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Statistical Inference for the Parameter of Rayleigh Distribution Through Fuzzy Membership Function

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Abstract. There are number of inference techniques that can be used for the estimation of the unknown parameters which are based on precise crisp data, but there are many situations where we deal with imprecise and vague data. In this situation the classical mathematical tools cannot help us to estimate the parameters. This impreciseness can be covered by introducing fuzzy concepts. This study deals with the maximum likelihood estimation for the parameters of Rayleigh distribution using Newton Raphson algorithm. A real-life data set analysis is presented by considering set of luminous intensity of light emitting diodes and finding of this study illustrates that proposed inferential technique is useful to deal with fuzziness of data when statistical inference of Rayleigh distribution is carried out for imprecise or fuzzy data.

Key Words: Maximum likelihood estimation, Parameter, Rayleigh distribution, Fuzzy data, Membership function.

1. Introduction

Rayleigh distribution is a continuous probability distribution for non negative random variables and considered as an appropriate model to calculate the lifetime of the units. This distribution plays significant role in real life applications as it relates to a number of distributions such as generalized extreme value, Weibull and Chi-square distributions.Now a days, Rayleigh distribution has a wide range of applications in reliability theory and also in survival analysis. It is also applicable in communication engineering, clinical studies and applied statistics due to its reliable and consistent results. It is specified by the probability density function as,

$$f_Z(z) = \begin{cases} \frac{z}{\alpha^2} e^{-\left(\frac{z^2}{\alpha^2}\right)}, & \alpha > 0, z > 0 \\ 0, & z < 0. \end{cases} \tag{1.1}$$

where α denotes the parameter of Rayleigh distribution and z is a random variable from Rayleigh distribution. Similarly, its distribution function is given as,

$$F_Z(z) = 1 - e^{-\left(\frac{z^2}{2\alpha^2}\right)}, \tag{1.2}$$

Many authors have studied inferential issues of Rayleigh distribution for different kind of data sets in which Siddiqui [13] studied inference problem related to Rayleigh distribution which includes estimation and hypothesis testing from a set of observation of radio signals, Dey and Das [3] predicted interval for a Rayleigh distribution using Bayesian approach, Soliman and Aboud [14] studied Bayesian inference using record values from Rayleigh model with application, Asgharzade and Azizpour [1] studied Bayesian inference for Rayleigh distribution under hybrid censoring, Liao and Gui [5] suggested the statistical inference of the Rayleigh distribution based on progressively type II censored competing risks data and Dey and Das [4] studied the statistical inference of the Rayleigh distribution under type II progressive censoring with binomial removals. The Rayleigh distribution has proved very significant results while studying data censoring schemes, common in most life testing experiments but in our daily life we often have to dealt with some measurements which are sometimes precise and sometimes not. In the situation of such kind of impreciseness, the crisp mathematical tools cannot be utilized to obtain effective results, for example data of height, weight, human vision, emotions, weather and the lifetime of an object cannot be determined precisely because certain kind of fuzziness or impreciseness is always present in these variables. To model this impreciseness fuzzy logic is used. This paper also emphasize on developing inferential technique for parameter of Rayleigh distribution when data is imprecise or fuzzy so that significant result could be obtained from real life applications of Rayleigh distribution when data is not available in precise form. Contrary to classical logic, fuzzy logic is defined as a multi-valued logic in which the truth lies in the interval of 0 and 1, where 0 represents completely false and 1 represents completely true. Classical logic deals only with two values and these two values are either true or false but fuzzy logic is a many-valued logic in which truth is represented in a closed interval $\left[0, 1\right]$ and the values in this interval shows different degrees of truth. As in classical set theory, characteristic functions are used for the representation of crisp sets, similarly membership functions are used for the representation of fuzzy sets and are defined as a possibility distribution for fuzzy sets. The most commonly used membership functions are trapezoidal and triangular membership function.

