sigma_0 = \psi_0 \lambda \psi_1 \psi_2 \cdots \psi_{2^n-1} \), \( \sigma_1 = \psi_0 \mu_0 \psi_1 \mu_1 \cdots \psi_0 \mu_{2^n-1} \), \( \sigma_2 = \psi_0 \nu_0 \psi_1 \nu_1 \cdots \psi_0 \nu_{2^n-1} \), \( \sigma_3 = \psi_0 \lambda_0 \psi_1 \lambda_1 \cdots \psi_0 \lambda_{2^n-1} \), \( \sigma_4 = \psi_0 \sigma_0 \psi_1 \sigma_1 \cdots \psi_0 \sigma_{2^n-1} \), \( \sigma_5 = \psi_0 \mu_0 \nu_0 \nu_1 \cdots \psi_0 \mu_{2^n-1} \), \( \sigma_6 = \psi_0 \lambda_0 \mu_0 \lambda_1 \cdots \psi_0 \lambda_{2^n-1} \), \( \sigma_7 = \psi_0 \sigma_0 \lambda_0 \sigma_1 \cdots \psi_0 \sigma_{2^n-1} \)

**Theorem 2.3.** [5] Let \( A = \psi(t_1, t_2, \ldots, t_{2^m}) \), where \( t_i \in F_{2^m}, m \in N_0 \) and \( p \) denotes a prime. Then

\[
A^{pn} = \sum_{j=1}^{2^m} \nu_j^m I_{pn}
\]
3. Results

3.1. The Structure of $Z(V_*(F_{2k}M_{2n+1}))$.

Theorem 3.2. The center of $Z(V_*(F_{2k}M_{2n+1}))$ is isomorphic to $C_2^{(2n-2, 5-1)k}$, i.e.

$$Z(V_*(F_{2k}M_{2n+1})) \cong C_2^{(2n-2, 5-1)k}.$$

Proof. Let $\chi = \sum_{a=0}^{2^n-1} \lambda_{\bar{a}}\psi_{\bar{a}} + \sum_{a=0}^{2^n-1} \gamma_{\bar{a}}\psi_{\bar{a}}\bar{\lambda}$ be an element of $V_*(F_{2k}M_{2n+1})$ where $\sum_{a=0}^{2^n-1} \lambda_{\bar{a}} = 1$. Let $V = V_*(F_{2k}M_{2n+1})$ for simplicity. Consider the set $C_\omega \psi = \{v \in V : \nu v = v \nu \psi \}$. Now $\nu v = v \nu \psi = 0$ if and only if $\psi \{\sum_{\bar{a}=0}^{2^n-1} \lambda_{\bar{a}}\psi_{\bar{a}} + \sum_{\bar{a}=0}^{2^n-1} \gamma_{\bar{a}}\psi_{\bar{a}}\bar{\lambda} - \{\sum_{\bar{a}=0}^{2^n-1} \lambda_{\bar{a}}\psi_{\bar{a}} + \sum_{\bar{a}=0}^{2^n-1} \gamma_{\bar{a}}\psi_{\bar{a}}\bar{\lambda}\} = 0$ which is true if and only if

$$\gamma_0 = \gamma_{2n-1}, \gamma_1 = \gamma_{1+2n-1}, \ldots, \gamma_{2n-1-1} = \gamma_{2n-1}.$$ Therefore,

$$C_\omega \psi = \{\omega = \sum_{\bar{a}=0}^{2^n-1} \lambda_{\bar{a}}\psi_{\bar{a}} + \sum_{\bar{a}=0}^{2^n-1-1} \gamma_{\bar{a}}\{((\psi_{\bar{a}} + \psi_{\bar{a}}\bar{\lambda}) + \psi_{\bar{a}}\bar{\lambda})\} \}

Z(V) = \{\omega \in C_\psi \omega v = \nu \omega \nu v \in V\}$. Take arbitrary $v \in V v = \sum_{\bar{a}=0}^{2^n-1} t_{\bar{a}}((\psi_{\bar{a}} + \sum_{\bar{a}=0}^{2^n-1} u_{\bar{a}}((\psi_{\bar{a}} + \bar{\lambda}) + \psi_{\bar{a}}\bar{\lambda})\} \}

