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Coincidence Point Results in Ordered Metric Spaces and its Application

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Abstract.: In this paper, we establish some coincidence point theorems of nonlinear rational contractions in a complete partially ordered metric space. Some consequences of the results are presented in terms of integral contractions and few examples are illustrated to support the results. As an application, we proved the existence of a unique solution to an integral equation.

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1. Introduction

In nonlinear analysis, the classical Banach contraction principle plays a vital role to acquire the unique solution of many existing results. It is very important and popular tool in various disciplines of applied mathematical analysis and scientific applications. In metric fixed point theory, the Banach contraction principle [7] has been improved in different directions with respected to the underlying contraction condition and also by imposing some weaker conditions on the space, some of which were in [11, 12, 13, 14, 17, 18, 24, 33, 34, 36] and the references therein.

Later, it has been generalized and improved in various directions. First the existence of fixed point for the mapping in partial order sets has been studied by Ran and Reurings [25] and applied their results to matrix equations. These results of [25] were again generalized and extended in partially ordered sets by Nieto et al. [21, 22] and provided some applications to differential equations. While a series of research papers have been done by several authors from more generalizations of a metric space, some of such generalizations are from rectangular metric spaces, cone metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, etc. Prominent works of the existence and uniqueness of the results on fixed point, coincidence point and common fixed point for the mappings in various spaces, can be found from [1, 2, 3, 4, 5, 6, 8, 9, 10, 15, 16, 19, 20, 23, 35, 37], which generate natural interest to establish usable fixed point theorems. Recently, Seshagiri Rao et al. [26, 27, 28, 29, 30, 31, 32] have explored some fixed point and coupled fixed point results in partially ordered metric spaces for monotone mappings satisfying rational type contractions.

The aim of this paper is to prove some coincidence point and common fixed point results in the frame work of a complete partially ordered metric space for a pair of self-mappings satisfying a generalized rational type contractive condition. Examples of the results, some consequences in terms of integral contractions are discussed. As an application, the existence of a unique solution to an integral equation is proved in this paper. These results generalize and extend the results of [21, 22, 25] and the result of Sharma and Yuel [33] in partially ordered metric spaces.

2. Mathematical Preliminaries

The succeeding definitions are frequently used in our study.

Definition 2.1. [28] *The triple $(\mathscr{P}, \Im, \preceq)$ is called a partially ordered metric space, if $(\mathscr{P}, \preceq)$ is a partially ordered set together with $(\mathscr{P}, \Im)$ is a metric space.*

Definition 2.2. [28] *If $(\mathscr{P}, \Im)$ is a complete metric space, then triple $(\mathscr{P}, \Im, \preceq)$ is called complete partially ordered metric space.*

Definition 2.3. [29] *A point $\varrho \in A$, where A is a non-empty subset of a partially ordered set $(S, \preceq)$ is called a common fixed (coincidence) point of two self-mappings $\tilde{a}$ and $\wp$, if $\tilde{a}\varrho = \wp\varrho = \varrho$ $(\tilde{a}\varrho = \wp\varrho)$.*

Definition 2.4. [27] *The two self-mappings $\tilde{a}$ and $\wp$ defined over a subset A of a partially ordered set S are called commuting, if $\tilde{a}\wp\varrho = \wp\tilde{a}\varrho$ for all $\varrho \in A$.*

Definition 2.5. [30] *Two self-mappings $\tilde{a}$ and $\wp$ defined over $A \subset S$ are compatible, if for any sequence $\{\varrho_n\}$ with $\lim_{n\to+\infty} \tilde{a}\varrho_n = \lim_{n\to+\infty} \wp\varrho_n = \mu$ for some $\mu \in A$, then $\lim_{n\to+\infty} \Im(\wp\tilde{a}\varrho_n, \tilde{a}\wp\varrho_n) = 0$.*

Definition 2.6. [31] *Two self-mappings $\tilde{a}$ and $\wp$ defined over $A \subset S$ are said to be weakly compatible, if they commute only at their coincidence points (i.e., if $\tilde{a}\varrho = \wp\varrho$ then $\tilde{a}\wp\varrho = \wp\tilde{a}\varrho$).*

Definition 2.7. [32] *Let A be a non-empty subset of a partially ordered set $(S, \preceq)$. If every two elements of A are comparable then it is called well ordered set.*

Definition 2.8. [26] *Let $\tilde{a}$ and $\wp$ be two self-mappings defined over a partially ordered set $(\mathscr{P}, \preceq)$. A mapping $\wp$ is called monotone $\tilde{a}$-nondecreasing, if*

$$\tilde{a}\varrho \preceq \tilde{a}\ell \text{ implies } \wp\varrho \preceq \wp\ell, \; for \; all \; \varrho, \ell \in \mathscr{P}.$$

3. Main Results

We begin with the following coincidence theorem in this section.