Several researches have been carried out for the estimation of the parameters when the nature of the data was fuzzy. Shafiq [12] considered the Bayesian as well as classical estimation techniques for Pareto distribution for fuzzy life time observations. Pak [11] applied Maximum Likelihood estimation (MLE) and Bayesian estimation for the parameters of Lindley distribution when the data was fuzzy. Merovci and Elbatal [6] proposed theory and applications of a three parameter model called Weibull Rayleigh distribution. Pak et al. [9, 10] used different methods to estimate the parameters of Rayleigh distribution in case of type II fuzzy censored data and presented the parameter estimation and reliability function of Rayleigh distribution by using Bayesian approach with fuzzy life time data. Noyanim and Ugomma [7] studied different methods for the parameter estimation of Weibull distribution. Similarly, Basharat et al. [2] presented the inference for the distribution of linear combination of two independent exponential random variables using fuzzy data set.

The main objective of this paper is to develop the inference technique for the Rayleigh distribution when the data is available in the form of fuzzy sets. The rest of the paper is organized as, section 2 consists of calculation of maximum likelihood function and Newton Raphson algorithm by using fuzzy logic. In section 3 life time data of light emitting diodes have been used for illustration. Some concluding remarks are given in section 4.

2. Fuzzy Data and Maximum Likelihood Function

Maximum likelihood estimation is a method which is used for the estimation of unknown parameters of the distribution from a given data. In this method each possible value which a parameter can have is considered and then for every possible value, the respective probabilities are calculated. Maximum likelihood estimation selects those values of parameters at which the likelihood function is maximum. In our present study, we have obtained the maximum likelihood estimate from the given set of fuzzy data.

Suppose the information about the random variable z is known exactly, the complete data maximum likelihood estimation of Rayleigh distribution can be obtained as,

$$L(\alpha) = \left(\frac{1}{\alpha^2}\right)^n \prod_{i=1}^{n} z_i e^{\left(\frac{z^2}{\alpha^2}\right)}, \tag{2.1}$$

Consider when the existing information about z is not known exactly, but it may rather encoded as fuzzy numbers with $\tilde{Z}_1, \tilde{Z}_2, ..., \tilde{Z}_n$ and their corresponding membership functions can be represented as $\mu_{\tilde{Z}_1}(.), \mu_{\tilde{Z}_2}(.), ..., \mu_{\tilde{Z}_n}(.)$.

The joint possibility distribution for all given membership functions can be represented as,

$$\mu_{\tilde{Z}}(z) = \mu_{\tilde{Z}_1}(z) \times \mu_{\tilde{Z}_2}(z) \times ... \times \mu_{\tilde{Z}_n}(z) \tag{2.2}$$

By assuming that the joint membership function can be decomposed as in equation (2.2), the log likelihood of the observed data can be obtained as,

$$L(\alpha) = \int f(z;\alpha)\mu_{\tilde{Z}_i}(z)dz, \tag{2.3}$$

$$L(\alpha) = \left(\frac{1}{\alpha^2}\right)^n \prod_{i=1}^{n} \int z e^{\left(\frac{z^2}{\alpha^2}\right)} \mu_{\tilde{Z}_i}(z)dz,$$

Similarly, the log likelihood function of the observed data is represented as,

$$\log L(\alpha) = -2n \log \alpha + \sum_{i=1}^{n} \log \int z e^{\left(\frac{z^2}{\alpha^2}\right)} \mu_{\tilde{Z}_i}(z)dz\ . \tag{2.4}$$

By maximizing the log likelihood function in equation(2.4) and equating it to zero, the unknown parameter can be obtained as

$$\frac{\partial \log L(\alpha)}{\partial \alpha} = \frac{-2n}{\alpha} + \sum_{i=1}^{n} \left[\frac{\int \frac{z^3}{\alpha^3} e^{\left(\frac{-z^2}{2\alpha^2}\right)} \mu_{\tilde{Z}_i}(z)dz}{\int z e^{\left(\frac{-z^2}{2\alpha^2}\right)} \mu_{\tilde{Z}_i}(z)dz} \right] = 0\ . \tag{2.5}$$

Since obtaining the maximum likelihood estimate of the parameter from equation (2.5) is rather complex as no closed form of the solution to the given likelihood equation can be obtained. Therefore, an iterative method can be used for the estimation of the parameter. In this context Newton Raphson algorithm has been used to estimate the parameter.