Now $\omega v = v \omega$ if and only if $\Gamma(\omega)\Gamma(v) = \Gamma(v)\Gamma(\omega) = 0$ This implies that

$$\left(\begin{array}{cc}
\lambda & \omega \\
\omega & \lambda'
\end{array}\right) \left(\begin{array}{ccc}
t_0 & t_1 & t_0 \\
t_1' & t_1 & t_0
\end{array}\right) = \left(\begin{array}{ccc}
t_0 & t_1 & t_0 \\
t_1' & t_1 & t_0
\end{array}\right) \left(\begin{array}{cc}
\lambda & \omega \\
\omega & \lambda'
\end{array}\right) = 0$$

which gives

$$\lambda_1 = \lambda_{1+2n-1}, \lambda_3 = \lambda_{3+2n-1}, \ldots, \lambda_{2n-1-1} = \lambda_{2n-1}.$$ Thus, we have centre as follows: $r_0 + r_1 \{\psi + \psi^{2n-1+1} + r_2 \{\psi^3 + \psi^{2n-1+3} + \ldots + r_{2n-1-1} \{\psi^{2n-1} + \psi^{2n-2} + r_2 \psi^{4} + \ldots + r_{2n-2} \psi^{2n-2} \} = 1 + r_2 + r_4 + \ldots + r_{2n-2}$. Now we prove that elements of center of $V$ are also elements of $V_*(F_{2k}M_{2n+1})$, for this consider an element $m$ from center of $V$, then

$$\Gamma(m) = \left(\begin{array}{cc}
\lambda & \omega \\
\omega & \lambda'
\end{array}\right)$$

where the above circulant matrices are defined below

$$j = circ(r_0, r_1, r_2, \ldots, r_{2n-1}, r_1, r_{2n-1+2}, \ldots, r_{2n-1-1})$$

and $\omega = circ(\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{2n-1-1}, \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{2n-1})$.

For unitary element: $m^* = m^{-1}$ iff $\Gamma(m^*) = \Gamma(m^{-1})$ iff $(\Gamma(m))^{-1}$ iff $(\Gamma(m))^{T} = \Gamma(m)$ iff $(\Gamma(m))^{T} = I$. 

Saima Parveen, Asia Inam and Farwa Idrees
Consider
\[
(\Gamma(m))(\Gamma(m))^T = \begin{pmatrix} j & \omega \\ \omega & j \end{pmatrix} \begin{pmatrix} j & \omega \\ \omega & j \end{pmatrix}
\]
\[
= \alpha I_8
\]
because by Theorem 2.3, we have, \(\alpha = j^2 + \omega^2 = I\).

So \(Z(V) \subset V_+(F_{2^k}M_{2^n+1})\) where \(F\) is field having characteristic 2 so we have, Hence \(C_2^{(2^n-2,5^2-1)k} \cong Z(V_+(F_{2^k}M_{2^n+1}))\). \(\square\)

**3.3. The Structure of \(Z(V_+(F_{2^k}(M_{2^n+1} \times C_2)))\).**

**Theorem 3.4.** The center of \(Z(V_+(F_{2^k}(M_{2^n+1} \times C_2)))\) is isomorphic to \(C_2^{(2^n-2,5^2-1)k}\), i.e.

\[
C_2^{(2^n-2,5^2-1)k} \cong Z(V_+(F_{2^k}(M_{2^n+1} \times C_2))).
\]