Theorem 3.1. *Let $(\mathscr{P}, \Im, \preceq)$ be a complete partially ordered metric space. Suppose the mappings $\tilde{a}, \wp : \mathscr{P} \to \mathscr{P}$ are continuous, $\wp$ is a monotone $\tilde{a}$-nondecreasing, and $\wp(\mathscr{P}) \subseteq \tilde{a}(\mathscr{P})$ satisfies*

$$\Im(\wp\varrho, \wp\ell) \leq \alpha \frac{\Im(\tilde{a}\varrho, \wp\varrho)\left[1 + \Im(\tilde{a}\ell, \wp\ell)\right]}{1 + \Im(\tilde{a}\varrho, \tilde{a}\ell)} + \beta\left[\Im(\tilde{a}\varrho, \wp\varrho) + \Im(\tilde{a}\ell, \wp\ell)\right] + \gamma\Im(\tilde{a}\varrho, \tilde{a}\ell), \tag{3. 1}$$

for all ϱ, ℓ in $\mathscr{P}$ for which $\tilde{a}(\varrho) \neq \tilde{a}(\ell)$ are comparable and there exist $\alpha, \beta, \gamma \in [0, 1)$ such that $0 \leq \alpha + 2\beta + \gamma < 1$. If there exists $\varrho_0 \in \mathscr{P}$ such that $\tilde{a}\varrho_0 \preceq \wp\varrho_0$ and the mappings h and $\wp$ are compatible, then $\tilde{a}$ and $\wp$ have a coincidence point in $\mathscr{P}$.

Proof. Suppose there exists a point $\varrho_0 \in \mathscr{P}$ such that $\tilde{a}\varrho_0 \preceq \wp\varrho_0$. Form the hypotheses, choose a point $\varrho_1 \in \mathscr{P}$ such that $\tilde{a}\varrho_1 = \wp\varrho_0$. As $\wp\varrho_1 \in \tilde{a}(\mathscr{P})$, then there exists a point $\varrho_2 \in \mathscr{P}$ such that $\tilde{a}\varrho_2 = \wp\varrho_1$. Thus, repeating the same process, we obtain a sequence $\{\varrho_n\}$ in $\mathscr{P}$ such that $\tilde{a}\varrho_{n+1} = \wp\varrho_n$ for all $n \geq 0$.

Also, from the hypotheses we get $\tilde{a}\varrho_0 \preceq \wp\varrho_0 = \tilde{a}\varrho_1$ and then the monotone property of $\wp$ implies that $\wp\varrho_0 \preceq \wp\varrho_1$. As by the similar argument, we obtain that

$$\wp\varrho_0 \preceq \wp\varrho_1 \preceq \ldots \preceq \wp\varrho_n \preceq \wp\varrho_{n+1} \preceq \ldots .$$

Now, we distinguish the following two cases:

Case:1 If for some n, $\Im(\wp\varrho_n, \wp\varrho_{n+1}) = 0$ then $\wp\varrho_{n+1} = \wp\varrho_n$. Thus, $\wp\varrho_{n+1} = \wp\varrho_n = \tilde{a}\varrho_{n+1}$. Therefore, ϱ_{n+1} is a coincidence point of $\wp$ and $\tilde{a}$.

Case:2 If $\Im(\wp\varrho_n, \wp\varrho_{n+1}) \neq 0$ for all $n \in \mathbb{N}$, then from (3. 1), we have

$$\begin{aligned}\Im(\wp\varrho_{n+1}, \wp\varrho_n) \leq\; & \alpha \frac{\Im(\tilde{a}\varrho_{n+1}, \wp\varrho_{n+1})\left[1 + \Im(\tilde{a}\varrho_n, \wp\varrho_n)\right]}{1 + \Im(\tilde{a}\varrho_{n+1}, \tilde{a}\varrho_n)} \\ & + \beta\left[\Im(\tilde{a}\varrho_{n+1}, \wp\varrho_{n+1}) + \Im(\tilde{a}\varrho_n, \wp\varrho_n)\right] + \gamma\Im(\tilde{a}\varrho_{n+1}, \tilde{a}\varrho_n),\end{aligned}$$

this implies that

$$\begin{aligned}\Im(\wp\varrho_{n+1}, \wp\varrho_n) \leq\; & \alpha\Im(\wp\varrho_n, \wp\varrho_{n+1}) + \beta\left[\Im(\wp\varrho_n, \wp\varrho_{n+1}) + \Im(\wp\varrho_{n-1}, \wp\varrho_n)\right] \\ & + \gamma\Im(\wp\varrho_n, \wp\varrho_{n-1}).\end{aligned}$$

Therefore,

$$\Im(\wp\varrho_{n+1}, \wp\varrho_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) \Im(\wp\varrho_n, \wp\varrho_{n-1}).$$

