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Implementation of Numerical Integration Simpson 3/8 Rule to Develop a Numerical Simpson Iterative Method for Solving Non-Linear Equations

Umair Khalid Qureshi[1]

[1] Department of Business Administration,
Shaheed Benazir Bhutto University, Sanghar, Sindh, Pakistan.
[1] Email: umair.khalidsng_@sbbusba.edu.pk (Corresponding Author)

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Abstract: This research paper has developed a numerical iterative method by using Simpson 3/8 rule for solving non-linear equations, which equations are studied in different sciences and engineering fields. The developed technique has quadratic converge. This paper reflects the idea and more better results from the work of the authors of [7-8]. Examples are given to show that the proposed iterative method is better than compared methods. Our developed method is compared with the Newton Raphson method and the Trapezoidal method. C++/MATLAB was used to compute the numerical results. It can be observed from the results that the simpson iterative method is better than the Newton Raphson method, and the Trapezoidal method in terms of iteration and accuracy perception.

AMS (MOS) Subject Classification Codes: —
Key Words: numerical integration, quadrature rule, Simpson 3/8th rule, second order methods, application problems, results.

1. INTRODUCTION

Integration has numerous applications in the field of science and engineering, and it is an influential tool to estimating the area under the curve and convert derivative into the integral function given as

$$\int f(x)dx \tag{1. 1}$$

Indeed, there are situations where analytical techniques have failed to obtain the required results; in such cases, we move towards numerical techniques. A numerical technique for solving the integration is known as quadrature rule. Quadrature rule is a procedure for determining the area under the curve of f(x) using Newton-Cotes method [1-2]. Newton cotes method was invented by Newton and Cotes and it is also knowns as quadrature rules.

Quadrature rules have five elementary rules for resolving numerical integration i.e. trapezoidal rule, simpson 1/3rd rule, simpson 3/8th rule, booles rule and weddles rule [3-4]. Later, researchers are trying to establish various methods for resolving nonlinear equations by quadrature type concept [5-7]. Similarly, this paper has suggested a numerical iteration method for solving nonlinear equations. The idea of the proposed iteration method comes from [7] and [8] given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n + f(x_n))} \tag{1. 2}$$

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 2f'(x_n + f(x_n)) + f'(x_n + 2f(x_n))} \tag{1. 3}$$

Both ((1.2) and (1. 3)) iteration methods are based on the trapezoidal quadrature rule and numerical techniques. In the same way, the proposed numerical iteration method depends on the simpson 3/8th rule and numerical techniques. The developed numerical iteration method was assessed against the newton raphson method and the trapezoidal method [8-9]. The numerical results demonstrate that the proposed method is better than the existing second order iterated methods in terms of the accuracy of the iterations in solving non-linear equations.

2. PROPOSED METHODOLOGY

This section established a numerical method for solving the nonlinear equation of the form f(x)=0 where x is real, using the simpson 3/8th rule. Let the simpson 3/8th rule for a function f(x)=0 and n = 3, be given by

$$\int_{x_o}^{x} f(x)dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] \tag{2. 4}$$

Assuming that the derivative $f'(x)$ is instead integrated, then:

$$\int_{x_o}^{x} f'(x)dx = \frac{3h}{8}[f'(x_0) + 3f'(x_1) + 3f'(x_2) + f'(x_3)] \tag{2. 5}$$

Therefore, the above integration can be written as:

$$f(x) = f(x_o) + \frac{3h}{8}[f'(x_o) + 3f'(x_1) + 3f'(x_2) + f'(x_3)] \tag{2. 6}$$

Since n = 3, then h by simpson rule, can be written as:

$$h = \frac{x - x_o}{3} \tag{2. 7}$$

h substituted in Eq. (2.1) gives:

$$f(x) = f(x_o) + \frac{x - x_o}{8}[f'(x_o) + 3f'(x_1) + 3f'(x_2) + f'(x_3)] \tag{2. 8}$$

To solve the equation f(x)=0, we are solving Eq. (2.2) for x, obtaining:

$$x = x_o - \frac{8f(x_o)}{f'(x_o) + 3f'(x_1) + 3f'(x_2) + f'(x_3)} \tag{2. 9}$$

The basic numerical technique is that $x_1 = x_o + h$, $x_2 = x_o + 2h$, $x_3 = x_o + 3h$, which is used in Eq. 2.9, to get:

$$x = x_o - \frac{8f(x_o)}{f'(x_o) + 3f'(x_o + h) + 3f'(x_o + 2h) + f'(x_o + 3h)} \tag{2. 10}$$

From reference [12], h=f(x); using this in Eq. 2.10, we have:

$$x = x_o - \frac{8f(x_o)}{f'(x_o) + 3f'(x_o + f(x_o)) + 3f'(x_o + 2f(x_o)) + f'(x_o + 3f(x_o))} \tag{2. 11}$$

Where xo is an initial guess, and which is near to the root 'x' of f(x)=0. Therefore, in general Eq.2.11, gives the iteration scheme in Eq.(2.12):

$$x_{n+1} = x_n - \frac{8f(x_n)}{f'(x_n) + 3f'(x_n + f(x_n)) + 3f'(x_n + 2f(x_n)) + f'(x_n + 3f(x_n))} \tag{2. 12}$$

Hence, Eq. 2.12 is a numerical iterated method for solving nonlinear equations f(x)=0.