**Proof:** \(\chi = \sum_{a=0}^{2^n-1} \lambda_a(\psi^a, 1) + \sum_{a=0}^{2^n-1} \gamma_a(\psi^a, \lambda, 1) + \sum_{a=0}^{2^n-1} e_a(\psi^a, \lambda, t) + \sum_{a=0}^{2^n-1} f_a(\psi^a, \lambda, t)\)

be an element of \(V_+(F_{2^k}(M_{2^n+1} \times C_2))\) where \(\sum_{a=0}^{2^n-1} \lambda_a = 1\) and let \(V' = V_+(F_{2^k}(M_{2^n+1} \times C_2))\) for simplicity. Consider the set \(C_0(\psi, 1) = \{v \in V : (\psi, 1)v = v(\psi, 1)\}\). Now \((\psi, 1)v = v(\psi, 1) = 0\) if and only if \((\psi, 1)\{\sum_{a=0}^{2^n-1} \lambda_a(\psi^a, 1) + \sum_{a=0}^{2^n-1} \gamma_a(\psi^a, \lambda, 1) + \sum_{a=0}^{2^n-1} e_a(\psi^a, \lambda, t) + \sum_{a=0}^{2^n-1} f_a(\psi^a, \lambda, t)\} = 0\) if and only if \(\gamma_0 = \gamma_2\) and \(\gamma_1 = \gamma_{1+2}\) and \(\gamma_{2-1} = \gamma_{2n-1}\) and \(f_0 = f_2\)

where \(z = 2^n-1\). Therefore, \(C_0(\psi, 1) = \{\omega = \sum_{a=0}^{2^n-1} \lambda_a(\psi^a, 1) + \sum_{a=0}^{2^n-1} e_a(\psi^a, \lambda, t) + \sum_{a=0}^{2^n-1} \gamma_a(\psi^a, \lambda, 1) + \sum_{a=0}^{2^n-1} f_a(\psi^a, \lambda, t)\} = \{a \in C_0(\psi, 1) | (\psi, 1)v = v(\psi, 1)\}\)

\[
Z(v) = \{\omega \in C_0(\psi, 1) | \omega v = v \omega \forall v \in V\}
\]

Take arbitrary \(v \in V\) \(\sum_{a=0}^{2^n-1} t_a(\psi^a, 1) + \sum_{a=0}^{2^n-1} u_a(\psi^a, \lambda, 1) + \sum_{a=0}^{2^n-1} v_a(\psi^a, \lambda, t, 1) + \sum_{a=0}^{2^n-1} w_a(\psi^a, \lambda, t, 1)\)

\(\omega v = v \omega\) if and only if \(\Gamma(\omega)\Gamma(v) = \Gamma(v)\Gamma(\omega) = 0\)

This implies that
Thus, we have centre as follows:

\[
\begin{pmatrix}
\mu_0 & \mu_1 & \mu_1' & \mu_0' \\
\mu_0' & \mu_1' & \mu_1 & \mu_0 \\
\mu_0 & \mu_1 & \mu_1' & \mu_0' \\
\mu_0' & \mu_1' & \mu_1 & \mu_0 \\
\end{pmatrix} - \begin{pmatrix}
\mu_0 & \mu_1 & \mu_1' & \mu_0' \\
\mu_0' & \mu_1' & \mu_1 & \mu_0 \\
\mu_0 & \mu_1 & \mu_1' & \mu_0' \\
\mu_0' & \mu_1' & \mu_1 & \mu_0 \\
\end{pmatrix} = 0
\]

\[
j_0 = \text{circ}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{2n-1}), j_0' = \text{circ}(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{2n-1})
\]

\[
\omega_0 = \text{circ}(s_0, s_1, s_2, \ldots, s_z, s_{z+1}, \ldots, s_{2n-1}), j_1 = \text{circ}(j_0, j_1, j_2, \ldots, j_{2n-1})
\]

\[
\omega_1 = \text{circ}(f_0, f_1, f_2, \ldots, f_{2n-1})
\]

Therefore,

\[
\Gamma(\omega)\Gamma(v) - \Gamma(v)\Gamma(\omega) = 0
\]

Then this implies that

\[
\lambda_1 = \lambda_{1+z}, \lambda_3 = \lambda_{3+z}, \ldots, \lambda_{2n-1} = \lambda_{2n-1}, e_1 = e_{1+z}, e_3 = e_{3+z}, \ldots, e_{2n-1} = e_{2n-1}
\]