Inductively, we obtain that

$$\Im(\wp\varrho_{n+1}, \wp\varrho_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right)^n \Im(\wp\varrho_1, \wp\varrho_0).$$

Let $k = \frac{\beta+\gamma}{1-\alpha-\beta} < 1$ and from triangular inequality for $m \geq n$, we have

$$\begin{aligned}\Im(\wp\varrho_m, \wp\varrho_n) &\leq \Im(\wp\varrho_m, \wp\varrho_{m-1}) + \Im(\wp\varrho_{m-1}, \wp\varrho_{m-2}) + ... + \Im(\wp\varrho_{n+1}, \wp\varrho_n)\\ &\leq \left(k^{m-1} + k^{m-2} + ... + k^n\right) \Im(\wp\varrho_1, \wp\varrho_0)\\ &\leq \frac{k^n}{1-k} \Im(\wp\varrho_1, \wp\varrho_0),\end{aligned}$$

as $m, n \to +\infty$, $\Im(\wp\varrho_m, \wp\varrho_n) \to 0$, this implies that $\{\wp\varrho_n\}$ is a Cauchy sequence in $\mathscr{P}$. Since $\mathscr{P}$ is complete then there exists some $\mu \in \mathscr{P}$ such that $\lim_{n\to+\infty} \wp\varrho_n = \mu$.

Further, the continuity of $\wp$ implies that

$$\lim_{n\to+\infty} \wp(\wp\varrho_n) = \wp\left(\lim_{n\to+\infty} \wp\varrho_n\right) = \wp\mu.$$

Since, $\tilde{a}\varrho_{n+1} = \wp\varrho_n$ then $\lim_{n\to+\infty} \tilde{a}\varrho_{n+1} = \mu$.

Further, from the compatibility of a pair of mappings $(\wp, \tilde{a})$, we have

$$\lim_{n\to+\infty} \Im(\wp\tilde{a}\varrho_n, \tilde{a}\wp\varrho_n) = 0.$$

Moreover, the triangular inequality of $\Im$, we have

$$\Im(\wp\mu, \tilde{a}\mu) = \Im(\wp\mu, \wp\tilde{a}\varrho_n) + \Im(\wp\tilde{a}\varrho_n, \tilde{a}\wp\varrho_n) + \Im(\tilde{a}\wp\varrho_n, \tilde{a}\mu).$$

On taking $n \to +\infty$ in the above inequality and from the continuity of $\wp$, $\tilde{a}$, we obtain that $\Im(\wp\mu, \tilde{a}\mu) = 0$, which implies that $\wp\mu = \tilde{a}\mu$. Hence, μ is a coincidence point. □

We have the following results as special cases of Theorem 3.1.

Corollary 3.2. *Let $(\mathscr{P}, \Im, \preceq)$ be a complete partially ordered metric space. The mappings $\tilde{a}, \wp : \mathscr{P} \to \mathscr{P}$ are continuous, $\wp$ is monotone $\tilde{a}$-nondecreasing, $\wp(\mathscr{P}) \subseteq \tilde{a}(\mathscr{P})$ satisfies the following contraction conditions*

(i).

$$\Im(\wp\varrho, \wp\ell) \leq \alpha \frac{\Im(\tilde{a}\varrho, \wp\varrho)\left[1 + \Im(\tilde{a}\ell, \wp\ell)\right]}{1 + \Im(\tilde{a}\varrho, \tilde{a}\ell)} + \beta\left[\Im(\tilde{a}\varrho, \wp\varrho) + \Im(\tilde{a}\ell, \wp\ell)\right],$$

for some $\alpha, \beta \in [0, 1)$ such that $0 \leq \alpha + 2\beta < 1$,

(ii).

$$\Im(\wp\varrho, \wp\ell) \leq \beta\left[\Im(\tilde{a}\varrho, \wp\varrho) + \Im(\tilde{a}\ell, \wp\ell)\right] + \gamma\Im(\tilde{a}\varrho, \tilde{a}\ell),$$

where $\beta, \gamma \in [0, 1)$ such that $0 \leq 2\beta + \gamma < 1$.

for all ϱ, ℓ in $\mathscr{P}$ for which $\tilde{a}(\varrho) \neq \tilde{a}(\ell)$ are comparable. If there exists a point $\varrho_0 \in \mathscr{P}$ such that $\tilde{a}\varrho_0 \preceq \wp\varrho_0$ and the mappings h and $\wp$ are compatible, then $\tilde{a}$ and $\wp$ have a coincidence point in $\mathscr{P}$.