2.1. **Newton Raphson algorithm.** It is an iterative method which is employed to find the precise approximations of the root. It consists of a function, its derivative and an initial guess. In this algorithm, the solution of the likelihood equation is obtained through an iterative procedure.

Let $\hat{\alpha}$ be the root to estimate, then the Newton Raphson algorithm is given as,

$$\hat{\alpha}^{(n+1)} = \alpha^n - \frac{\frac{\partial L(\alpha)}{\partial \alpha}}{\frac{\partial L^2(\alpha)}{\partial \alpha^2}}\ . \tag{2.6}$$

The second order partial derivative w.r.t α is required for the computation of Newton Raphson method which is obtained as follows

$$\frac{\partial^2 L(\alpha;\tilde{z})}{\partial \alpha^2} = \frac{-2n}{\alpha^2} + \sum_{i=1}^{n} \left\{ \frac{\int \left(\frac{z^2}{\alpha^2} - 3\right) \frac{z^3}{\alpha^4} e^{\left(\frac{-z^2}{2\alpha^2}\right)} \mu_{\tilde{Z}_i}(z)dz}{\int \frac{z^3}{\alpha^3} e^{\left(\frac{-z^2}{2\alpha^2}\right)} \mu_{\tilde{Z}_i}(z)dz} - \left[\frac{\int \frac{z^3}{\alpha^3} e^{\left(\frac{-z^2}{2\alpha^2}\right)} \mu_{\tilde{Z}_i}(z)dz}{\int \frac{z^3}{\alpha^3} e^{\left(\frac{-z^2}{2\alpha^2}\right)} \mu_{\tilde{Z}_i}(z)dz} \right]^2 \right\} . \tag{2.7}$$

This iteration procedure continues until the convergence obtained.

3. Application Example

For the numerical illustration, a case study of light emitting diodes manufacturing process was considered, where the focus of manufacturing process was to observe the luminous amounts of light emitting diodes. We have considered the data set of 15 light emitting diodes which was originally used by Pak et al. [8]. The luminous intensity of a each light emitting diode contain some amount of impreciseness, so the luminous amounts of light emitting diodes (LED) were presented in the form of lower and upper bounds as well as a point estimate, which are represented as,

(2.163, 2.738, 3.068), (5.972, 6.353, 8.150), (1.032, 1.971, 2.642),
(0.628, 0.964, 1.735), (2.995, 3.442, 5.066), (3.766, 5.814, 6.212),
(0.974, 1.839, 2.045), (4.352, 5.206, 5.988), (3.920, 4.762, 6.121),
(1.375, 2.195, 3.086), (0.618, 0.839, 2.217), (4.575, 6.050, 6.734),
(1.027, 1.218, 3.116), (6.279, 8.156, 9.435), (2.821, 3.409, 5.272).

From the above data of fifteen light emitting diodes, each triplet represents a triangular fuzzy number $\tilde{Z}_i$ and known as a possibility distribution associated with an unknown value of $\tilde{Z}_i$. We have applied Newton Raphson algorithm to obtain the maximum likelihood estimates. The membership function for each triplet of fuzzy number has been calculated separately by using triangular membership function which is given as

$$\mu_{\tilde{z}_i}(z) = \begin{cases} 0, & z \leq p \\ \frac{z-p}{q-p}, & p \leq z \leq q \\ \frac{r-z}{r-q}, & q \leq z \leq r \\ 0, & z \geq r \end{cases}$$