Thus, we have centre as follows:

\[
Z(V) = r_0((1, 1) + r_1((\psi^1, 1) + (\psi^{2+1}, 1)) + r_2((\psi^2, 1) + (\psi^{3+1}, 1)) + \ldots + r_{z-1}((\psi^{z-1}, 1) + (\psi^{2z-1}, 1)) + r_z((\psi^z, 1) + \ldots + r_{2n-2}(\psi^{2n-2}, 1) + s_0(1, 1) + s_1((\psi, 1) + (\psi^{1+z}, 1)) + s_2((\psi^2, 1) + (\psi^{2+z}, 1)) + \ldots + + s_{2n-2}(\psi^{2n-2}, 1) + \sum_{i=0}^{z-1} f_i((\psi^{i+1}, 1) + (\psi^{i+z}, 1)) + \sum_{i=0}^{2n-2} f_i((\psi^{i+1}, 1) + (\psi^{i+z}, 1))
\]

where, \( r_0 = 1 + r_2 + r_4 + \ldots + r_{2n-2} + s_0 + s_2 + s_4 + \ldots + s_{2n-2} \). Now we prove that elements of center of \( V \) are also elements of \( V_s(F_{2^b}(M_{2n+1} \times C_2)) \), for this consider an element \( m \) from center of \( V \), then

\[
\Gamma(m) = \begin{pmatrix}
\mu_0 & \mu_1 & \mu_1' & \mu_0' \\
\mu_0' & \mu_1' & \mu_1 & \mu_0 \\
\mu_0 & \mu_1 & \mu_1' & \mu_0' \\
\mu_0' & \mu_1' & \mu_1 & \mu_0 \\
\end{pmatrix}
\]

\[
j_0 = \text{circ}(r_0, r_1, r_2, \ldots, r_z, r_{z+1}, \ldots, r_{2n-1}), j_1 = \text{circ}(s_0, s_1, s_2, \ldots, s_z, s_{z+1}, s_{z+2}, \ldots, s_{2n-1}),
\]

\[
\omega_0 = \text{circ}(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_{z-1}, \gamma_0, \gamma_1, \gamma_2, \gamma_{z-1})
\]

For unitary element, \( m^* = m^{-1} \) iff

\[
\Gamma(m^*) = \Gamma(m)^{-1} \iff (\Gamma(m))^T = (\Gamma(m))^{-1}
\]

Hence

\[
C_{2^2}^{(2^{n-2} - 5)} = Z(V_s(F_{2^b}(M_{2n+1} \times C_2))).
\]
3.5. The center of $Z(V_c(F_{2^n}(⟨M_{2^{n+1}} × C_2⟩ × C_2)))$. In this section we describe the center of $Z(V_c(F_{2^n}(⟨M_{2^{n+1}} × C_2⟩ × C_2)))$.