Proof. The proof follows Theorem 3.1 by setting $\gamma = 0$ and $\alpha = 0$. □

Corollary 3.3. *Let $(\mathscr{P}, \Im, \preceq)$ be a complete partially ordered metric space. Suppose that $\wp : \mathscr{P} \to \mathscr{P}$ be a mapping such that for all comparable $\varrho, \ell \in \mathscr{P}$, the contraction condition in Theorem 3.1 (or Corollary 3.2) is satisfied.*

Assume that $\wp$ satisfies the following hypotheses:

(i). *$\wp$ is continuous,*
(ii). *$\wp(\wp\varrho) \preceq \wp\varrho$ for all $\varrho \in \mathscr{P}$.*

If there exists a point $\varrho_0 \in \mathscr{P}$ such that $\varrho_0 \preceq \wp\varrho_0$, then $\wp$ has a fixed point in $\mathscr{P}$.

Proof. Follow from Theorem 3.1 by taking $\tilde{a} = I_{\mathscr{P}}$ (the identity map). □

Relaxing the continuity criteria on $\wp$ in Theorem 3.1 is still valid by assuming the following hypothesis in $\mathscr{P}$:

$$\text{for any nondecreasing sequence } \{\varrho_n\} \subset S \text{ such that } \varrho_n \to \varrho \text{ then } \varrho_n \preceq \varrho \text{ for } n \in \mathbb{N}. \quad (3.\ 2)$$

Theorem 3.4. *In Theorem 3.1, assume that S satisfies the condition (3. 2). If $\tilde{a}(\mathscr{P})$ is a complete subset of $\mathscr{P}$, then $\wp$ and $\tilde{a}$ have a coincidence point in $\mathscr{P}$. Further, if $\wp$ and $\tilde{a}$ are weakly compatible, then $\wp$ and $\tilde{a}$ have a common fixed point in $\mathscr{P}$. Moreover, the set of common fixed points of $\wp$ and $\tilde{a}$ are well ordered if and only if $\wp$ and $\tilde{a}$ have one and only one common fixed point in $\mathscr{P}$.*

Proof. Assume that $\tilde{a}(\mathscr{P})$ is a complete subset of $\mathscr{P}$. From the proof of Theorem 3.1, we have a Cauchy sequence $\{\tilde{a}\varrho_n\} \subset \tilde{a}(\mathscr{P})$ and for some $\tilde{a}u \in \tilde{a}(\mathscr{P})$ such that

$$\lim_{n\to+\infty} \wp\varrho_n = \lim_{n\to+\infty} \tilde{a}\varrho_n = \tilde{a}u.$$

Notice that the sequences $\{\wp\varrho_n\}$ and $\{\tilde{a}\varrho_n\}$ are nondecreasing from which we get $\wp\varrho_n \preceq \tilde{a}u$ and $\tilde{a}\varrho_n \preceq \tilde{a}u$ and, also the monotone property of $\wp$ implies that $\wp\varrho_n \preceq \wp u$ for all n. Hence, by limiting case of it, we obtain that $\tilde{a}u \preceq \wp u$.

Suppose that $\tilde{a}u \prec \wp u$. Construct a sequence $\{u_n\} \subset S$ by $u_0 = u$ and $\tilde{a}u_{n+1} = \wp u_n$ for all $n \in \mathbb{N}$. From the proof of Theorem 3.1, the sequence $\{\tilde{a}u_n\}$ is nondecreasing and Cauchy sequence such that $\lim\limits_{n\to+\infty} \tilde{a}(u_n) = \lim\limits_{n\to+\infty} \wp u_n = \tilde{a}v$ for some $v \in \mathscr{P}$. Thus from the hypotheses, we have $\sup \tilde{a}u_n \preceq \tilde{a}v$ and $\sup \wp u_n \preceq \tilde{a}v$ for all $n \in \mathbb{N}$.

Therefore,

$$\tilde{a}\varrho_n \preceq \tilde{a}u \preceq \tilde{a}u_1 \preceq ... \preceq \tilde{a}u_n \preceq ... \leq \tilde{a}v.$$

Now, we have the following cases:
Case:1 If $\tilde{a}\varrho_{n_0} = \tilde{a}u_{n_0}$ for some $n_0 \geq 1$ then

$$\tilde{a}\varrho_{n_0} = \tilde{a}u = \tilde{a}u_{n_0} = \tilde{a}u_1 = \wp u.$$

Thus, u is a coincidence point of $\wp$ and $\tilde{a}$.
Case:2 For all $n \in \mathbb{N}$, $\tilde{a}\varrho_{n_0} \neq \tilde{a}u_{n_0}$ then from (3. 1), we have

$$\begin{aligned}\Im(\tilde{a}\varrho_{n+1}, \tilde{a}u_{n+1}) &= \Im(\wp\varrho_n, \wp u_n)\\ &\leq \alpha \frac{\Im(\tilde{a}\varrho_n, \wp\varrho_n)\,[1+\Im(\tilde{a}u_n, \wp u_n)]}{1+\Im(\tilde{a}\varrho_n, \tilde{a}u_n)} + \beta\,[\Im(\tilde{a}\varrho_n, \wp\varrho_n) + \Im(\tilde{a}u_n, \wp u_n)]\\ &\quad + \gamma\Im(\tilde{a}\varrho_n, \tilde{a}u_n).\end{aligned}$$

Taking limit as $n \to +\infty$ in the above inequality, we obtain that

$$\begin{aligned}\Im(\tilde{a}u, \tilde{a}v) &\leq \gamma\Im(\tilde{a}u, \tilde{a}v)\\ &< \Im(\tilde{a}u, \tilde{a}v), \text{ since } \gamma < 1.\end{aligned}$$

Therefore,

$$\tilde{a}u = \tilde{a}v = \tilde{a}u_1 = \wp u,$$

which shows that u is a coincidence point of $\wp$ and $\tilde{a}$.