where p, q and r are three parameters of triangular fuzzy number. For the given data set, triangular membership function for first fuzzy number $\widetilde{z_1} = (2.163, 2.738, 3.068)$ is constructed as;

$$\mu_{\tilde{z}_i}(z) = \begin{cases} 0, & z \leq 2.163 \\ \frac{z-2.163}{2.738-2.163}, & 2.163 \leq z \leq 2.738 \\ \frac{3.068-z}{3.068-2.738}, & 2.738 \leq z \leq 3.068 \\ 0, & z \geq 3.068 \end{cases} \tag{3. 1}$$

Similarly, all the membership functions of remaining triangular fuzzy numbers have been calculated. After construction of membership functions, the maximum likelihood estimates of the unknown parameter of Rayleigh distribution are calculated by using Newton Raphson algorithm. The parameter is estimated by using different values of scale parameter α as an initial guess. The initial guesses used in calculations are $\alpha = 5, \alpha = 4$ and $\alpha = 3$. Iterations in Newton Raphson algorithm have been repeated till convergence of estimates. The required computations are obtained by using MATLAB(R 2014) software. The numerical results are given in table 1.

Table 1: Estimates of α for different initial guess

N	α	Estimate of α
15	5	2.992
15	4	2.9927
15	3	2.993

The above table 1 shows that initial guesses $\alpha = 5, \alpha = 4$ and $\alpha = 3$ yields the approximately same value of the estimate for the parameter of Rayleigh distribution for the given fuzzy data set of luminous intensity of light emitting diodes.

The figure 1 and 2 below represent the probability density function and cumulative distribution function of Rayleigh distribution at estimated values of parameter $\alpha = 2.992, 2.9927$ and 2.993. It is shown from the figure 1 and 2 that Rayleigh distribution has positively skewed shape for all the values of parameter α and cumulative distribution rises to unity with increasing in values of Rayleigh random variable z.

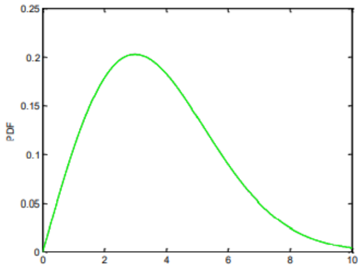

Figure 1: Probability density function of Rayleigh distribution at $\alpha = 2.992, 2.9927$ and 2.993.

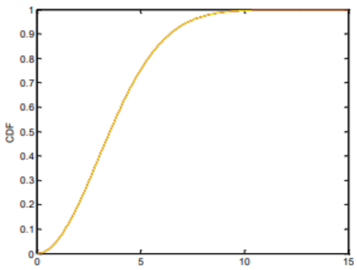

Figure 2: Cumulative distribution function of Rayleigh distribution at $\alpha = 2.992, 2.9927$ and 2.993.

4. Conclusion

There are a number of classical techniques that are used to deal with precise information, but in many situations the exact and precise measurements cannot be obtained. This impreciseness can be covered by introducing fuzzy concepts in other mathematical techniques. This study illustrates the application of Rayleigh distribution for imprecise or fuzzy data. The Rayleigh distribution has been used to obtain the estimate of unknown parameter in case of fuzzy data. The estimates of the unknown parameter have been obtained by using maximum likelihood estimation method via Newton Raphson algorithm. As a Rayleigh distribution has a wide range of applications in reliability theory, survival analysis, physical sciences and is considered as an effective distribution to model lifetime data. In all these filed, Rayleigh distribution generates very reliable estimates for precise or complete data sets using ordinary inferential classical techniques but the problem arises when the data is not precise or exact and these classical inferential techniques are not suitable to produce significant results. Therefore to deal with such kind of situations, our proposed inference technique for Rayleigh distribution served as an effective tool to obtain considerable conclusions from a variety of incomplete data problems where Rayleigh distribution is considered as an appropriate probability distribution.

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