**Lemma 3.6.** The center of $Z(V_c(F_{2^n}(⟨M_{2^{n+1}} × C_2⟩ × C_2)))$ have elements of the form 
$$\begin{align*}
&\{r_0((1, 1), 1) + r_1(\langle ψ, 1, 1 \rangle + (\langle ψ^{+1}, 1, 1 \rangle) + r_3(\langle ψ^3, 1, 1 \rangle + (\langle ψ^{+3}, 1, 1 \rangle) + \ldots + r_{2^n-1}(\langle ψ^{2^n-1}, 1, 1 \rangle) + (\langle ψ^{2^n-1}, 1, 1 \rangle) + r_2(\langle ψ^2, 1, 1 \rangle) + r_4(\langle ψ^4, 1, 1 \rangle) + \ldots + r_{2^n-2}(\langle ψ^{2^n-2}, 1, 1 \rangle) + \ldots + r_{2^n-1}(\langle ψ^{2^n-1}, 1, 1 \rangle) + (\langle ψ^{2^n-1}, 1, 1 \rangle)\} + \ldots +
&\{s_0((1, 1), 1) + s_1(\langle ψ, 1, 1 \rangle + (\langle ψ^{+1}, 1, 1 \rangle) + s_3(\langle ψ^3, 1, 1 \rangle + (\langle ψ^{+3}, 1, 1 \rangle) + \ldots + s_{2^n-1}(\langle ψ^{2^n-1}, 1, 1 \rangle) + (\langle ψ^{2^n-1}, 1, 1 \rangle)\} + \ldots +
&\{t_0((1, 1), 1) + t_1(\langle ψ, 1, 1 \rangle + (\langle ψ^{+1}, 1, 1 \rangle) + t_3(\langle ψ^3, 1, 1 \rangle + (\langle ψ^{+3}, 1, 1 \rangle) + \ldots +
&t_{2^n-1}(\langle ψ^{2^n-1}, 1, 1 \rangle) + (\langle ψ^{2^n-1}, 1, 1 \rangle)\}) + \ldots +
&\{u_0((1, 1), 1) + u_1(\langle ψ, 1, 1 \rangle + (\langle ψ^{+1}, 1, 1 \rangle) + u_3(\langle ψ^3, 1, 1 \rangle + (\langle ψ^{+3}, 1, 1 \rangle) + \ldots +
&u_{2^n-1}(\langle ψ^{2^n-1}, 1, 1 \rangle) + (\langle ψ^{2^n-1}, 1, 1 \rangle)\}) + \ldots +
&\{v_0((1, 1), 1) + v_1(\langle ψ, 1, 1 \rangle + (\langle ψ^{+1}, 1, 1 \rangle) + v_3(\langle ψ^3, 1, 1 \rangle + (\langle ψ^{+3}, 1, 1 \rangle) + \ldots +
&v_{2^n-1}(\langle ψ^{2^n-1}, 1, 1 \rangle) + (\langle ψ^{2^n-1}, 1, 1 \rangle)\}).
\end{align*}$$

where, $r_0 = 1 + r_2 + r_4 + \ldots + r_{2^n-2} + s_0 + s_2 + s_4 + \ldots + s_{2^n-2} + t_0 + t_2 + t_4 + \ldots + t_{2^n-2} + u_0 + u_2 + u_4 + \ldots + u_{2^n-2}$.

**Proof.**

$$\begin{align*}
χ = &\sum_{a=0}^{2^n-1} λ_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} γ_a(\langle ψ^a, 1, 1 \rangle) \\
+ &\sum_{a=0}^{2^n-1} e_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} g_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} h_a(\langle ψ^a, 1, 1 \rangle) + \ldots \\
&\sum_{a=0}^{2^n-1} J_a(\langle ψ^a, 1, 1 \rangle) + \ldots \\
&\sum_{a=0}^{2^n-1} J_a(\langle ψ^a, 1, 1 \rangle) = 1
\end{align*}$$

and $V' = V_c(F_{2^n}(⟨M_{2^{n+1}} × C_2⟩ × C_2))$ for simplicity, and $n = 2^n - 1$. Consider the set $C_v(\langle ψ, 1, 1 \rangle) = \{v \in V : \langle ψ, 1, 1 \rangle v = v(\langle ψ, 1, 1 \rangle)\}$. Now $\langle ψ, 1, 1 \rangle v = v(\langle ψ, 1, 1 \rangle)$ is 0 if and only if
$$\begin{align*}
&\langle ψ, 1, 1 \rangle \big(\sum_{a=0}^{2^n-1} λ_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} γ_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} e_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} g_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} h_a(\langle ψ^a, 1, 1 \rangle) + \ldots
&\sum_{a=0}^{2^n-1} J_a(\langle ψ^a, 1, 1 \rangle) + \ldots
&\sum_{a=0}^{2^n-1} J_a(\langle ψ^a, 1, 1 \rangle) = 1
\end{align*}$$