Now, assume that $\wp$ and $\tilde{a}$ are weakly compatible and let w be a coincidence point. Then we have

$$\wp w = \wp\tilde{a}\mathring{b} = \tilde{a}\wp\mathring{b} = \tilde{a}w, \text{ since } w = \wp\mathring{b} = \tilde{a}\mathring{b}, \text{ for some } \mathring{b} \in S.$$

From (3. 1), we have

$$\begin{aligned}\Im(\wp\mathring{b}, \wp w) &\leq \alpha \frac{\Im(\tilde{a}\mathring{b}, \wp\mathring{b})\,[1+\Im(\tilde{a}w, \wp w)]}{1+\Im(\tilde{a}\mathring{b}, \tilde{a}w)} + \beta\left[\Im(\tilde{a}\mathring{b}, \wp\mathring{b}) + \Im(\tilde{a}w, \wp w)\right] + \gamma\Im(\tilde{a}\mathring{b}, \tilde{a}w)\\ &\leq \gamma\Im(\wp\mathring{b}, \wp w).\end{aligned}$$

As $\gamma < 1$, we get $\Im(\wp\mathring{b}, \wp w) = 0$ this implies that $\wp\mathring{b} = \wp w = \tilde{a}w = w$. Therefore, w is a common fixed point of $\wp$ and $\tilde{a}$.

Lastly, assume that the set of common fixed points of $\wp$ and $\tilde{a}$ is well ordered. Next, to show that the common fixed point of $\wp$ and $\tilde{a}$ is unique. Let $u \neq v$ be two common fixed points of $\wp$ and $\tilde{a}$. Then from (3. 1), we have

$$\begin{aligned}\Im(u, v) &\leq \alpha \frac{\Im(\tilde{a}u, \wp u)\,[1+\Im(\tilde{a}v, \wp v)]}{1+\Im(\tilde{a}u, \tilde{a}v)} + \beta\,[\Im(\tilde{a}u, \wp u) + \Im(\tilde{a}v, \wp v)] + \gamma\Im(\tilde{a}u, \tilde{a}v)\\ &\leq \gamma\,\Im(u, v)\\ &< \Im(u, v), \text{ since } \gamma < 1,\end{aligned}$$

which is a contradiction. Thus, $u = v$. Conversely, suppose $\wp$ and $\tilde{a}$ have only one common fixed point then the set of common fixed points of $\wp$ and $\tilde{a}$ being a singleton is well ordered. □

Besides, in Corollary 3.2 relaxing the continuity criteria on $\wp$ and satisfying the hypothesis of Theorem 3.4, one can obtains the coincidence point, common fixed point and its uniqueness of $\wp$ and h in $\mathscr{P}$.

Corollary 3.5. *Let $(\mathscr{P}, \Im, \preceq)$ be a complete partially ordered metric space. Suppose that $\wp : \mathscr{P} \to \mathscr{P}$ be a mapping such that for all comparable $\varrho, \ell \in \mathscr{P}$, the contraction condition* (3. 1) *is satisfied.*

Suppose that the following hypotheses are satisfied (i). *if* $\{\varrho_n\}$ *is a nondecreasing sequence in* S *with respect to* $\preceq$ *such that* $\varrho_n \to \varrho \in \mathscr{P}$ *as* $n \to +\infty$*, then* $\varrho_n \preceq \varrho$*, for all* $n \in \mathbb{N}$ *and*

(ii). $\wp(\wp\varrho) \preceq \wp\varrho$ *for all* $\varrho \in \mathscr{P}$.

If there exists a point $\varrho_0 \in \mathscr{P}$ *such that* $\varrho_0 \preceq \wp\varrho_0$*, then* $\wp$ *has a fixed point in* $\mathscr{P}$.

Proof. Follow from Theorem 3.4 by taking $h = I_{\mathscr{P}}$ (the identity map). □

Some other consequences of the main result for the self mappings involving in the integral type contractions are as follows:

Let Θ denote the set of all functions $\zeta : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(a). each ζ is Lebesgue integrable function on every compact subset of $[0, +\infty)$ and

(b). for any $\epsilon > 0$, we have $\int_0^\epsilon \zeta(t)dt > 0$ for $t \in [0, +\infty)$.