3. CONVERGENCE ANALYSIS

This section gives the key results of this paper. It can be observed in this section that the proposed method has quadratic convergence.

Proof

Using Taylor series, we expand $f(x_n)$, $f'(x_n)$, $f'(x_n + f(x_n))$, $f'(x_n + 2f(x_n))$ and $f'(x_n + 3f(x_n))$ only up to second order terms about a point 'a', leading to the following:

$$f(x_n) = f'(a)(e_n + ce_n^2) \tag{3. 13}$$

Or

$$f'(x_n) = f'(a)(1 + 2ce_n) \tag{3. 14}$$

Or

$$f'(x_n + f(x_n)) = f'(a)(1 + f'(x_n) + 2(e_n + f(x_n))(1 + f'(x_n))c) \tag{3. 15}$$

Eq. 3.13 and Eq. 3.14 substituted in Eq. 3.15, gives:

$$f'(x_n + f(x_n)) = f'(a)(1 + f'(a) + 2ce_n + 6ce_n f'(a) + 2ce_n f'^2(a)) \tag{3. 16}$$

Now,

$$f'(x_n + 2f(x_n)) = f'(a)[(e_n + 2f(x_n)) + c(e_n + 2f(x_n))^2] \tag{3. 17}$$

Taking derivative,

$$f'(x_n+2f(x_n)) = f'(a)[(1+2f'(x_n)) + 2c(e_n+2f(x_n))(1+2f'(x_n))] \quad (3.\ 18)$$

Substituting Eq. 3.13 and Eq. 3.14 in Eq. 3.18, we get Eq. 3.19:

$$f'(x_n+2f(x_n)) = f'(a)(1+2f'(a)+2ce_n+12ce_nf'(a)+8ce_nf'^2(a)) \quad (3.\ 19)$$

Finaly,

$$f'(x_n+3f(x_n)) = f'(a)[(e_n+3f(x_n)) + c(e_n+3f(x_n))^2] \quad (3.\ 20)$$

Taking derivative,

$$f'(x_n+3f(x_n)) = f'(a)[(1+3f'(x_n)) + 2c(e_n+3f(x_n))(1+3f'(x_n))] \quad (3.\ 21)$$

Substituting Eq. 3.13 and Eq. 3.14 in Eq. 3.21, we get Eq. 3.22:

$$f'(x_n+3f(x_n)) = f'(a)[1+3f'(a)+2ce_n(1+6f'(a))] \quad (3.\ 22)$$

Note $c = \frac{f'(a)}{2f'(a)}$

From Eq. 3.14, Eq. 3.15, Eq. 3.16, Eq.3.19 and Eq.3.22, we have:

$$f'(x_n)+3f'(x_n+f(x_n))+3f'(x_n+2f(x_n))+f'(x_n+3f(x_n)) = 8f'(a)(1+4/3f'(a)+ce_n(2+33/4f'(a)) \quad (3.\ 23)$$

Now using Eq. 3.13 and Eq. 3.23 in the developed method, we get

$$e_{n+1} = e_n - \frac{8e_nf'(a)(1+ce_n)}{8f'(a)(1+4/3f'(a)+ce_n(2+33/4f'(a))} \quad (3.\ 24)$$

or

$$e_{n+1} = e_n - e_n(1+ce_n)(1+4/3f'(a)+ce_n(2+33/4f'(a))^{-1} \quad (3.\ 25)$$

or

$$e_{n+1} = e_n - e_n(1+ce_n)(1-4/3f'(a)-ce_n(2+33/4f'(a)) \quad (3.\ 26)$$

To simplify eq.(3.26), we get

$$e_{n+1} = -4/3e_nf'(a) + ce_n^2(1+37/3f'(a)) \quad (3.\ 27)$$

Finally, f(x)=0 used in Eq. 3.1 then the result put in Eq. 3.27, yields Eq. 3.28, we get

$$e_{n+1} = -4/3e_n^2f''(a) + ce_n^2(1+37/3f'(a)) \quad (3.\ 28)$$

or

$$e_{n+1} = e_n^2[-4/3f''(a) + c(1+37/3f'(a))] \quad (3.\ 29)$$

Hence, Eq. 3.29 confirms the assertion that the proposed Simpson Iteration method converges quadratically.