which is true if and only if $γ_0 = γ_1 = γ_{1+1} = \ldots = γ_{2^n-1} = γ_{2^n-1} = 0$, $f_1 = f_{1+1} = \ldots = f_{2^n-1} = f_{2^n-1}$, $h_0 = h_1 = h_{1+1} = \ldots = h_{2^n-1} = h_{2^n-1}$, and $J_0 = J_1 = J_{1+1} = \ldots = J_{2^n-1} = J_{2^n-1}$ where $2^n-1 = 1$. Therefore,
$$\begin{align*}
C_v(\langle ψ, 1, 1 \rangle) = \{ω = \sum_{a=0}^{2^n-1} λ_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} γ_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} e_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} g_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} h_a(\langle ψ^a, 1, 1 \rangle) + \ldots
&\sum_{a=0}^{2^n-1} J_a(\langle ψ^a, 1, 1 \rangle) + \ldots
&\sum_{a=0}^{2^n-1} J_a(\langle ψ^a, 1, 1 \rangle) = 1
\end{align*}$$

Since center of "V" is a subset of centralizer, therefore we have
$$Z(\langle ψ, 1, 1 \rangle) = \{ω ∈ C_v(\langle ψ, 1, 1 \rangle)|ωv = ωv \forall v \in V\}$$

Take arbitrary $v ∈ V$ as follows
$$v = \sum_{a=0}^{2^n-1} t_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} u_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} v_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} w_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} ω_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} θ_a(\langle ψ^a, 1, 1 \rangle) + \sum_{a=0}^{2^n-1} σ_a(\langle ψ^a, 1, 1 \rangle).$$
Now $\omega \nu = \nu \omega$ if and only if $\Gamma(\omega)\Gamma(\nu) - \Gamma(\nu)\Gamma(\omega) = 0$

This implies that

$$
\begin{pmatrix}
  j_0 & j_0 & j_1 & j_2 & j_3 & j_3 & j_3 & j_3 \\
j_0 & j_0 & j_1 & j_2 & j_3 & j_3 & j_3 & j_3 \\
j_1 & j_1 & j_0 & j_0 & j_3 & j_2 & j_2 & j_2 \\
j_2 & j_2 & j_3 & j_3 & j_0 & j_1 & j_1 & j_1 \\
j_3 & j_3 & j_2 & j_2 & j_1 & j_0 & j_0 & j_0 \\
j_3 & j_3 & j_2 & j_2 & j_1 & j_0 & j_0 & j_0 \\
j_3 & j_3 & j_2 & j_2 & j_1 & j_0 & j_0 & j_0 \\
j_3 & j_3 & j_2 & j_2 & j_1 & j_0 & j_0 & j_0 \\
\end{pmatrix}
- \begin{pmatrix}
  \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 \\
  \mu_1 & \mu_1 & \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_0 \\
  \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_0 \\
  \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_0 \\
  \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_0 \\
  \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_0 \\
  \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_0 \\
  \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_1 & \mu_0 & \mu_0 & \mu_0 \\
\end{pmatrix} = 0
$$

Where the matrices are defined below.

$$
\begin{align*}
  u_0 &= \text{circ}(t_0, t_1, t_2, t_3, \ldots, t_{2n-1}), \\
  u_1 &= \text{circ}(u_0, u_1, u_2, u_3, \ldots, u_{2n-1}), \\
  \nu_0 &= \text{circ}(\omega_0, \omega_1, \omega_2, \ldots, \omega_{2n-1}), \\
  \nu_1 &= \text{circ}(\beta_0, \beta_1, \beta_2, \ldots, \beta_{2n-1}), \\
  \nu_2 &= \text{circ}(\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{2n-1}), \\
  \nu_3 &= \text{circ}(\delta_0, \delta_1, \delta_2, \ldots, \delta_{2n-1}).
\end{align*}
$$

Then this implies that

$$
\begin{align*}
  \lambda_1 &= \lambda_{1+2}, \\
  \lambda_3 &= \lambda_{3+2}, \\
  \lambda_{2-1} &= \lambda_{2-1}, \\
  \lambda_{2+1} &= \lambda_{2+1}, \\
  g_1 &= g_{1+2}, \\
  g_3 &= g_{3+2}, \\
  g_{2-1} &= g_{2-1}, \\
  I_1 &= I_{1+2}, \\
  I_3 &= I_{3+2}, \\
  I_{2-1} &= I_{2-1}
\end{align*}
$$

which gives the result.