Corollary 3.6. *Let* $(\mathscr{P}, \Im, \preceq)$ *be a complete partially ordered metric space. Suppose that the mappings* $\wp, \tilde{a} : \mathscr{P} \to \mathscr{P}$ *are continuous,* $\wp$ *is a monotone* $\tilde{a}$*-nondecreasing,* $\wp(\mathscr{P}) \subseteq \tilde{a}(\mathscr{P})$ *satisfies*

$$\begin{aligned} \int_0^{\Im(\wp\varrho,\wp\ell)} \zeta(t)dt \leq \alpha \int_0^{\frac{\Im(\tilde{a}\varrho,\wp\varrho)[1+\Im(\tilde{a}\ell,\wp\ell)]}{1+\Im(\tilde{a}\varrho,\tilde{a}\ell)}} \zeta(t)dt + \beta \int_0^{\Im(\tilde{a}\varrho,\wp\varrho)+\Im(\tilde{a}\ell,\wp\ell)} \zeta(t)dt \\ + \gamma \int_0^{\Im(\tilde{a}\varrho,\tilde{a}\ell)} \zeta(t)dt, \end{aligned} \tag{3. 3}$$

for all ϱ, ℓ *in* $\mathscr{P}$ *for which* $\tilde{a}\varrho \neq \tilde{a}\ell$ *are comparable,* $\zeta \in \Theta$ *and where* $\alpha, \beta, \gamma \in [0, 1)$ *such that* $0 \leq \alpha + 2\beta + \gamma < 1$. *If there exists a point* $\varrho_0 \in \mathscr{P}$ *such that* $\tilde{a}\varrho_0 \preceq \wp\varrho_0$ *and the mappings* $\tilde{a}$ *and* $\wp$ *are compatible, then* $\tilde{a}$ *and* $\wp$ *have a coincidence point in* $\mathscr{P}$.

We have the following consequences from Corollary 3.6 by setting $\gamma = 0$ and $\beta = 0$.

Corollary 3.7. *Let* $(\mathscr{P}, \Im, \preceq)$ *be a complete partially ordered metric space. Suppose that the self-mappings* $\tilde{a}, \wp$ *on* $\mathscr{P}$ *are continuous,* $\wp$ *is a monotone* $\tilde{a}$*-nondecreasing,* $\wp(\mathscr{P}) \subseteq \tilde{a}(\mathscr{P})$ *such that*

$$\int_0^{\Im(\wp\varrho,\wp\ell)} \zeta(t)dt \leq \alpha \int_0^{\frac{\Im(\tilde{a}\varrho,\wp\varrho)[1+\Im(\tilde{a}\ell,\wp\ell)]}{1+\Im(\tilde{a}\varrho,\tilde{a}\ell)}} \zeta(t)dt + \beta \int_0^{\Im(\tilde{a}\varrho,\wp\varrho)+\Im(\tilde{a}\ell,\wp\ell)} \zeta(t)dt, \tag{3. 4}$$

for all ϱ, ℓ *in* $\mathscr{P}$ *for which* $\tilde{a}\varrho \neq \tilde{a}\ell$ *are comparable,* $\zeta \in \Theta$ *and for some* $\alpha, \beta \in [0, 1)$ *with* $0 \leq \alpha + 2\beta < 1$. *If there exists a point* $\varrho_0 \in \mathscr{P}$ *such that* $\tilde{a}\varrho_0 \preceq \wp\varrho_0$ *and the mappings* $\wp$ *and* $\tilde{a}$ *are compatible, then* $\tilde{a}$ *and* $\wp$ *have a coincidence point in* $\mathscr{P}$.

Corollary 3.8. *Let* $(\mathscr{P}, \Im, \preceq)$ *be a complete partially ordered metric space. Suppose that the self-mappings* $\tilde{a}, \wp$ *on* $\mathscr{P}$ *are continuous,* $\wp$ *is a monotone* $\tilde{a}$*-nondecreasing,* $\wp(\mathscr{P}) \subseteq \tilde{a}(\mathscr{P})$ *such that*

$$\int_0^{\Im(\wp\varrho,\wp\ell)} \zeta(t)dt \leq \beta \int_0^{\Im(\tilde{a}\varrho,\wp\varrho)+\Im(\tilde{a}\ell,\wp\ell)} \zeta(t)dt + \gamma \int_0^{\Im(\tilde{a}\varrho,\tilde{a}\ell)} \zeta(t)dt, \tag{3. 5}$$

for all ϱ, ℓ *in* $\mathscr{P}$ *for which* $\tilde{a}\varrho \neq \tilde{a}\ell$ *are comparable,* $\zeta \in \Theta$ *and for some* $\beta, \gamma \in [0, 1)$ *with* $0 \leq 2\beta + \gamma < 1$. *If there exists a point* $\varrho_0 \in \mathscr{P}$ *such that* $\tilde{a}\varrho_0 \preceq \wp\varrho_0$ *and the mappings* $\wp$ *and* $\tilde{a}$ *are compatible, then* $\tilde{a}$ *and* $\wp$ *have a coincidence point in* $\mathscr{P}$.