4. NUMERICAL RESULTS

In this part, the developed Simpson iterated method is comparing with Newton Raphson Method and Trapezoidal method. The results of proposed numerical method are examined by C++ with stopping criteria in computer programming as:

$$|x_{n+1} - x_n| < \epsilon \; where \; \epsilon > 10^{10}, \quad (4.\ 30)$$

Furthermore, the proposed Simpson iteration method is applied with five physical problems, such as

$a)\ sin^2x - x^2 + 1 = 0$
$b)\ 2x - lnx - 7 = 0$
$c)\ x^2 - e^x = 0$
$d)\ 2x^2 - 5x - 2 = 0$
$e)\ e^{-x} - cosx = 0$

C++/MATLAB is used to compute the numerical results. The results obtained from the proposed iterative method were compared with not only the Newton Raphson method but also Trapezoidal method. The results are presented in Table 1

Table 1: Results obtained by the proposed, Newton Raphson, and Trapezoidal methods

Functions	Initial guess	Method	Iterations	Root	AE
$sin^2x - x^2 + 1 = 0$	$x_0 = 1$	Newton Raphson Method	7		2.22045e-016
		Trapezoidal method	6	1.40449	2.22045e-016
		Simpson iterated method	6		2.22045e-016
$x^2 - e^x = 0$	$x_0 = 2$	Newton Raphson Method	6		1.01070e-010
		Trapezoidal method	5	0.703467	1.39852e-008
		Simpson iterated method	5		2.29049e-008
$e^{-x} - cosx = 0$	$x_0 = 4$	Newton Raphson Method	4		1.23767e-008
		Trapezoidal method	4	4.72129	1.89182e-013
		Simpson iterated method	4		6.19140e-011
$2x^2 - 5x - 2 = 0$	$x_0 = 0$	Newton Raphson Method	6		5.55112e-017
		Trapezoidal method	6	0.35079	5.97350e-010
		Simpson iterated method	6		4.99600e-016
$2x - lnx - 7 = 0$	$x_0 = 6$	Newton Raphson Method	5		8.88178e-016
		Trapezoidal method	8	4.219906	8.88178e-016
		Simpson iterated method	5		8.88178e-016

5. CONCLUSION

This article has proposed a Simpson iterative method for solving non-linear equations. The proposed iterative method is derived from the Simpson 3/8 quadrature rule. It has been proved that the proposed method has quadratic convergence. Furthermore. It can be observed from the results that the proposed method is better in terms of accuracy and number of iterations as compared to Newton Raphson method and the Trapezoidal method but proposed new method in each iteration is a little expansive than Newton-Raphson method.

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REFERENCES

[1] S.Akram and Q. U. Ann., *Newton Raphson Method,* International Journal of Scientific and Engineering Research **6**, (2015) 456-462.

[2] A., A. Khatri, A. Shaikh and K. A. Abro, *Closed Newton Cotes Quadrature Rules with Derivatives,* Mathematical Theory and Modeling **9**, No. 5, (2019).

[3] A. A. Mallah, A. A. Shaikh, S. Qureshi, *Trapezoidal Second Order Iterated Method for Solving Nonlinear Problems,* University of Sindh Journal of Information and Communication Technology (USJICT), **2**, No. 2, (2018).

[4] M. A. Perhiyar, S. F. Shah and A. A. Shaikh, *Modified Trapezoidal Rule Based Different Averages for Numerical Integration,* Mathematical Theory and Modeling **9**, No. 9, (2019).

[5] U. K. Qureshi, A New Accelerated Third-Order Two-Step Iterative Method for Solving Nonlinear Equations, Mathematical Theory and Modeling, **8**, No. 5, (2018).

[6] U. K. Qureshi, Z. A. Kalhoro, A. A. Shaikh and A. R. Nagraj, *Modied Bracketing Method for Solving Nonlinear Problems with Second Order of Convergence,* Punjab University Journal of Mathematics **51**, No. 3, (2018), pp. 145-151.

[7] U. K. Qureshi, I. A. Bozdar, A. Pirzada, and M. B. Arain, *Quadrature Rule Based Iterative Method for the Solution of Non-Linear Equations,* Proceedings of the Pakistan Academy of Sciences, A. Physical and Computational Sciences **56**, No. 1, (2019), pp. 39-43.

[8] K. Rajput, A. A. Shaikh and S. Qureshi, *Comparison of Proposed and Existing Fourth Order Schemes for Solving Non-linear Equations,* Asian Research Journal of Mathematics, **15**, No. 2, (2019)1-7.

[9] M., M. Shaikh, S. Chandio and A. Soomro, *A Modified Four-point Closed Mid-point Derivative Based Quadrature Rule for Numerical Integration,* Sindh University Research Journal-SURJ (Science Series) **48**, No. 2 (2016).

[10] Z. Weijing and H. Li, *Midpoint Derivative-Based Closed Newton-Cotes Quadrature,* Hindawi Publishing Corporation Abstract and Applied Analysis**2013** (2013).