3.7. The Structure of $Z(V_*(F_{2n}((M_{2n+1} \times C_2) \times C_2)))$.

**Lemma 3.8.** $Z(V)$ is a unitary unit subgroup.
Proof. Consider an arbitrary element \( m \) from center of \( V \), then

\[
\Gamma(m) = \begin{pmatrix}
\ell_0 & \ell_1 & \ell_2 & \ell_3 \\
0 & \ell_0 & \ell_1 & \ell_2 \\
\ell_1 & 0 & \ell_0 & \ell_1 \\
\ell_2 & \ell_3 & \ell_0 & \ell_1 \\
\ell_3 & \ell_2 & \ell_1 & \ell_0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

where the above circulant matrices are defined below

\[
\begin{align*}
\ell_0 &= \text{circ}(r_0, r_1, r_2, \ldots, r_z), \quad \ell_1 = \text{circ}(s_0, s_1, s_2, \ldots, s_z), \\
\ell_2 &= \text{circ}(t_0, t_1, t_2, \ldots, t_z), \quad \ell_3 = \text{circ}(u_0, u_1, u_2, \ldots, u_z), \\
\end{align*}
\]

and \( \ell_3 = \text{circ}(J_0, J_1, J_2, \ldots, J_{z-1}, J_0, J_1, J_2, \ldots, J_{z-1}) \) For unitary element; \( m^* = m^{-1} \)

iff \( \Gamma(m^*) = \Gamma(m)^{-1} \) iff \( (\Gamma(m))^{T} = (\Gamma(m))^{-1} \) iff

\[
(\Gamma(m))^{T} \Gamma(m) = I.
\]

Consider

\[
(\Gamma(m))^{T} (\Gamma(m)) = \begin{pmatrix}
\ell_0 & \ell_0 & \ell_0 & \ell_0 & \ell_0 & \ell_0 \\
0 & \ell_0 & \ell_0 & \ell_0 & \ell_0 & \ell_0 \\
\ell_1 & \ell_1 & \ell_1 & \ell_1 & \ell_1 & \ell_1 \\
\ell_2 & \ell_2 & \ell_2 & \ell_2 & \ell_2 & \ell_2 \\
\ell_3 & \ell_3 & \ell_3 & \ell_3 & \ell_3 & \ell_3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha
\end{pmatrix}
\]

where, \( \alpha = \ell_0^2 + \ell_1^2 + \ell_2^2 + \ell_3^2 \) by using Theorem 2.3, we have

\( \alpha = \ell_0^2 + \ell_1^2 + \ell_2^2 + \ell_3^2 \)
Let us denote again $r_u$ where,

$\sum V_z(V) = z(V, F_{2^k}((M_{2^{n+1}} \times C_2) \times C_2))$. Thus $Z(V) \subset V_n(F_{2^k}((M_{2^{n+1}} \times C_2) \times C_2))$. 

Theorem 3.9. $Z(V_n(F_{2^k}((M_{2^{n+1}} \times C_2) \times C_2))) \cong C_{2^k}(2^{n+1} + 2^n - 3)$, where $Z(V_n(F_{2^k}((M_{2^{n+1}} \times C_2) \times C_2)))$. is center of unitary unit subgroup $V_n(F_{2^k}((M_{2^{n+1}} \times C_2) \times C_2))$. 

\[\begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & I & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & I & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & I & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & I \\
\end{bmatrix}\]

Hence $m \in Z(V)$ is an element of $V_n(F_{2^k}((M_{2^{n+1}} \times C_2) \times C_2))$. Thus $Z(V) \subset V_n(F_{2^k}((M_{2^{n+1}} \times C_2) \times C_2))$. 

\[\begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & I & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & I & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & I & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & I \\
\end{bmatrix}\]

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REFERENCES