Now, we give the examples for the main Theorem 3.1.

Example 3.9. *Define a metric* $d : \mathscr{P} \times \mathscr{P} \to [0, +\infty)$ *by* $\Im(\varrho, \ell) = |\varrho - \ell|$*, where* $\mathscr{P} = [0, 1]$ *with usual order* $\leq$*. Let* $\wp$ *and* $\tilde{a}$ *be two self mappings on* $\mathscr{P}$ *such that* $\wp\varrho = \frac{\varrho^2}{2}$ *and* $\tilde{a}\varrho = \frac{2\varrho^2}{1+\varrho}$*, then* $\wp$ *and* $\tilde{a}$ *have a coincidence point in* $\mathscr{P}$*.*

Proof. Note that $(\mathscr{P}, \Im)$ is a complete metric space and thus, $(\mathscr{P}, \Im, \leq)$ be a complete partially ordered metric space with respect to usual order $\leq$. Let $\varrho_0 = 0 \in \mathscr{P}$ then $\tilde{a}\varrho_0 \leq \wp\varrho_0$ and also note that $\wp$ and $\tilde{a}$ are continuous, $\wp$ is monotone $\tilde{a}$-nondecreasing and $\wp(\mathscr{P}) \subseteq \tilde{a}(\mathscr{P})$.

Now consider the following for any ϱ, ℓ in $\mathscr{P}$ with $\varrho < \ell$,

$$\begin{aligned}
\Im(\wp\varrho, \wp\ell) = \frac{1}{2}|\varrho^2 - \ell^2| = \frac{1}{2}(\varrho + \ell)|\varrho - \ell| &\leq \frac{2(\varrho + \ell + \varrho\ell)}{(1+\varrho)(1+\ell)}|\varrho - \ell| \\
&\leq \alpha \frac{2\varrho^2|3-\varrho|\left[(1+\ell) + \ell^2|3-\ell|\right]}{4(1+\varrho)(1+\ell) + 2|\varrho - \ell|(\varrho + \ell + \varrho\ell)} \\
&\quad + \frac{\beta}{2}\frac{\varrho^2(1+\ell)|\varrho - 3| + \ell^2(1+\varrho)|\ell - 3|}{(1+\varrho)(1+\ell)} + \gamma\frac{2(\varrho + \ell + \varrho\ell)}{(1+\varrho)(1+\ell)}|\varrho - \ell| \\
&\leq \alpha \frac{\frac{\varrho^2|\varrho-3|}{2(1+\varrho)} \cdot \frac{2(1+\ell)+\ell^2|3-\ell|}{2(1+\ell)}}{1 + \frac{2|\varrho-\ell|(\varrho+\ell+\varrho\ell)}{(1+\varrho)(1+\ell)}} + \beta\left[\frac{\varrho^2|\varrho-3|}{2(1+\varrho)} + \frac{\ell^2|\ell-3|}{2(1+\ell)}\right] \\
&\quad + \gamma\frac{2(\varrho + \ell + \varrho\ell)}{(1+\varrho)(1+\ell)}|\varrho - \ell| \\
&\leq \alpha \frac{\Im(\tilde{a}\varrho, \wp\varrho)\left[1 + \Im(\tilde{a}\ell, \wp\ell)\right]}{1 + \Im(\tilde{a}\varrho, \tilde{a}\ell)} + \beta\left[\Im(\tilde{a}\varrho, \wp\varrho) + \Im(\tilde{a}\ell, \wp\ell)\right] + \gamma\Im(\tilde{a}\varrho, \tilde{a}\ell).
\end{aligned}$$

Then, the contraction condition in Theorem 3.1 holds by selecting proper values of α, β, γ in $[0, 1)$ such that $0 \leq \alpha + 2\beta + \gamma < 1$. Therefore, $\wp$ and $\tilde{a}$ have a coincidence point $0 \in \mathscr{P}$. □

Example 3.10. *Define a distance function* $\Im : \mathscr{P} \times \mathscr{P} \to [0, +\infty)$ *by* $\Im(\varrho, \ell) = |\varrho - \ell|$*, where* $\mathscr{P} = [0, 1]$ *with usual order* $\leq$*. Let* $\wp$ *and* $\tilde{a}$ *be two self mappings on* $\mathscr{P}$ *such that* $\wp\varrho = \varrho^3$ *and* $\tilde{a}\varrho = \varrho^4$*, then* $\wp$ *and* $\tilde{a}$ *have two coincidence points* $0, 1$ *in* $\mathscr{P}$ *with* $\varrho_0 = \frac{1}{4}$*.*

4. Applications

Now our aim is to give an existence theorem for a solution of the following integral equation.

$$\hbar(\varrho) = \int_0^M \mu(\varrho, \ell, \hbar(\ell))d\ell + g(\varrho), \ \varrho \in [0, M], \tag{4. 6}$$

where $M > 0$. Let $\mathscr{P} = C[0, M]$ be the set of all continuous functions defined on $[0, M]$. Now, define $\Im : \mathscr{P} \times \mathscr{P} \to \mathbb{R}^+$ by

$$\Im(u, v) = \sup_{\varrho \in [0,M]} \{|u(\varrho) - v(\varrho)|\}$$

then, $(\mathscr{P}, \leq)$ is a partially ordered set. Now, we prove the following result.

Theorem 4.1. *Suppose the following hypotheses holds:*

(i). $\mu : [0,M] \times [0,M] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous,*

(ii). *for each* $\varrho, \ell \in [0,M]$, *we have*

$$\mu(\varrho, \ell, \int_0^M \mu(\varrho, \mathring{b}, \hbar(\mathring{b}))d\mathring{b} + g(\varrho)) \le \mu(\varrho, \ell, \hbar(\ell)),$$

(iii). *there exists a continuous function* $N : [0,M] \times [0,M] \to [0,+\infty]$ *such that*

$$|\mu(\varrho, \ell, a) - \mu(\varrho, \ell, b)| \le N(\varrho, \ell)|a - b| \text{ and}$$

(iv).

$$\sup_{\varrho \in [0,M]} \int_0^M N(\varrho, \ell)d\ell \le \gamma$$

for some $\gamma < 1$. *Then, the integral equations* (4. 6) *has a solution* $a \in C[0,M]$.

Proof. Define $P : C[0,M] \to C[0,M]$ by

$$\wp w(\varrho) = \int_0^M \mu(\varrho, \ell, w(\varrho))d\varrho + g(\varrho), \ \varrho \in [0,M].$$

Now, we have

$$\begin{aligned}
\wp(\wp w(\varrho)) &= \int_0^M \mu(\varrho, \ell, \wp w(\varrho))d\varrho + g(\varrho) \\
&= \int_0^M \mu(\varrho, \ell, \int_0^M \mu(\varrho, \mathring{b}, w(\mathring{b}))d\mathring{b} + g(\varrho))d\varrho + g(\varrho) \\
&\le \int_0^M \mu(\varrho, \ell, w(\mathring{b}))d\mathring{b} + g(\varrho) \\
&= \wp w(\varrho)
\end{aligned}$$

Thus, we have $\wp(\wp\varrho) \le \wp\varrho$ for all $\varrho \in C[0,M]$. For any $\varrho^*, \ell^* \in C[0,M]$ with $\varrho \le \ell$, we have

$$\begin{aligned}
\Im(\wp\varrho^*, \wp\ell^*) &= \sup_{\varrho \in [0,M]} |\wp\varrho^*(\varrho) - \wp\ell^*(\ell)| \\
&= \sup_{\varrho \in [0,M]} |\int_0^M \mu(\varrho, \ell, \varrho^*(\varrho)) - \mu(\varrho, \ell, \ell^*(\varrho))d\varrho| \\
&\le \sup_{\varrho \in [0,M]} |\int_0^M \mu(\varrho, \ell, \varrho^*(\varrho)) - \mu(\varrho, \ell, \ell^*(\varrho))|d\varrho \\
&\le \sup_{\varrho \in [0,M]} |\int_0^M N(\varrho, \ell)|\varrho^*(\varrho) - \ell^*(\varrho)|d\varrho \\
&\le \sup_{\varrho \in [0,M]} |\varrho^*(\varrho) - \ell^*(\varrho)| \sup_{\varrho \in [0,M]} \int_0^M N(\varrho, \ell)d\varrho \\
&= \Im(\varrho^*, \ell^*) \sup_{\varrho \in [0,M]} \int_0^M N(\varrho, \ell)d\varrho \\
&\le \gamma \Im(\varrho^*, \ell^*).
\end{aligned}$$

Moreover, $\{\varrho_n^*\}$ is a nondecreasing sequence in $C[0, M]$ such that $\varrho_n^* \to \varrho^*$, then $\varrho_n^* \leq \varrho^*$ for all $n \in \mathbb{N}$. Thus all the required hypotheses of Corollary 3.5 are satisfied. Thus, there exists a solution $a \in [0, M]$ of the integral equation (4. 6). □

5. CONCLUSION

In this work, we have proved some coincidence point results for two self mappings satisfying a generalized rational contraction condition in a complete partially ordered metric space. Furthermore, some consequences of the main result in terms of integral type contractions are given in the same context. Two numerical examples are presented to justify the main result. In the last, one application of integral equation in terms of obtaining the unique solution is discussed using the main result.

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